

Optimal storage capacity of quantum Hopfield neural networks

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Quantum neural networks form one pillar of the emergent field of quantum machine learning. Here quantum generalizations of classical networks realizing associative memories—capable of retrieving patterns, or memories, from corrupted initial states—have been proposed. It is a challenging open problem to analyze quantum associative memories with an extensive number of patterns and to determine the maximal number of patterns the quantum networks can reliably store, i.e., their storage capacity. In this work, we propose and explore a general method for evaluating the maximal storage capacity of quantum neural network models. By generalizing what is known as Gardner’s approach in the classical realm, we exploit the theory of classical spin glasses for deriving the optimal storage capacity of quantum networks with quenched pattern variables. As an example, we apply our method to an open-system quantum associative memory formed of interacting spin-1/2 particles realizing coupled artificial neurons. The system undergoes a Markovian time evolution resulting from a dissipative retrieval dynamics that competes with a coherent quantum dynamics. We map out the nonequilibrium phase diagram and study the effect of temperature and Hamiltonian dynamics on the storage capacity. Our method opens an avenue for a systematic characterization of the storage capacity of quantum associative memories.

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I. INTRODUCTION

Neural networks (NNs) constitute a powerful machine-learning paradigm to solve computationally demanding tasks [1–4], ranging from pattern recognition to deep learning. Motivated by the advancements in quantum information and quantum many-body systems there has been an increasing interest in designing and characterizing *quantum* NNs [5], which form a backbone of quantum machine learning [6]. Research efforts to harness the potential power of quantum NNs are focusing on quantum algorithms and quantum circuit settings [7–17], e.g., through so-called feed-forward quantum NNs, as well as on many-body physics and condensed matter scenarios. For instance, suitably tailored spin-boson systems have been analyzed for the accomplishment of NNs tasks [18–24].

Our work focuses on Hopfield-type NNs [25] generalized via *open* quantum systems [26,27]. Classical Hopfield-type NNs can implement associative memories and belong to the class of attractor NNs, which can be modeled as classical spin systems subject to thermal fluctuations. The retrieval mechanism follows a classical nonequilibrium dynamics, memories correspond to stationary solutions of a stochastic dynamics,

and patterns are written in the interconnections among neurons via proper learning prescriptions [28].

A key figure of merit of Hopfield-type NNs is the *storage capacity*, $\alpha \equiv p/N$, quantifying the number p of patterns that can be stored in a network of N constituents. The storage capacity varies, depending on how the interconnections among neurons are parametrized in terms of the patterns, i.e., depending on the chosen learning rule. For example, adopting the widely known Hebb’s prescription and random patterns, the NN can store on the order of $0.138N$ memories [29]; for models with very correlated patterns, some prescriptions permit storing order of $N^2/\ln(N^2)$ patterns. The problem of evaluating the optimal storage capacity has been formulated by Gardner and coworkers in a series of seminal works [30–33] for deterministic dynamics and subsequently generalized for stochastic dynamics [34,35]. The paradigm, often referred to as *Gardner’s program*, consists in requiring a set of patterns to be stationary solutions of the dynamics, while the learning rule is left as free parameter. The quantity of interest is the typical volume of parameter space fulfilling the condition of retrieval. This represents a statistical mechanics model that can be tackled via, e.g., spin glass techniques. By combining Gardner’s program with the general framework of quantum maps [36], recent findings formally show the possibility of a quantum advantage in the storage capacity of quantum NNs over their classical counterparts. However, applications to specific models do not provide conclusive results with regard to enhanced quantum storage advantage [37–39], and a general method to derive the storage capacity of concrete instances of quantum Hopfield-type NN families is lacking. In this paper we introduce such a technique: In the spirit of Gardner’s

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program, we require the quantum dynamical evolutions to display a number of stationary solutions, or stable quantum memories, without specifying any learning rule. The typical fractional volume of quantum evolutions fulfilling such a constraint represents a generalized partition function. This enables us to map the evaluation of the optimal storage capacity of a quantum NN to a classical statistical mechanics model. We apply this method to an open quantum spin-based associative memory, introduced in Refs. [26,27], and analyzed in the limit of vanishing storage capacity only. The dynamics of the model is governed by a quantum master equation, where the classical out-of-equilibrium dynamics is embedded via the dissipative contribution, while a coherent term permits to account for quantum effects. We establish for the first time, as far as we know, that robust storage of an extensive number of patterns is possible under Hamiltonian perturbations and furthermore consistently recover the classical NN behavior in absence of coherent dynamics. Our method can be further exploited for (i) studying the optimal storage capacity of a wider class of QNN models, some of them potentially revealing a quantum advantage, or (ii) selecting optimal parameters for maximizing storage capacity of quantum associative memories that have been proposed for near-term experimental realizations [40].

II. THE METHOD: GENERALIZED GARDNER'S PROGRAM

Our generalization of the Gardner's program to open quantum evolutions representing associative memories allows one to determine the maximal asymptotic number of stationary states, or quantum memories, which can be stored, irrespective of the concrete set of model parameters, here representing the learning rule. To do so, we map the constraints for the desired quantum memories to be stationary states into an auxiliary statistical mechanics model, the learning rule playing the role of degrees of freedom. We consider a quantum system of N components described by a set of local operators $\mathbf{x} = (x_1, \dots, x_N)$, whose discrete time evolution is given by $\mathbf{x}(t + dt) = \mathbf{f}_\theta[\mathbf{x}(t)]$. The one-step dynamical generator, $\mathbf{f}_\theta[\cdot]$, is specified by a set of parameters θ . In the spirit of Gardner's program, we want to determine the maximal storage capacity α_c , i.e., the maximum value of the number p of stationary states of the above-defined dynamical generator. First, we identify a set of order parameters $\{O^\mu\}_{\mu=1, \dots, p}$, defined as expectation values of some macroscopic operators, $O^\mu = \langle g_\mu(\mathbf{x}) \rangle$. Here g_μ identifies a generic function, which allows one to pass from the microscopic description in terms of N local operators to some macroscopic ones, acting on an extensive subset of the N components. It is chosen such that $O^\mu \in [0, 1]$, with $O^\mu \neq 0$ signaling the (partial) storage of the μ th pattern. Second, we consider the case where a finite number, say, r , of additional scalar quantities is needed to derive a closed set of equations of motion of the order parameters. Under this assumption, the evolution of the order parameter $\mathbf{O}^\mu = (O_1^\mu, \dots, O_r^\mu)$ reads $\mathbf{O}^\mu(t + dt) = \langle g_\mu(\mathbf{f}_\theta[\mathbf{x}(t)]) \rangle \approx \mathbf{F}_\theta^\mu[\mathbf{O}^\mu(t)]$ for $\mu = 1, \dots, p$. The stationary states $\mathbf{O}^{\mu,*}$ are then determined by the stable fixed-point solutions (denoted by $*$) of the map $\mathbf{F}_\theta^\mu[\cdot]$,

satisfying

$$\mathbf{O}^{\mu,*} = \mathbf{F}_\theta^\mu[\mathbf{O}^{\mu,*}], \quad \left| \frac{\partial \mathbf{F}_\theta^\mu}{\partial \mathbf{O}^\mu} \right|_{\mathbf{O}^\mu = \mathbf{O}^{\mu,*}} < 1, \quad (1)$$

where the last condition guarantees stability. A given θ -parametrized model admits p stable stationary states if at least one fixed-point solution fulfills $O^\mu \geq \epsilon > 0$, $\forall \mu$, namely if the number of solutions of Eqs. (1), with a value larger than ϵ ,

$$\mathcal{N}_\theta^\mu = \int_\epsilon^1 d^p \mathbf{O} \frac{\delta(\mathbf{O}^\mu - \mathbf{F}_\theta^\mu[\mathbf{O}])}{\left| 1 - \frac{\partial \mathbf{F}_\theta^\mu}{\partial \mathbf{O}^\mu} \right|^{-1}} \Theta\left(1 - \left| \frac{\partial \mathbf{F}_\theta^\mu}{\partial \mathbf{O}^\mu} \right|\right), \quad (2)$$

is nonvanishing, $\forall \mu$. In Eq. (2), we introduced the step function $\Theta(x) = 1$, $\forall x \geq 0$, and 0 otherwise. We want to derive the maximum number of these solutions, regardless of a given θ parametrization. Thus, we consider the typical fractional volume of the space of the parameters θ fulfilling the condition $\Pi_\mu \mathcal{N}_\theta^\mu \neq 0$. For large storage capacity values, such a volume can be written as

$$V(\epsilon) = \frac{\int_D d\theta \Pi_\mu \Theta(\mathcal{N}_\theta^\mu)}{\int_D d\theta}, \quad (3)$$

where D identifies the space of the parameters θ . Thus, evaluation of the optimal storage capacity reduces to calculating $V(\epsilon)$. As illustrated in Fig. 1(a), as the number of stationary solutions approaches the maximum value $\alpha_c N$, the volume shrinks, implying that fewer and fewer typical parametrizations θ yield a finite number of solutions $\Pi_\mu \mathcal{N}_\theta^\mu \neq 0$. The point of a vanishing volume signals that α_c has been reached, meaning that there are no longer typical parametrizations θ fulfilling the stationarity condition for all p patterns.

III. OPEN QUANTUM HOPFIELD MODEL

We will now apply this method to a quantum Hopfield-type NN model, of which we outline here the main properties. The classical Hopfield NN is described [28,41] as an Ising model exhibiting all-to-all connectivity of N binary spins, $\{s_i = \pm 1\}_{i=1}^N$, according to the energy $E\{s_i\} = -\frac{1}{2N} \sum_{i,j} J_{ij} s_i s_j$. Here J_{ij} represents the coupling matrix encoding a set of p binary configurations $\{\xi_i^\mu = \pm 1\}_{i=1, \mu=1}^{N,p}$, which correspond to the patterns. Different learning prescriptions can be constructed in such a way that patterns represent minima of the energy function E . The storage mechanism is accomplished through a single spin-flip stochastic dynamics, or Glauber process, on considering the NN in contact with a thermal bath at temperature $T = 1/\beta$, with $k_B = 1$.

To generalize the Hopfield NN to the quantum realm, as illustrated by Fig. 1(b), we replace the N Ising spins with an open quantum spin-1/2 system, described by Pauli operators σ_i^a , $a = x, y, z$, $\forall i = 1, \dots, N$. The system state ρ evolves according to a Markovian master equation in Lindblad form [42], $\dot{\rho} = \mathcal{L}_Q[\rho] + \mathcal{L}_{\text{Hopf}}[\rho]$. Here $\mathcal{L}_{\text{Hopf}}$ induces a purely dissipative spin-flip dynamics, which stems from the classical Glauber process, and is given by the jump operators [26,27],

$$\Gamma_{n,\pm} = f_{n,\pm} \sigma_n^\pm, \quad f_{n,\pm} = \frac{\exp(\pm \beta/2 \Delta E_n)}{\sqrt{2} \cosh(\beta \Delta E_n)}. \quad (4)$$

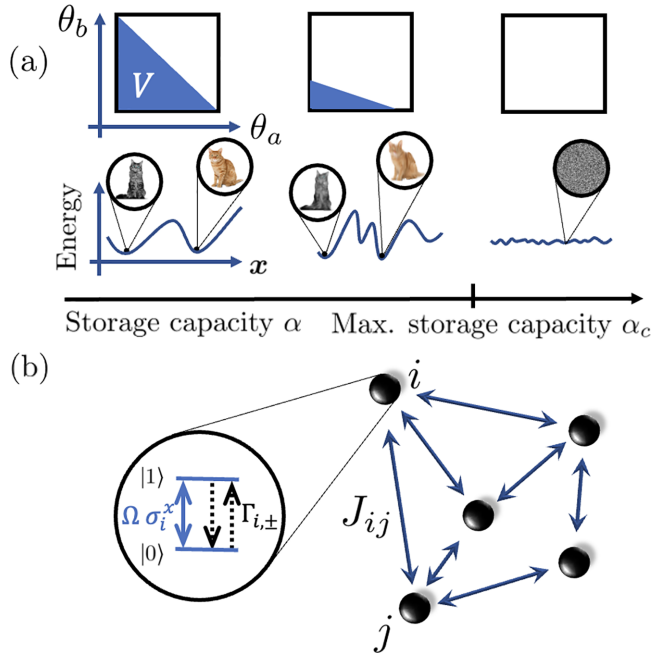


FIG. 1. (a) Schematics of maximal storage capacity. The first row represents the volume of NNs [Eq. (3)] admitting a set of patterns as stable stationary states. This volume is evaluated with respect to the phase space of the model parameters θ , corresponding to the learning rule. On increasing the number of stored patterns, the volume shrinks, as the set of suited models loses degeneracy. In Hopfield-type NNs, patterns represent minima of an energy landscape, as depicted in the second row. Pattern retrieval is generically captured by a low-temperature Glauber dynamics and works below the maximal capacity α_c . Conversely, above α_c , typical learning rules allowing the retrieval of any of the p patterns cease to exist. (b) The open quantum Hopfield NN of spin 1/2-particles undergoing a Markovian quantum dynamics: Coherent dynamics, induced by a transverse field of strength Ω , competes with dissipative spin-flip dynamics with operator-valued rates $\Gamma_{i,\pm}$ [Eq. (4)]. These depend on the spin-spin coupling matrix J_{ij} and encode the learning rule, i.e., the model parametrization θ_k in (a).

The spin-1/2 ladder operators $\sigma_n^\pm$ account for single spin flips, and operator-valued rates $f_{n,\pm}$ are determined by the energy difference induced by a single flip. Note that the energy function of the Hopfield NN becomes an operator, $E = \frac{1}{2\sqrt{N}} \sum_{ij} J_{ij} \sigma_i^z \sigma_j^z$, as does the energy difference for flipping the n th spin, $\Delta E_n = \frac{1}{\sqrt{N}} \sum_{j \neq n} J_{nj} \sigma_j^z$. With these definitions, Hopfield-type NN dynamics is thus governed by the generator

$$\mathcal{L}_{\text{Hopf}}[\cdot] = \sum_{n,\tau=\pm} \Gamma_{n,\tau} \cdot \Gamma_{n,\tau}^\dagger - \frac{1}{2} \{ \Gamma_{n,\tau}^\dagger \Gamma_{n,\tau}, \cdot \}, \quad (5)$$

so that at low values of both temperature T and capacity α , depending on the initial configuration, the system is able to thermally relax into either of the patterns, which play the role of stationary states. Quantum effects can be included by coherent Hamiltonian dynamics $\mathcal{L}_Q[\cdot] = -i[H, \cdot]$, where we choose $H = \Omega \sum_i \sigma_i^x$, corresponding to a homogeneous transverse field. This dynamical evolution is designed so as to compete with the dissipative dynamics, which can lead the

system into one of the pattern configurations, whereas the quantum drive permanently rotates the system out of it.

A. Dynamics of macroscopic overlap operators

The retrieval of patterns is quantified by the finite value assumed by the macroscopic overlaps between the spin configuration and the patterns, $M_a^\mu = \frac{1}{N} \sum_i \xi_i^\mu \sigma_i^a$, $a = x, y, z$. A finite set of closed equations of motion (EoMs) for the overlap operators M_a^μ can be derived by applying some approximations that we outline here, leaving the details in Appendix A. Proceeding similarly to Ref. [35], we employ (i) a mean-field approximation, $\langle M_a^\mu M_b^\nu \rangle \approx \langle M_a^\mu \rangle \langle M_b^\nu \rangle$, allowing us to focus on evolution of expectation values of operators only. For a lighter notation we omit the bracket $\langle \cdot \rangle$, as it is understood that from now on we deal with expectation values only. We restrict our analysis (ii) to a regime of high overlap along the z direction, i.e., $\langle M_z^\mu \rangle \approx 1$, and we assume (iii) a homogeneous distribution of the misalignment between patterns and spins, i.e., $\langle \sigma_i^z \rangle = \xi_i^\mu \langle M_z^\mu \rangle$. Last, for the internal field, or local energy, of the pattern configurations, $h_i^\mu = \xi_i^\mu / \sqrt{N} \sum_{j \neq i} J_{ij} \xi_j^\mu$, we employ (iv) a spatial homogeneity approximation, assuming that h_i does not depend strongly on the site index, $h_i^\mu \approx \frac{1}{N} \sum_j h_j^\mu$. Under these assumptions, we obtain the closed set of EoMs,

$$\begin{aligned} \dot{M}_z^\mu &= -M_z^\mu + \frac{1}{N} \sum_i \tanh(\beta h_i^\mu M_z^\mu) + 2\Omega M_y^\mu \\ \dot{M}_y^\mu &= -2\Omega M_z^\mu - \frac{M_y^\mu}{2} \frac{1}{N} \\ &\quad \times \sum_i \left\{ 1 + \frac{\beta^2}{2} [1 + M_z^\mu \tanh(\beta h_i^\mu M_z^\mu)] \right. \\ &\quad \times \left. [1 - \tanh(\beta h_i^\mu M_z^\mu)^2] \right\}, \end{aligned} \quad (6)$$

with the evolution of M_x^μ completely decoupled (see Appendix A). The EoMs of the overlaps along z and y are coupled via Ω -dependent terms, allowing us to recover the classical case for vanishing Ω .

IV. STORAGE CAPACITY OF OPEN QUANTUM HOPFIELD MODELS

We now compute the maximal capacity α_c of the quantum Hopfield NN, at given temperature T and quantum drive Ω . To this end, first, we set the patterns as stable stationary solutions of the dynamics by requiring a minimal finite value m for the overlap M_z^μ , applying the general condition (1) to the dynamical map (6). This yields $\dot{M}_z^{\mu,*} = \dot{M}_y^{\mu,*} = 0$, with $M_z^{\mu,*} > m$. The stability of the latter is guaranteed by the Jacobian of the time derivatives with respect to the overlaps of Eq. (6) being negative definite. Thus, by Eq. (2), the number of solutions of a coupling-dependent model equipped with p patterns as stable stationary configurations reads

$$\mathcal{N}^\mu = \int_m^1 dM_z^\mu \int_{-1}^1 dM_y^\mu \frac{\delta(\dot{\mathbf{M}}^\mu)}{\left| \frac{\partial \dot{\mathbf{M}}^\mu}{\partial \mathbf{M}^\mu} \right|^{-1}} \Theta \left(-\frac{\partial \dot{\mathbf{M}}^\mu}{\partial \mathbf{M}^\mu} \right), \quad (7)$$

with $\mu = 1, \dots, p$. The explicit expression of the latter can be read in Appendix B. To derive the maximum number of these solutions, regardless of any learning rule, we consider the typical volume of quantum Hopfield models that can store p patterns in the space of the couplings J_{ij} . This volume reads

$$V = \int \prod_{i \neq j} \{dJ_{ij}\} \prod_{\mu} \mathcal{N}^{\mu} \prod_i \delta \left(\sum_j J_{ij}^2 - N \right), \quad (8)$$

where we enforced a spherical normalization constraint for the coupling constants, so that the latter are nonextensive in N , i.e., $J_{ij} = \mathcal{O}(1)$.

The volume defined by Eq. (8) can be viewed as the partition function of a classical statistical mechanics model, depending on a given set of p patterns. By taking the latter as independent and identically distributed random variables, we introduce disorder in $V(\{\xi^{\mu}\})$. To obtain general statements on the capacity, we calculate the quenched pattern average of the volume $\bar{V} = \exp(\langle \log(V) \rangle_{\xi})$, where $\langle \cdot \rangle_{\xi}$ is the average over disordered pattern configurations. To deal with the latter, we employ methods from the classical theory of disordered systems and spin glasses. In particular, we apply the replica trick [43], summarizing here the main steps, and leaving the details in Appendix C. Let us start by considering n copies of the system, of which the pattern-averaged volume reads $\langle \langle V^n \rangle \rangle_{\xi}$. Employing the analytical continuation of n to the real numbers, through the limit $n^{-1} \log \langle \langle V^n \rangle \rangle \xrightarrow{n \rightarrow 0} \langle \log(V) \rangle$, we will then recover $\bar{V}$. On performing the pattern average of the replicated volume V^n , we have to evaluate

$$\left\langle \exp \left(-\frac{i}{\sqrt{N}} \sum_{\alpha} \hat{h}_i^{\mu, \alpha} \xi_i^{\mu} \sum_{j \neq i} J_{ij}^{\alpha} \xi_j^{\mu} \right) \right\rangle_{\xi} =: \langle \langle e^z \rangle \rangle_{\xi}, \quad (9)$$

where all variables except the patterns are replicated. Here α denotes the replica index, and $\hat{h}_i^{\mu, \alpha}$ is a Lagrange multiplier enforcing the definition of the local field $h_i^{\mu, \alpha}$. This average is difficult to perform for non-Gaussian pattern distributions. For going ahead, we assume to deal with an extremely diluted network [35,44]: For each spin, we consider a finite number C of matrix elements J_{ij} , where $C < \ln(N)$. This means that, after an appropriate reordering, $J_{i,j > C} = 0 \forall i$. Notice that all normalization factors of affected sums are changed appropriately as $1/\sqrt{N} \rightarrow 1/\sqrt{C}$. With this assumption, the pattern average can be performed in terms of a cumulant expansion, $\langle \langle e^z \rangle \rangle_{\xi} = e^{\sum_k c_k}$, c_k being the k th cumulant. One can show [44] (see also Appendix C) that the pattern average of the replicated volume reduces to the one of the second cumulant, which reads

$$\langle \langle z^2 \rangle \rangle_{\xi} = -\frac{1}{2C} \sum_{\alpha \beta i j} \hat{h}_i^{\mu \alpha} [\hat{h}_i^{\mu \beta} J_{ij}^{\alpha} J_{ij}^{\beta} + \hat{h}_j^{\mu \beta} J_{ij}^{\alpha} J_{ij}^{\beta}]. \quad (10)$$

Here an interaction among the replicated systems is induced, their coupling matrix being referred to as replica matrix. Correlations among replicas are quantified by the Edwards-Anderson-like order parameters [43], $q_i^{\alpha \beta} = \frac{1}{C} \sum_j J_{ij}^{\alpha} J_{ij}^{\beta}$, which we assume to be symmetric with respect to replicas, $q_i^{\alpha \beta} = q_i$, $\forall \alpha \neq \beta$, and to sites $q_i = q$, $\forall i$. The latter assumption is reminiscent of spatial homogeneity of the local energy

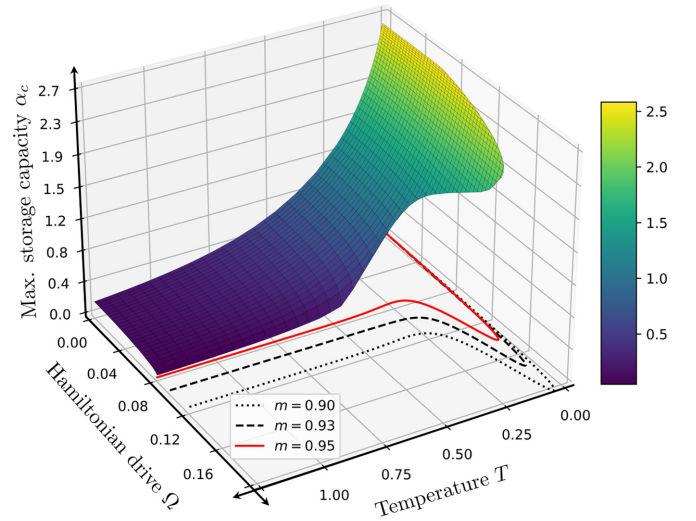


FIG. 2. Map of the maximal storage capacity α_c in the parameter space of temperature and quantum drive (T, Ω) for a fixed minimal overlap $m = 0.95$. The lines (solid, dashed, and dotted) in the $\alpha_c = 0$ plane correspond to projections of the maximal Hamiltonian drive $\Omega_c(T)$ of nonvanishing storage capacity for different minimal overlap values ($m = 0.95, 0.93, 0.9$).

h_i^{μ} . Under these assumptions, the replica limit $n \rightarrow 0$ is performed, and $\langle \langle V^n \rangle \rangle_{\xi}$ is given in terms of multiple, nested, and non-Gaussian integrals of the form $\int do e^{NCS(o)}$, where o identify replicated variables. Finally, we employ the saddle-point method, according to which, in the thermodynamic limit, $\int do e^{NCS(o)} \sim e^{NCS(o^*)}$, with o^* given by the saddle-point equations $\frac{d}{do} S(o)|_{o=o^*} = 0$. The details of the calculation can be found in Appendix D. The resulting pattern-averaged volume we find reads

$$\bar{V} = \exp \left[\frac{NC}{2(1-q)} \left(1 - \frac{\alpha}{\alpha_c(m, T, \Omega)} \right) \right]. \quad (11)$$

The maximal capacity $\alpha_c(m, T, \Omega)$ is reached in the limit of maximal replica correlation $q \rightarrow 1$.

V. RESULTS

We compute the maximal capacity for a fixed minimal overlap, m , on varying the temperature T and the quantum drive Ω . The results are displayed in Fig. 2 for a minimal overlap of $m = 0.95$. In the classical limit $\Omega = 0$, at vanishing temperature we consistently obtain the classical result originally derived by Gardner [30,33], i.e., $\alpha_c \xrightarrow{T, \Omega \rightarrow 0} [\int_0^{\sqrt{2} \text{erf}^{-1}(m)} \frac{dt}{\sqrt{2\pi}} e^{-\frac{1}{2} t^2} t^2]^{-1} \xrightarrow{m \rightarrow 1} 2$, and at finite temperatures, the classical results given in Ref. [35], assuming large overlaps $(1-m) \ll 1$. We find that the maximal storage capacity decreases monotonically on increasing both temperature and coherent drive, which indicates that both quantities introduce noise with respect to the pattern retrieval capability. For any fixed temperature, there exists a critical value, $\Omega_c(T)$, at which a discontinuous crossover from a finite to zero maximal storage capacity takes place. The corresponding transition lines $\Omega_c(T)$ are displayed in Fig. 2 as projections for different values of minimal overlap. For large temperatures,

the maximal capacity decreases as $\alpha_c \sim \beta^2$, and the critical Hamiltonian drive is given by a temperature-independent constant $\Omega_c(T) = \frac{1}{2}\sqrt{\frac{1}{2}(\frac{1}{m} - 1)}$. Finally, for $\Omega \ll 1$ and finite temperature, the Hamiltonian drive has a quadratic perturbative effect on classical storage capacity $[\alpha_c(\Omega = 0) - \alpha_c(\Omega)] \sim \Omega^2$; see Appendix E for more details. We have thus established the robustness of storage capability on perturbing the retrieval mechanism via a Hamiltonian drive.

VI. CONCLUSIONS AND OUTLOOK

We have introduced a general method to assess the storage capacity of quantum Hopfield-type NNs and have benchmarked it by applying it to an open driven-dissipative quantum NN model, which acts as an associative memory. Our technique, which relies on an extension of Gardner's program and spin-glass techniques for classical NNs to the quantum realm, is applicable to a wider class of quantum associative memories [23,37,40]. It will be interesting to identify QNNs that allow for storage of quantum-mechanical patterns and assess their maximum storage capacity, as well as to investigate the potential of many-body systems, such as cavity QED systems endowing associative memory behavior, which have been proposed for near-term experimental realizations [40]. Moreover, formulating quantum associative memories capable of storing quantum states and understanding their storage capacity are of direct interest to the field of quantum error correction, in which quantum memories realized via engineered open quantum many-body systems are under exploration [45].

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APPENDIX A: DERIVATION OF THE MEAN-FIELD EQUATIONS

In this section we will present the derivation of the mean-field equations (6) that describe the macroscopic time evolution of the open quantum Hopfield model introduced in the main text. The goal is thus to obtain a closed set of EoMs for the overlap operators defined as

$$M_a^\mu = \frac{1}{N} \sum_i^N \xi_i^\mu \sigma_i^a, \quad a \in \{x, y, z\}. \quad (\text{A1})$$

This represents the starting point for obtaining the optimal storage capacity of the model, as shown in the next sections.

Given a Lindblad equation with a Hamiltonian H and a set of jump operators $\{\Gamma\}$, the time evolution of any operator O is given by the following equation of motion:

$$\frac{d}{dt}O = i[H, O] + \sum_{n,\tau} \left(\Gamma_{n,\tau}^\dagger O \Gamma_{n,\tau} - \frac{1}{2} \{ \Gamma_{n,\tau}^\dagger \Gamma_{n,\tau}, O \} \right). \quad (\text{A2})$$

For the model we consider, the jump operators are chosen so as to perform a stochastic Hopfield-type dynamics parameterized by the couplings J_{ij} . As shown by Eq. (4), the jumps operator read

$$\Gamma_{n,\pm} = f_{n,\pm} \sigma_n^\pm, \quad f_{n,\pm} = \frac{\exp(\pm\beta/2\Delta E_n)}{\sqrt{2 \cosh(\beta\Delta E_n)}}, \quad (\text{A3})$$

where $\Delta E_n = \frac{1}{\sqrt{N}} \sum_{j \neq n} J_{nj} \sigma_j^z$ represents the energy difference for flipping of the n th spin. With these definitions, the dissipative part of the Lindblad equations induces a spin-flip dynamics that, stemming from the classical Glauber process, endows a retrieval dynamics. As described in the main text, explicitly shown by Eq. (8), and consistently with previous works [35,44], we take J_{ij} to obey a spherical normalization. The Hamiltonian that we consider is given by a transverse field, $H = \Omega \sum_i \sigma_i^x$. With such a choice, the dissipative term competes with the Hamiltonian one, this possibly giving rise to quantum effects.

First, we will obtain the equations of motion of all the degrees of freedom σ_i^a , $a \in \{x, y, z\}$ of the open quantum Hopfield model, with general coupling matrix J_{ij} . We begin by treating the equation of motion of the Pauli- z operator of the i th spin, σ_i^z . We note that the latter commutes with the jump process on any other spin j , i.e., $[\sigma_i^z, \Gamma_{j,\pm}] = 0$ for $i \neq j$, as $\Gamma_{j,\pm}$ depends only on the operators $\{\sigma_k^z\}_{k \neq j}$ and $\sigma_j^\pm$. This simplifies the EoM for σ_i^z , which takes the form

$$\frac{d}{dt} \sigma_i^z = i\Omega [\sigma_i^x, \sigma_i^z] + \sum_{\tau=\pm} \frac{\exp[\beta(\tau \frac{1}{\sqrt{N}} \sum_{j \neq i} J_{ij} \sigma_j^z)]}{2 \cosh(\frac{\beta}{\sqrt{N}} \sum_{j \neq i} J_{ij} \sigma_j^z)} (\sigma_i^{-\tau} \sigma_i^z \sigma_i^\tau - \frac{1}{2} \{ \sigma_i^{-\tau} \sigma_i^\tau, \sigma_i^z \}). \quad (\text{A4})$$

Now we apply the identity $(\sigma_i^{-\tau} \sigma_i^z \sigma_i^\tau - \frac{1}{2} \{ \sigma_i^{-\tau} \sigma_i^\tau, \sigma_i^z \}) = -\sigma_i^z + \tau$ and obtain

$$\frac{d}{dt} \sigma_i^z = 2\Omega \sigma_i^y - \sigma_i^z + \tanh\left(\frac{\beta}{\sqrt{N}} \sum_{j \neq i} J_{ij} \sigma_j^z\right). \quad (\text{A5})$$

For any finite Ω , the set of EoMs for $\{\sigma_i^z\}_{i=1,\dots,N}$ does not close, and we need to consider also the EoMs for $\sigma_i^{x/y}$ or, equivalently, for $\sigma_i^\pm$. We will proceed with the latter and consider the rate operator $f_{n,\pm}$ introduced by Eq. (A3). This quantity is an operator-valued rate function that does not commute with $\sigma_k^\pm$ for $i \neq k$. As $\sigma_k^\pm$ anticommutes with σ_k^z , it holds instead

$$\sigma_k^\pm f_\pm(\{\sigma_j^z\}_{j \neq i}) = f_\pm(\sigma_1^z, \sigma_2^z, \dots, -\sigma_k^z, \dots, \sigma_{i-1}^z, \sigma_{i+1}^z, \dots, \sigma_N^z) \sigma_k^\pm. \quad (\text{A6})$$

The alteration of the rate $f_{n,\pm}$ when commuted with $\sigma_k^\pm$ has an effect of order $\mathcal{O}(N^{-1/2})$ with respect to the original $f_{n,\pm}$, and the altered rate reads

$$f_{n,\pm}^{(k)} = f_\pm(\sigma_1^z, \sigma_2^z, \dots, -\sigma_k^z, \dots, \sigma_{i-1}^z, \sigma_{i+1}^z, \dots, \sigma_N^z) = \frac{\exp\left[\pm \frac{\beta}{2\sqrt{N}}(\sum_{j \neq i} J_{ij} \sigma_j^z) \mp \frac{\beta}{\sqrt{N}} J_{ik} \sigma_k^z\right]}{\sqrt{2 \cosh\left(\frac{\beta}{\sqrt{N}} \sum_{j \neq i} J_{ij} \sigma_j^z - 2 \frac{\beta}{\sqrt{N}} J_{ik} \sigma_k^z\right)}}. \quad (\text{A7})$$

In the thermodynamic limit (TDL), where $J_{ik}/\sqrt{N} \rightarrow 0$, we can perform a series expansion of $f_{n,\pm}^{(k)}$. At the 0th order the non-commutativity of Eq. (A7) is effectively omitted, as $f_{n,\pm}^{(k)} = f_{n,\pm} + \mathcal{O}(N^{-1/2})$. The 0th order term contributes to the equation of motion as

$$\sum_{\tau,k} f_{k,\tau}^\dagger f_{k,\tau} \left(\sigma_k^{-\tau} \sigma_i^\pm \sigma_k^\tau - \frac{1}{2} \{ \sigma_k^{-\tau} \sigma_k^\tau, \sigma_i^\pm \} \right) = -\frac{1}{2} \delta_{ik} \sum_{\tau} f_{k,\tau}^\dagger f_{k,\tau} \{ \sigma_i^{-\tau} \sigma_i^\tau, \sigma_i^\pm \} = -\frac{1}{2} \sigma_i^\pm, \quad (\text{A8})$$

where the identities $f_{n,\tau}^\dagger = f_{n,\tau}$ and $f_{n,\tau}^2 + f_{-n,\tau}^2 = 1$ were employed. Notice that also the identity $f_{n,\tau}^2 - f_{-n,\tau}^2 = \tanh(\tau \frac{\beta}{\sqrt{N}} \sum_{j \neq i} J_{ij} \sigma_j^z)$ holds. From the above expression, we can see that at 0th order only the diagonal part, $k = i$, is present. Moreover, the latter does not contribute to higher-order terms, as $[\sigma_i^\pm, f_{n,\tau}] = 0$. Indeed, for what concerns higher orders $n = 1, 2, \dots$, in $\mathcal{O}(N^{-1/2})^n$, only off-diagonal terms are nonvanishing, leading to a total contribution that is of order $\mathcal{O}(1)$ for $n = 1$. Higher-order corrections than this first-order contribution in $\mathcal{O}(N^{-1/2})$ do vanish in the TDL. Following these observations, we rewrite the dissipator term distinguishing the 0th order, diagonal term from the higher-order, off-diagonal one,

$$\begin{aligned} & \sum_{\tau,k} f_{k,\tau}^\dagger \sigma_k^{-\tau} \sigma_i^\pm \sigma_k^\tau f_{k,\tau} - \frac{1}{2} \{ f_{k,\tau}^\dagger f_{k,\tau} \sigma_k^{-\tau} \sigma_k^\tau, \sigma_i^\pm \} \\ &= -\frac{1}{2} \sigma_i^\pm + \sum_{\tau,k \neq i} \left(f_{k,\tau} f_{k,\tau}^{(i)} - \frac{1}{2} f_{k,\tau}^2 - \frac{1}{2} [f_{k,\tau}^{(i)}]^2 \right) \sigma_k^{-\tau} \sigma_k^\tau \sigma_i^\pm = -\frac{1}{2} \sigma_i^\pm - \frac{1}{4} \sum_{\tau,k \neq i} [f_{k,\tau} - f_{k,\tau}^{(i)}]^2 (\tau \sigma_k^z - 1) \sigma_i^\pm. \end{aligned} \quad (\text{A9})$$

Now we perform the expansion of $f_{k,\tau}^{(i)}$ to first order in $N^{-1/2}$, and the relevant expanded term reads

$$\begin{aligned} [f_{k,\tau} - f_{k,\tau}^{(i)}]^2 &= \left\{ \frac{\tau \beta J_{ki}}{\sqrt{N}} \sigma_i^z \frac{\exp\left[\tau \frac{\beta}{2\sqrt{N}} (\sum_{j \neq k} J_{kj} \sigma_j^z)\right]}{\sqrt{2 \cosh\left(\frac{\beta}{\sqrt{N}} \sum_{j \neq k} J_{kj} \sigma_j^z\right)}} - \frac{\beta J_{ki}}{\sqrt{N}} \sigma_i^z \tanh\left(\frac{\beta}{\sqrt{N}} \sum_{j \neq k} J_{kj} \sigma_j^z\right) \frac{\exp\left[\tau \frac{\beta}{2\sqrt{N}} (\sum_{j \neq k} J_{kj} \sigma_j^z)\right]}{\sqrt{2 \cosh\left(\frac{\beta}{\sqrt{N}} \sum_{j \neq k} J_{kj} \sigma_j^z\right)}} + \mathcal{O}(1/N) \right\}^2 \\ &= \frac{\beta^2 J_{ki}^2}{N} \left[\tau - \tanh\left(\beta \sum_{j \neq k} J_{kj} \sigma_j^z\right) \right]^2 f_{k,\tau}^2 + \mathcal{O}(N^{-3/2}). \end{aligned} \quad (\text{A10})$$

Here the leading term is of order $\mathcal{O}(N^{-1})$, and there are $N - 1$ of such off-diagonal contributions in the sum. They jointly contribute to the same order of the diagonal term. All higher-order terms of the expansion can be neglected in the TDL. On performing the sum over τ , the dissipator applied to the ladder operators becomes

$$\begin{aligned} & \sum_{\tau,k} f_{k,\tau}^\dagger f_{k,\tau} \left(\sigma_k^{-\tau} \sigma_i^\pm \sigma_k^\tau - \frac{1}{2} \{ \sigma_k^{-\tau} \sigma_k^\tau, \sigma_i^\pm \} \right) \\ &= -\frac{1}{2} \sigma_i^\pm - \frac{\beta^2}{4N} \sigma_i^\pm \sum_{k \neq i} J_{ki}^2 \left[1 + \tanh^2\left(\beta \sum_{j \neq k} J_{kj} \sigma_j^z\right) + 2 \sigma_i^z \tanh\left(\beta \sum_{j \neq k} J_{kj} \sigma_j^z\right) - \tanh\left(\beta \sum_{j \neq k} J_{kj} \sigma_j^z\right) \right. \\ & \quad \times \left. \left[\sigma_i^z + \sigma_i^z \tanh^2\left(\beta \sum_{j \neq k} J_{kj} \sigma_j^z\right) + 2 \tanh\left(\beta \sum_{j \neq k} J_{kj} \sigma_j^z\right) \right] \right] \\ &= -\frac{\sigma_i^\pm}{2} \left\{ 1 + \frac{\beta^2}{2N} \sum_{k \neq i} J_{ki}^2 \left[1 + \sigma_i^z \tanh\left(\beta \sum_{j \neq k} J_{kj} \sigma_j^z\right) \right] \left[1 - \tanh^2\left(\beta \sum_{j \neq k} J_{kj} \sigma_j^z\right) \right] \right\} =: -\frac{\sigma_i^\pm}{2} C_i, \end{aligned} \quad (\text{A11})$$

where we have introduced the positive semidefinite operator C_i as

$$C_i = 1 + \frac{\beta^2}{2N} \sum_{k \neq i} J_{ki}^2 \left[1 + \sigma_i^z \tanh \left(\beta \sum_{j \neq k} J_{kj} \sigma_j^z \right) \right] \left[1 - \tanh^2 \left(\beta \sum_{j \neq k} J_{kj} \sigma_j^z \right) \right] \geq 0. \quad (\text{A12})$$

We can now write the equations of motion of the $3N$ Pauli matrices in a compact form. They read

$$\begin{aligned} \frac{d}{dt} \sigma_i^z(t) &= -\sigma_i^z(t) + \tanh \left[\frac{\beta}{\sqrt{N}} \sum_{j \neq i} J_{ij} \sigma_j^z(t) \right] + 2\Omega \sigma_i^y(t), \\ \frac{d}{dt} \sigma_i^y(t) &= -\frac{C_i}{2} \sigma_i^y(t) - 2\Omega \sigma_i^z(t), \\ \frac{d}{dt} \sigma_i^x(t) &= -\frac{C_i}{2} \sigma_i^x(t), \end{aligned} \quad (\text{A13})$$

where the last equation is completely decoupled from the others. Moreover, C_i being positive, the expectation values of all $\sigma_i^x(t)$ will vanish at long times. For this reason, we will focus only on the set of EoMs of $\sigma_i^{z/y}(t)$.

From Eq. (A13), we will now construct the EoMs of the relevant overlap operators, which are defined by Eq. (A1). The expectation values of the operators $M_\mu^{z/y}$ evolve according to

$$\begin{aligned} \frac{d}{dt} \langle M_z^\mu(t) \rangle &= -\langle M_z^\mu(t) \rangle + 2\Omega \langle M_y^\mu(t) \rangle \\ &+ \frac{1}{N} \sum_i \xi_i^\mu \left\langle \tanh \left[\frac{\beta}{\sqrt{N}} \sum_{j \neq i} J_{ij} \sigma_j^z(t) \right] \right\rangle, \end{aligned} \quad (\text{A14})$$

$$\frac{d}{dt} \langle M_y^\mu(t) \rangle = -\frac{1}{2N} \sum_i \xi_i^\mu \langle C_i \sigma_i^y(t) \rangle - 2\Omega \langle M_z^\mu(t) \rangle. \quad (\text{A15})$$

Notice, however, that this set of equations does not close within the quantities $M_\mu^{z/y}$. In order to find a set of equations that is closed, we will perform a series of suited approximations, adopted also in the classical context [35]. We begin by considering the mean-field approximation regarding the expectation values of operators, i.e., $\langle \sigma_j^z(t) \sigma_k^z(t) \rangle \approx \langle \sigma_j^z(t) \rangle \langle \sigma_k^z(t) \rangle$. This leads to

$$\left\langle \tanh \left[\frac{\beta}{\sqrt{N}} \sum_{j \neq i} J_{ij} \sigma_j^z(t) \right] \right\rangle \approx \tanh \left[\frac{\beta}{\sqrt{N}} \sum_{j \neq i} J_{ij} \langle \sigma_j^z(t) \rangle \right]. \quad (\text{A16})$$

As a next step, we are going to restrict the regime of possible solutions to the case $\langle M_z^\nu(t > t^*) \rangle \approx 1$, where the average spin configuration $\langle \sigma_i^z(t) \rangle$ becomes locally close to the ν th pattern in the stationary state, for almost all spins at large times. Here we refer to t^* as such a large timescale, and we will not be interested in the dynamics before this time. Following these assumptions, we introduce the approximation

$$\langle \sigma_i^z(t > t^*) \rangle \approx \xi_i^\nu \langle M_z^\nu(t > t^*) \rangle. \quad (\text{A17})$$

In the following, we omit the expectation value brackets $\langle \cdot \rangle$, and the time argument $(t > t^*)$. Let us introduce the definition of local energy, h_i^μ , of the i th spin while the whole system is

in the pattern configuration μ . It reads

$$h_i^\mu := \frac{\xi_i^\mu}{\sqrt{N}} \sum_{j \neq i} J_{ij} \xi_j^\mu, \quad (\text{A18})$$

and it can be understood as a local energy, as its sum over the spin sites corresponds to the negative classical Hopfield energy, $E\{s_i = \xi_i^\mu\} = -\sum_i h_i^\mu$.

Employing these approximations and notation, the EoMs (A15) can be written as

$$\begin{aligned} \frac{d}{dt} M_z^\mu &= -M_z^\mu + 2\Omega M_y^\mu + \frac{1}{N} \sum_i \tanh(\beta h_i^\mu M_z^\mu), \\ \frac{d}{dt} M_y^\mu &= -2\Omega M_z^\mu - \frac{1}{2N} \sum_i \xi_i^\mu C_i \sigma_i^y \\ &= -2\Omega M_z^\mu - \frac{M_y^\mu}{2} - \frac{\beta^2}{4N^2} \sum_{i,k \neq i} \xi_i^\mu \sigma_i^y J_{ki}^2 \\ &\quad \times [1 + M_z^\mu \tanh(\beta h_k^\mu M_z^\mu)] [1 - \tanh^2(\beta h_k^\mu M_z^\mu)]. \end{aligned} \quad (\text{A19})$$

Notice that the second line of the above expression does not close yet with respect to the overlap expectation values. In order to achieve this, we further employ a homogeneity approximation with respect to the site dependency of the local energy. This approximation reads $1/N \sum_i h_i^\mu \approx h_i^\mu \forall i$, meaning that we assume that the local energies in the pattern configurations have similar values. This assumption is reasonable for a system exhibiting pattern retrieval, as the local energies in one of the pattern configuration corresponds to an energy minima, to which all site contributions are of equal importance. Thus, we perform the replacement $[1 + M_z^\mu \tanh(\beta h_k^\mu M_z^\mu)] [1 - \tanh^2(\beta h_k^\mu M_z^\mu)] \rightarrow 1/N \sum_j [1 + M_z^\mu \tanh(\beta h_j^\mu M_z^\mu)] [1 - \tanh^2(\beta h_j^\mu M_z^\mu)]$ for every spin. Then, the spherical normalization of the coupling matrix (8) is used explicitly. The EoM of the y overlap reads now

$$\begin{aligned} \frac{d}{dt} M_y^\mu &\approx -2\Omega M_z^\mu - \frac{M_y^\mu}{2} - \frac{\beta^2 M_y^\mu}{4N} \\ &\quad \times \sum_i [1 + M_z^\mu \tanh(\beta h_i^\mu M_z^\mu)] [1 - \tanh^2(\beta h_i^\mu M_z^\mu)]. \end{aligned} \quad (\text{A20})$$

In order to write the EoMs in a compact form, we introduce the functions A and B , depending on the z overlap as well as on all local energies and temperature,

$$\begin{aligned} A(M_z^\mu) &= \frac{1}{N} \sum_i \tanh(\beta h_i^\mu M_z^\mu), \\ B(M_z^\mu) &= 1 + \frac{\beta^2}{2N} \sum_i [1 + M_z^\mu \tanh(\beta h_i^\mu M_z^\mu)] \\ &\quad \times [1 - \tanh^2(\beta h_i^\mu M_z^\mu)]. \end{aligned} \quad (\text{A21})$$

Hence, the EoMs read

$$\begin{aligned}\frac{d}{dt}M_z^\mu &= -M_z^\mu + 2\Omega M_y^\mu + A(M_z^\mu), \\ \frac{d}{dt}M_y^\mu &= -2\Omega M_z^\mu - \frac{1}{2}M_y^\mu B(M_z^\mu).\end{aligned}\quad (\text{A22})$$

It is worth noticing that from the above equation we can clearly see that the presence of the quantum drive Ω couples the EoMs of M_z^μ and M_y^μ . In the limit $\Omega = 0$, M_y^μ vanishes and the classical Hopfield dynamics for M_z^μ is recovered.

APPENDIX B: CALCULATION OF THE NUMBER OF ATTRACTIVE SOLUTIONS

Following the main text, we aim to calculate the number of fixed-point solutions to the EoMs (6) that further admit a finite overlap in z direction, i.e., $M_z^\mu > m$. This number of solutions, $\mathcal{N}^\mu$, defined by Eq. (7) of the manuscript, reads

$$\mathcal{N}^\mu = \int_m^1 dM_z^\mu \int_{-1}^1 dM_y^\mu \frac{\delta(\dot{\mathbf{M}}^\mu)}{|\partial \dot{\mathbf{M}}^\mu|} \Theta\left(-\frac{\partial \dot{\mathbf{M}}^\mu}{\partial \mathbf{M}^\mu}\right), \quad (\text{B1})$$

where $\mathbf{M}^\mu = (M_z^\mu, M_y^\mu)^T$. The above equation involves a constraint of negative definiteness of the Jacobian, $\frac{\partial \dot{\mathbf{M}}^\mu}{\partial \mathbf{M}^\mu}$, of the EoMs, ensuring the stability of the fixed-point solutions. The Jacobian is given by the following 2×2 matrix

$$\frac{\partial \dot{\mathbf{M}}^\mu}{\partial \mathbf{M}^\mu} = \begin{bmatrix} A' - 1 & 2\Omega \\ -2\Omega - B'M_y^\mu/2 & -\frac{1}{2}B \end{bmatrix}, \quad (\text{B2})$$

and its eigenvalues read

$$\begin{aligned}\lambda_\pm &= \frac{1}{2} \left[A' - 1 - \frac{1}{2}B \right. \\ &\quad \left. \pm \sqrt{\left(A' - 1 + \frac{1}{2}B \right)^2 - 4\Omega B'M_y^\mu - 16\Omega^2} \right],\end{aligned}\quad (\text{B3})$$

where the notation $A' = \partial_{M_z^\mu} A$ is used as a shorthand to indicate the derivative by M_z^μ . The negative definiteness condition becomes therewith

$$\text{Max Re } \lambda_\pm = \text{Re } \lambda_+ < 0. \quad (\text{B4})$$

First we consider the case of a weak quantum drive, in which the root is real and the stability condition (B4) can be reformulated in terms of the determinant and trace of the matrix, as follows:

$$\begin{aligned}D_\mu &:= \det\left(\frac{\partial \dot{\mathbf{M}}^\mu}{\partial \mathbf{M}^\mu}\right) = \lambda_+ \lambda_- \\ &= -\frac{1}{2}(A' - 1)B + \Omega B'M_y^\mu + 4\Omega^2 > 0, \\ T_\mu &:= \text{Tr}\left(\frac{\partial \dot{\mathbf{M}}^\mu}{\partial \mathbf{M}^\mu}\right) = \lambda_+ + \lambda_- = A' - 1 - \frac{1}{2}B < 0.\end{aligned}\quad (\text{B5})$$

Note that in the classical limit $\Omega = 0$ the stability condition reads

$$\begin{aligned}1 > A' &= \frac{1}{N} \sum_i \beta h_i^\mu [1 - \tanh^2(\beta h_i^\mu M_z^\mu)] \\ &= \frac{1}{N} \sum_i x_i [1 - \tanh^2(x_i M_z^\mu)].\end{aligned}\quad (\text{B6})$$

This condition is fulfilled for $M_z^\mu \gtrsim 0.4478$, which is already guaranteed by our approximation of large overlap. Indeed, the validity of the EoMs is restricted to a regime where $M_z^\mu > m \rightarrow 1$. That implies that, in this regime, stability is always granted in the classical limit. Further, the condition for the trace $T_\mu < 0$ is also fulfilled for all Ω values, as $B > 0$. Consequently, the stability condition for finite quantum drive $\Omega \neq 0$ reduces to ensuring positivity of the determinant of the Jacobian, i.e., $D_\mu > 0$. In the case of strong quantum drive, the discriminate of the root in λ_+ may turn negative, in which case the stability condition modifies, and we demand the maximum of the real parts of the eigenvalues to be negative. In this case the condition becomes $T_\mu < 0$, which is again fulfilled in the regime that we are interested in. In the following, we continue to work in the low-quantum-drive regime, stressing that for larger quantum drive stability would still be ensured by the previous argument.

We can now focus on the calculation of the number of fixed-point solutions defined by Eq. (B1). It can be written as

$$\begin{aligned}\mathcal{N}_\mu &= \int_{-1}^1 dM_y^\mu \int_m^1 dM_z^\mu \delta(M_z^\mu) \delta(M_y^\mu) |D_\mu| \Theta(D_\mu) \\ &= \int_{-1}^1 dM_y^\mu \int_m^1 dM_z^\mu \int_{-\infty}^{\infty} \frac{d\lambda_\mu}{(2i\pi/N)^3} \int_{-\infty}^0 dD_\mu |D_\mu| \exp\left\{ N\lambda_{1,\mu} [-M_z^\mu + A(M_z^\mu) + 2\Omega M_y^\mu] \right. \\ &\quad \left. + N\lambda_{2,\mu} \left[-\frac{1}{2}M_y^\mu B(M_z^\mu) - 2\Omega M_z^\mu \right] + N\lambda_{\theta,\mu} \left[D_\mu - \frac{1}{2}(A' - 1)B + \Omega B'M_y^\mu + 4\Omega^2 \right] \right\}.\end{aligned}\quad (\text{B7})$$

In the second line we expressed the δ and Θ constraints in the integral, by writing them in Fourier space as

$$\begin{aligned}\theta(x - \kappa) &= \int_{-\kappa}^{\infty} dy \delta(x - y), \\ \delta(x - y) &= \int_{-\infty}^{\infty} \frac{d\lambda}{2i\pi} \exp[\lambda(x - y)].\end{aligned}\quad (\text{B8})$$

The integrals over the new variables λ_1 and λ_2 correspond thereby to the δ distributions $\delta(M_z)$ and $\delta(M_y)$, whereas the variable λ_θ corresponds to the Heaviside function. Later, the integrals over the overlap variables and the three Lagrange multipliers λ will be solved employing the saddle-point method. Note that the latter becomes exact in the TDL as the exponents of the integrand scale as N . Assuming that the

saddle-point value of $\lambda_{\theta,\mu}$ lies on the non-negative real axis, i.e., $\lambda_{\theta,\mu} \rightarrow |\lambda_{\theta,\mu}|$, the integral over D_μ can be performed, and it reads $\int_0^\infty dD_\mu D_\mu e^{-N|\lambda_{\theta,\mu}|D_\mu} = \frac{1}{N^2|\lambda_{\theta,\mu}|^2}$. As the latter evaluates to a term that is subleading in the system size N , it can be omitted in the TDL and will be dropped in the following. If the saddle-point value of $\lambda_{\theta,\mu}$ were negative, the number of solutions and therewith the volume and the capacity would vanish. Note that the integrals over λ are along the imaginary axis, therefore the saddle-point method involves the minimization of the exponent of the integral exponential. In particular for $\lambda_{\theta,\mu} \rightarrow |\lambda_{\theta,\mu}|$ the stability condition can be recovered easily as $\text{Min}_{\lambda_{\theta,\mu}} |\lambda_{\theta,\mu}| \det(\frac{\partial \tilde{M}^\mu}{\partial \mathbf{M}^\mu})$, and either has the solution $\lambda_{\theta,\mu} = 0$ if the stability condition is fulfilled or $\lambda_{\theta,\mu} = \pm\infty$ if the stability is violated. The latter would cause the capacity to vanish.

Considering further the result of the saddle-point method, the integrals are given in terms of the integrand at the saddle-point value. These resulting terms become factorizable over the sites i , if we assume that the saddle-point values do not depend strongly on the microscopic structure of the local energy, i.e., if we repeat the homogeneity approximation $h_i^\mu \approx \frac{1}{N} \sum_i h_i^\mu$, and likewise for the functions A and B that depend on $\{h_i^\mu\}$. Employing this treatment, $\mathcal{N}_\mu$ factorizes over the sites, i.e., $\mathcal{N}_\mu = \Pi_i \mathcal{N}_{i,\mu}$. By introducing the following definitions:

$$\begin{aligned} a_i &:= \tanh(\beta M_z^\mu h_i^\mu), \\ b_i &:= 1 + \frac{\beta^2}{2} [1 + M_z^\mu \tanh(\beta h_i^\mu M_z^\mu)] [1 - \tanh^2(\beta h_i^\mu M_z^\mu)], \end{aligned} \quad (\text{B9})$$

the quantity $\mathcal{N}_{i,\mu}$ reads

$$\mathcal{N}_{i,\mu} \sim \exp\left\{\lambda_{1,\mu} \left[-M_z^\mu + a_i(M_z^\mu) + 2\Omega M_y^\mu\right] + \lambda_{2,\mu} \left[-\frac{1}{2} M_y^\mu b_i(M_z^\mu) - 2\Omega M_z^\mu\right] + |\lambda_{\theta,\mu}| \left[-\frac{1}{2}(a_i' - 1)b_i + \Omega b_i' M_y^\mu + 4\Omega^2\right]\right\}, \quad (\text{B10})$$

where the values $\mathbf{M}$ and λ have to be understood as the saddle-point solutions. The latter expression, as well as the approximation leading to the factorization $\mathcal{N}_\mu = \Pi_i \mathcal{N}_{i,\mu}$ will be employed in the next section.

APPENDIX C: REPLICA CALCULATION

In the following, we are going to present the calculation of the (not normalized) volume of attractive quantum Hopfield models in the space of coupling matrices. The volume, defined by Eq. (8) in this paper, reads

$$V \sim \int \Pi_{i \neq j} \{dJ_{ij}\} \Pi_\mu \mathcal{N}_\mu \Pi_i \delta\left(\sum_j J_{ij}^2 - N\right), \quad (\text{C1})$$

where we have also replaced $\Pi_\mu \Theta(\mathcal{N}_\mu) \rightarrow \Pi_\mu \mathcal{N}_\mu$, following the assumption [35] that the number of solutions $\mathcal{N}_\mu$ approaches zero as $\alpha \rightarrow \alpha_c$. As the volume depends on a concrete set of stored patterns, we will consider averages over the latter. To this end, patterns are now assumed to be identically and independently distributed variables according to the distribution

$$P(\xi) = \frac{1}{2Np} \Pi_{i,\mu} [\delta(\xi_i^\mu + 1) + \delta(\xi_i^\mu - 1)]. \quad (\text{C2})$$

Furthermore, V can be considered as the partition function of a statistical mechanics model, and its average can be performed with respect to quenched disorder variables, i.e., the patterns. In this setting, one can focus on the pattern average of the corresponding cumulant generating function, or free energy, $\log V$. It can be computed employing the replica trick, based on the identity $n^{-1} \log \langle \langle V^n \rangle \rangle \xrightarrow{n \rightarrow 0} \langle \langle \log(V) \rangle \rangle$, where $\langle \langle \cdot \rangle \rangle$ identifies the quenched disorder average. This computation involves as a first step the calculation of the pattern average of the n times replicated volume $\langle \langle V^n \rangle \rangle_\xi$, and, eventually, the analytic continuation of n to the real numbers, so as to perform the limit $n \rightarrow 0$.

Let us start by evaluating the n times replicated volume $\langle \langle V^n \rangle \rangle_\xi$. To this end, we enforce the definition (A18) via an additional delta constraint,

$$\delta\left(h_i^\mu - \frac{\xi_i^\mu}{\sqrt{N}} \sum_{j \neq i} J_{ij} \xi_j^\mu\right) = \int \Pi_{i,\mu} \left\{ \frac{d\hat{h}_i^\mu}{2\pi} \right\} \exp\left[i \hat{h}_i^\mu \left(h_i^\mu - \frac{\xi_i^\mu}{\sqrt{N}} \sum_{j \neq i} J_{ij} \xi_j^\mu \right) \right], \quad (\text{C3})$$

and we add the according integrations $\int \Pi_{i,\mu} \{d\hat{h}_i^\mu\}$ over the real axis. Thus, the averaged, replicated volume is expressed as

$$\begin{aligned} \langle \langle V^n \rangle \rangle_\xi &= \int \Pi_{i \neq j, \alpha} \{dJ_{ij}^\alpha\} \Pi_{i,\mu,\alpha} \left\{ \frac{d\hat{h}_i^{\mu,\alpha} dh_i^{\mu,\alpha}}{2\pi} \right\} \Pi_{\mu,\alpha} \mathcal{N}_\mu^\alpha(\{h_i^{\mu,\alpha}\}) \Pi_{i,\mu,\alpha} e^{i \hat{h}_i^{\mu,\alpha} h_i^{\mu,\alpha}} \\ &\times \Pi_\mu \left\langle \exp\left(-i \sum_{i,\alpha} \hat{h}_i^{\mu,\alpha} \frac{\xi_i^\mu}{\sqrt{N}} \sum_{j \neq i} J_{ij}^\alpha \xi_j^\mu\right) \right\rangle_\xi \int \Pi_{i,\alpha} \left\{ \frac{dE_i^\alpha}{4i\pi} \right\} \exp\left[-\frac{1}{2} \sum_{i,j,\alpha} E_i^\alpha (J_{ij}^\alpha)^2 + \frac{1}{2} \sum_{i,\alpha} E_i^\alpha\right], \end{aligned} \quad (\text{C4})$$

where $\alpha = 1, 2, \dots, n$ is called the replica index and tracks the instances of replication of the partition function. Accordingly, all sums and products over α go from 1 to n . Further, the spherical constraint of the coupling matrix elements is expressed in Fourier domain, introducing the Lagrange parameter E_i^α . Note that the pattern average $\langle\langle \cdot \rangle\rangle$ only affects one factor of the volume integrand, i.e., the quantity

$$\Pi_\mu \left\langle \exp \left(-i \sum_{i,\alpha} \hat{h}_i^{\mu,\alpha} \frac{\xi_i^\mu}{\sqrt{N}} \sum_{j \neq i} J_{ij}^\alpha \xi_j^\mu \right) \right\rangle_\xi. \quad (\text{C5})$$

This average is hard to compute, as we deal with a highly non-Gaussian distribution P of Np independent binary random variables, as defined by Eq. (C2). Additionally, the quantity to be averaged involves products of all possible combinations of the μ th pattern values at different sites. Nonetheless, we are going to perform such an average by means of a cumulant expansion that will be cut off. Such a cutoff can be indeed justified in the limit of diluted networks.

To apply the dilution to the coupling matrix $J \in \mathbb{R}^{N \times N}$, we set a number of matrix elements to zero such that there are only NC finite elements in the matrix, with $C < N$. The latter can then be reordered such that $J_{i,j>C} = 0 \forall i$. We further choose $C < \log(N)$, which is referred to as logarithmic dilution, as it is proven [44] that in this case the cumulant expansion converges, in the TDL, with a finite number of terms. Technically, by diluting the network, we remove all the integrals over matrix entries which vanish by the dilution constraint itself. Accordingly, all the sums over the second index of J (mostly j) run now up to C , instead of N . Moreover, we modify as well the normalization of all sums that run over the second index of J as $1/\sqrt{N} \rightarrow 1/\sqrt{C}$. The cumulant expansion is then performed as

$$\left\langle \exp \left(-\frac{i}{\sqrt{C}} \sum_{i,\alpha} \hat{h}_i^{\mu,\alpha} \xi_i^\mu \sum_{j \neq i} J_{ij}^\alpha \xi_j^\mu \right) \right\rangle_\xi = \langle\langle e^z \rangle\rangle_\xi = e^{\sum_k^\infty c_k}, \quad (\text{C6})$$

where c_k denotes the k th cumulant. It can be shown that only cumulants up to order 2 are finite in the TDL for the diluted networks [44]. The first cumulant vanishes explicitly because P is symmetric. Consequently, one can perform the pattern average by inserting the second cumulant that corresponds to $c_2 = \langle\langle z^2 \rangle\rangle_\xi$, and reads

$$\begin{aligned} \langle\langle z^2 \rangle\rangle_\xi &= -\frac{1}{2C} \sum_{\alpha\beta,i} \hat{h}_i^{\mu\alpha} \hat{h}_i^{\mu\beta} \sum_j J_{ij}^\alpha J_{ij}^\beta \\ &\quad - \frac{1}{2C} \sum_{\alpha\beta,i} \hat{h}_i^{\mu\alpha} \sum_j \hat{h}_j^{\mu\beta} J_{ij}^\alpha J_{ji}^\beta. \end{aligned} \quad (\text{C7})$$

We simplify the second term by replacing $\hat{h}_j^{\mu\beta}$ by its average value over all sites, $\hat{h}_j^{\mu\beta} \rightarrow 1/N \sum_j \hat{h}_j^{\mu\beta}$. This approximation is in accordance with the previously employed approximation that assumes the conjugate variable of $\hat{h}_j^{\mu\beta}$, i.e., the local energy $h_j^{\mu\beta}$, to be homogeneous over the sites. On performing such an approximation the second cumulant reads

$$\langle\langle z^2 \rangle\rangle_\xi = -\frac{1}{2} \sum_{\alpha\beta,i} \hat{h}_i^{\mu\alpha} \hat{h}_i^{\mu\beta} q_i^{\alpha\beta} - \frac{1}{2N} \sum_{\alpha\beta,i,j} \hat{h}_i^{\mu\alpha} \hat{h}_j^{\mu\beta} r_j^{\alpha\beta}, \quad (\text{C8})$$

where we have defined

$$q_i^{\alpha\beta} := \frac{1}{C} \sum_j J_{ij}^\alpha J_{ij}^\beta, \quad (\text{C9})$$

$$r_i^{\alpha\beta} := \frac{1}{C} \sum_j J_{ij}^\alpha J_{ji}^\beta. \quad (\text{C10})$$

We also employ these definitions by means of new delta constraints,

$$\delta \left(q_i^{\alpha\beta} - \frac{1}{C} \sum_j J_{ij}^\alpha J_{ij}^\beta \right), \quad \delta \left(r_i^{\alpha\beta} - \frac{1}{C} \sum_j J_{ij}^\alpha J_{ji}^\beta \right), \quad (\text{C11})$$

and express them in Fourier space, with conjugate variables $Q_i^{\alpha\beta}$ and $R_i^{\alpha\beta}$. As a next step, the Gaussian integrals over $\hat{h}$ and J can be solved. Before doing so, we state the full replicated volume, where all δ constraints are expressed in Fourier space:

$$\begin{aligned} \langle\langle V^n \rangle\rangle_\xi &= \int \Pi_{\alpha,i} \left\{ \frac{dE_i^\alpha}{4i\pi} \right\} \Pi_{\alpha<\beta,i} \left\{ \frac{dQ_i^{\alpha\beta} dq_i^{\alpha\beta}}{2i\pi/C} \right\} \Pi_{\alpha\beta,i} \left\{ \frac{dR_i^{\alpha\beta} dr_i^{\alpha\beta}}{4i\pi/C} \right\} \exp \left(\frac{C}{2} \sum_{\alpha,i} E_i^\alpha + C \sum_{\alpha<\beta,i} Q_i^{\alpha\beta} q_i^{\alpha\beta} + \frac{C}{2} \sum_{\alpha\beta,i} R_i^{\alpha\beta} r_i^{\alpha\beta} \right) \\ &\quad \times \int \Pi_{\alpha,i \neq j} \{ dJ_{ij}^\alpha \} \exp \left(-\frac{1}{2} \sum_{\alpha,i,j} E_i^\alpha (J_{ij}^\alpha)^2 - \sum_{\alpha<\beta,i,j} Q_i^{\alpha\beta} J_{ij}^\alpha J_{ij}^\beta - \frac{1}{2} \sum_{\alpha\beta,i,j} R_i^{\alpha\beta} J_{ij}^\alpha J_{ji}^\beta \right) \int \Pi_{\alpha,\mu,i} \left\{ \frac{d\hat{h}_i^{\mu\alpha} d\hat{h}_i^{\mu\alpha}}{2\pi} \right\} \\ &\quad \times \Pi_{\alpha,\mu} \{ \mathcal{N}_\mu^\alpha(\{\hat{h}_i^{\mu\alpha}\}) \} \exp \left[i \sum_{\alpha,\mu,i} \hat{h}_i^{\mu\alpha} \hat{h}_i^{\mu\alpha} - \frac{1}{2} \sum_{\alpha,\mu,i} (\hat{h}_i^{\mu\alpha})^2 - \frac{1}{2} \sum_{\alpha\beta,\mu,i} \hat{h}_i^{\mu\alpha} \hat{h}_i^{\mu\beta} q_i^{\alpha\beta} - \frac{1}{2N} \sum_{\alpha\beta,\mu,i,j} \hat{h}_i^{\mu\alpha} \hat{h}_j^{\mu\beta} r_j^{\alpha\beta} \right] \\ &= \int \Pi_{\alpha,i} \left\{ \frac{dE_i^\alpha}{4i\pi} \right\} \Pi_{\alpha<\beta,i} \left\{ \frac{dQ_i^{\alpha\beta} dq_i^{\alpha\beta}}{2i\pi/C} \right\} \Pi_{\alpha\beta,i} \left\{ \frac{dR_i^{\alpha\beta} dr_i^{\alpha\beta}}{4i\pi/C} \right\} e^{CG}. \end{aligned} \quad (\text{C12})$$

In the last line we defined the action G that does not scale with the number of couplings per spin. We will further use the definition $\alpha = p/C$ for the capacity and reformulate the action employing the following definitions:

$$G := \frac{1}{2} \sum_{\alpha,i} E_i^\alpha + \sum_{\alpha < \beta, i} Q_i^{\alpha\beta} q_i^{\alpha\beta} + \frac{1}{2} \sum_{\alpha\beta, i} R_i^{\alpha\beta} r_i^{\alpha\beta} + G_J + \alpha G_h, \quad (C13)$$

$$G_J := \frac{1}{C} \log \left\{ \int \Pi_{\alpha, i \neq j} \{dJ_{ij}^\alpha\} \exp \left[-\frac{1}{2} \sum_{\alpha, i, j} E_i^\alpha (J_{ij}^\alpha)^2 - \sum_{\alpha < \beta, i, j} Q_i^{\alpha\beta} J_{ij}^\alpha J_{ij}^\beta - \frac{1}{2} \sum_{\alpha\beta, i, j} R_i^{\alpha\beta} J_{ij}^\alpha J_{ji}^\beta \right] \right\}, \quad (C14)$$

$$G_h := \log \left(\int \Pi_{\alpha, i} \left\{ \frac{dh_i^\alpha d\hat{h}_i^\alpha}{2\pi} \right\} \Pi_{i, \alpha} \{ \mathcal{N}^\alpha(\{h_i^\alpha\}) \} \exp \left[i \sum_{\alpha, i} h_i^\alpha \hat{h}_i^\alpha - \frac{1}{2} \sum_{\alpha, i} (\hat{h}_i^\alpha)^2 - \frac{1}{2} \sum_{\alpha\beta, i} \hat{h}_i^\alpha \hat{h}_i^\beta q_i^{\alpha\beta} - \frac{1}{2N} \sum_{\alpha\beta, i, j} \hat{h}_i^\alpha \hat{h}_j^\beta r_j^{\alpha\beta} \right] \right), \quad (C15)$$

where G_J and G_h are partial actions. Note that defining G_h involves a factorization over the pattern index μ for the variables h and $\hat{h}$ and their integrals.

We will now perform the Gaussian integral over J , h , and $\hat{h}$, so as to compute the partial actions G_J and G_h . We first employ the saddle-point method on the integration variables E_i^α , $Q_i^{\alpha\beta}$, $q_i^{\alpha\beta}$, $R_i^{\alpha\beta}$, and $r_i^{\alpha\beta}$ over which e^{CG} is to be integrated. Note that in the TDL, where $N \rightarrow \infty$, also $C < \log(N) \rightarrow \infty$ is chosen to diverge logarithmically. Consequently, the saddle-point method becomes exact in the TDL. We assume to find solutions of the corresponding saddle-point equations that feature a saddle point on the real axis. For these solutions, we assume replica symmetry and a site symmetry to be fulfilled as in the classical treatment [35],

$$\begin{aligned} E_i^\alpha &= E, & \forall \alpha, i \\ Q_i^{\alpha\beta} &= Q, & \forall \alpha \neq \beta, i \\ q_i^{\alpha\beta} &= q, & \forall \alpha \neq \beta, i \end{aligned}$$

$$\begin{aligned} R_i^{\alpha\beta} &= R, & \forall \alpha \neq \beta, i \\ r_i^{\alpha\beta} &= r, & \forall \alpha \neq \beta, i \\ R_i^{\alpha\alpha} &= S, & \forall \alpha, i \\ r_i^{\alpha\alpha} &= s, & \forall \alpha, i. \end{aligned} \quad (C16)$$

Now we employ this symmetry assumption to perform the two Gaussian integrals in G_J and G_h . We begin with the source-free integral over the J_{ij}^α variables by defining the replica matrix Λ that describes the coupling among the J_{ij}^α as

$$\begin{aligned} \Lambda_{ij,kl}^{\alpha,\beta} &:= \delta_{\alpha\beta} (\delta_{ij} \delta_{jl} E + \delta_{il} \delta_{jk} S) \\ &+ (1 - \delta_{\alpha\beta}) (\delta_{ij} \delta_{jl} Q + \delta_{il} \delta_{jk} R), \end{aligned} \quad (C17)$$

this being a rank-6 tensor representation, which can be reshaped into a matrix $\Lambda \in \mathbb{R}^{nNC \times nNC}$ taking $J_{ij}^\alpha = J_k$ as a vector component with superindex $k = (i, j, \alpha)$. One can determine the eigenvalues and their algebraic multiplicities by guessing the eigenvectors. The former are given by Table I. We can thus perform the Gaussian integral $\int d^{(nNC)} J e^{-1/2 \vec{J}^T \Lambda \vec{J}}$ and obtain the partial action G_J as

$$\begin{aligned} G_J &= -\frac{1}{2C} \log(\det \Lambda) + \text{const.} = -n \frac{N-1}{4} \left\{ \log(E - Q - R + S) + \log(E - Q + R - S) \right. \\ &+ \frac{\log[E + (n-1)Q - (n-1)R - S] - \log(E - Q + R - S)}{n} \\ &+ \left. \frac{\log[E + (n-1)Q + (n-1)R + S] - \log(E - Q - R + S)}{n} \right\}, \end{aligned} \quad (C18)$$

where all constants in the action, i.e., factors of the volume, can be omitted.

TABLE I. Eigenvalues of the replica matrix Λ

Eigenvalue	Multiplicity
$E - Q - R + S$	$(n-1) \frac{C(N-1)}{2}$
$E - Q + R - S$	$(n-1) \frac{C(N-1)}{2}$
$E + (n-1)Q + (n-1)R + S$	$\frac{C(N-1)}{2}$
$E + (n-1)Q - (n-1)R - S$	$\frac{C(N-1)}{2}$

Regarding the replica calculation, the analytical continuation of n to the real number can now be performed. Notice that only the limit $n \rightarrow 0$ is of interest for the replica calculation. As such, we can expand G_J to the first order in n and neglect all higher orders. The linear order in n will then contribute to the averaged volume, $\bar{V}$. We also perform the simplification $N-1 \approx N$ and obtain

$$\begin{aligned} G_J &= -\frac{Nn}{4} \left[\log(E - Q - R + S) + \log(E - Q + R - S) \right. \\ &+ \frac{Q + R}{E - Q + S - R} + \frac{Q - R}{E - Q - S + R} \left. \right] + \mathcal{O}(n^2). \end{aligned} \quad (C19)$$

The solutions for replica and site symmetric variables E, Q, R, S are now given by algebraic equations that are obtained by extremizing the partial action,

$$\begin{aligned}\tilde{G}_J &= G_J + \frac{nN}{2}E + \frac{n(n-1)N}{2}Qq + \frac{n(n-1)N}{2}Rr + \frac{nN}{2}Ss \\ &= G_J + \frac{Nn}{2}(E + Ss - Qq - Rr) + \mathcal{O}(n^2).\end{aligned}\quad (\text{C20})$$

The four stationarity equations read

$$\begin{aligned}0 &= \partial_E \tilde{G}_J \sim 1 + \frac{1}{2} \left[\frac{Q+R}{(E-Q+S-R)^2} + \frac{Q-R}{(E-Q-S+R)^2} - \frac{1}{E-Q+S-R} - \frac{1}{E-Q-S+R} \right], \\ 0 &= \partial_Q \tilde{G}_J \sim q + \frac{1}{2} \left[\frac{Q+R}{(E-Q+S-R)^2} + \frac{Q-R}{(E-Q-S+R)^2} \right], \\ 0 &= \partial_R \tilde{G}_J \sim r + \frac{1}{2} \left[-\frac{Q+R}{(E-Q+S-R)^2} + \frac{Q-R}{(E-Q-S+R)^2} \right], \\ 0 &= \partial_S \tilde{G}_J \sim s + \frac{1}{2} \left[\frac{Q+R}{(E-Q+S-R)^2} - \frac{Q-R}{(E-Q-S+R)^2} - \frac{1}{E-Q+S-R} + \frac{1}{E-Q-S+R} \right],\end{aligned}\quad (\text{C21})$$

and can be solved algebraically and reinserted in $\tilde{G}_J$. By doing so, this part of the action takes the form

$$\begin{aligned}\tilde{G}_J &= \frac{Nn}{2} \left\{ \log(1-q) + \frac{1}{2} \log \left[1 - \left(\frac{s-r}{1-q} \right)^2 \right] + \frac{q - r \frac{s-r}{1-q}}{(1-q) \left[1 - \left(\frac{s-r}{1-q} \right)^2 \right]} \right\} \\ &= \frac{Nn}{2} \left[\log(1-q) + \frac{1}{2} \log(1-x^2) + \frac{q - rx}{(1-q)(1-x^2)} \right],\end{aligned}\quad (\text{C22})$$

where we have used the definition $x := \frac{s-r}{1-q}$. To obtain the maximal capacity we will investigate the regime of large replica correlations, measured by q . The reason for this is that in this regime it is seemingly hard to find a parametrization of J such that the storage requirements are fulfilled. This corresponds to the regime of storing many patterns. Accordingly, we will consider the limit $q \rightarrow 1$, in which $\tilde{G}_J$ further simplifies as only the strongest diverging term must be considered,

$$\tilde{G}_J \xrightarrow{q \rightarrow 1} \frac{1 - rx}{(1-q)(1-x^2)} + \mathcal{O}[\log(1-q)]. \quad (\text{C23})$$

To proceed, we need to treat the second Gaussian integral over $\hat{h}_i^\alpha$ in the partial action G_h . As this integral has the local energy as a source term, we proceed in a different manner, introducing the new Gaussian variables t_0, t_1^α , and $t_{2,i}$. In this way, we lift the couplings among different sites and replicas of the $\hat{h}_i^\alpha$ and turn them into further source terms so as to obtain a diagonal self-coupling of $\hat{h}_i^\alpha$. Indeed, the partial action reads now

$$\begin{aligned}G_h &= \log \int \Pi_{\alpha,i} \left\{ \frac{d\hat{h}_i^\alpha dh_i^\alpha}{2\pi} \right\} \Pi_{i,\alpha} \{ \mathcal{N}_i^\alpha(h_i^\alpha) \} \exp \left[i \sum_{\alpha,i} h_i^\alpha \hat{h}_i^\alpha - \frac{1}{2} \sum_{\alpha,i} (\hat{h}_i^\alpha)^2 (1-q) - \frac{1}{2} \sum_{\alpha\beta,i} \hat{h}_i^\alpha \hat{h}_i^\beta q \right. \\ &\quad \left. - \frac{1}{2N} \sum_{\alpha\beta,ij} \hat{h}_i^\alpha \hat{h}_j^\beta r - \frac{1}{2N} \sum_{\alpha,ij} \hat{h}_i^\alpha \hat{h}_j^\alpha (s-r) \right] \\ &= \log \int \Pi_{\alpha,i} \left\{ \frac{d\hat{h}_i^\alpha dh_i^\alpha}{2\pi} \right\} \Pi_\alpha \{ \mathcal{N}^\alpha(\{h_i^\alpha\}) \} \frac{dt_0}{\sqrt{2\pi/N}} e^{-\frac{N}{2}(t_0)^2} \Pi_\alpha \left\{ \frac{dt_1^\alpha}{\sqrt{2\pi/N}} e^{-\frac{N}{2}(t_1^\alpha)^2} \right\} \Pi_i \left\{ \frac{dt_{2,i}}{\sqrt{2\pi}} e^{-\frac{1}{2}(t_{2,i})^2} \right\} \\ &\quad \times \exp \left[-\frac{1-q}{2} \sum_{\alpha,i} (\hat{h}_i^\alpha)^2 + i \sum_{\alpha,i} \hat{h}_i^\alpha (h_i^\alpha + \sqrt{r}t_0 + \sqrt{s-rt_1^\alpha} + \sqrt{qt_{2,i}}) \right] \\ &= \log \int \frac{dt_0}{\sqrt{2\pi/N}} e^{-\frac{N}{2}(t_0)^2} \Pi_i \left\{ \frac{dt_{2,i}}{\sqrt{2\pi}} e^{-\frac{1}{2}(t_{2,i})^2} \right\} \left[\int \frac{dt_1}{\sqrt{2\pi/N}} \Pi_i \left\{ e^{-\frac{1}{2}(t_1)^2} \frac{d\hat{h}_i dh_i}{2\pi} \mathcal{N}_i(h^i) \exp \left[-\frac{1-q}{2} (\hat{h}_i)^2 \right. \right. \right. \\ &\quad \left. \left. \left. + i \hat{h}_i (h_i + \sqrt{r}t_0 + \sqrt{s-rt_1} + \sqrt{qt_{2,i}}) \right] \right\} \right]^n,\end{aligned}\quad (\text{C24})$$

where we factorized all terms of replicated variables and applied the factorization approximation over sites for the number of solutions, $\mathcal{N}(\{h^i\}) = \prod_i \mathcal{N}_i(h^i)$, in the last step. To proceed forward, we prepare the limit $n \rightarrow 0$ by expanding to the first order in n and obtain

$$G_h = n \int \frac{dt_0}{\sqrt{2\pi/N}} e^{-\frac{N}{2}(t_0)^2} \Pi_i \left\{ \frac{dt_{2,i}}{\sqrt{2\pi}} e^{-\frac{1}{2}(t_{2,i})^2} \right\} \log \int \frac{dt_1}{\sqrt{2\pi/N}} \Pi_i \left\{ e^{-\frac{1}{2}(t_1)^2} \frac{d\hat{h}_i dh_i}{2\pi} \mathcal{N}_i(h^i) \right. \\ \left. \times \exp \left[-\frac{1-q}{2}(\hat{h}_i)^2 + i\hat{h}_i^\alpha (h_i + \sqrt{r}t_0 + \sqrt{s-r}t_1 + \sqrt{q}t_{2,i}) \right] \right\} + \mathcal{O}(n^2). \quad (C25)$$

Note that the variance of the Gaussian variable t_0 scales as $1/N$. Therefore we replace $t_0 \rightarrow 0$ and omit this integral in the TDL. Formally, this corresponds to a simple saddle-point approximation of the variable t_0 , as the non-Gaussian part of the integrand scales only to subleading order in N due to the logarithm. This could be seen explicitly if the integral over t_1 in the logarithm did not mix the $t_{2,i}$ variables and let the whole argument of the logarithm factorize. Indeed, that this is really the case can be shown by considering the saddle-point solution for the variables r and x applied to both parts of the action $G = G_h + \tilde{G}_J$ and utilizing $\sqrt{s-r} = x\sqrt{1-q}$. By doing so the transformation of the integration $s \rightarrow x$ adds only a term to the action that is of order $\mathcal{O}[\log(1-q)]$ and can be omitted, such that before applying the saddle-point method, the joint action reads

$$G/(nN) = \frac{1}{2} \left[\log(1-q) + \frac{1}{2} \log(1-x^2) + \frac{q-rx}{(1-q)(1-x^2)} \right] + \frac{1}{N} \int \Pi_i \left\{ \frac{dt_{2,i}}{\sqrt{2\pi}} e^{-\frac{1}{2}(t_{2,i})^2} \right\} \log \int \frac{dt_1}{\sqrt{2\pi/N}} \\ \times \Pi_i \left\{ e^{-\frac{1}{2}(t_1)^2} \frac{d\hat{h}_i dh_i}{2\pi} \mathcal{N}_i(h^i) \exp \left[-\frac{1-q}{2}(\hat{h}_i)^2 + i\hat{h}_i (h_i + x\sqrt{1-qt_1} + \sqrt{q}t_{2,i}) \right] \right\}. \quad (C26)$$

The stationarity condition for r reads $\partial_r G = 0$ and is equivalent to $x = 0$. As a result, the variable r drops out and the integral over t_1 can be performed trivially such that the action becomes

$$G/(nN) \\ = \frac{1}{2} \left[\log(1-q) + \frac{q}{1-q} \right] + \frac{\alpha}{N} \int \Pi_i \left\{ \frac{dt_i}{\sqrt{2\pi}} e^{-\frac{1}{2}(t_i)^2} \right\} \log \left(\int \Pi_i \left\{ \frac{d\hat{h}_i dh_i}{2\pi} \mathcal{N}_i(h^i) \exp \left[\frac{1-q}{2}(\hat{h}_i)^2 + i\hat{h}_i^\alpha (h_i + \sqrt{q}t_{2,i}) \right] \right\} \right) \\ = \frac{1}{2} \left[\log(1-q) + \frac{q}{1-q} \right] + \alpha \int \frac{dt}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} \log \left\{ \int \frac{d\hat{h} dh}{2\pi} \mathcal{N}(h) \exp \left[-\frac{1-q}{2}\hat{h}^2 + i\hat{h}(h + \sqrt{q}t) \right] \right\}, \quad (C27)$$

where the addressing of different sites by the index i yielded a summation over equal terms, which is consistent to the site symmetric approximation we performed multiple times before. Now we perform the integral over $\hat{h}$ and obtain

$$G/(nN) = \frac{1}{2} \frac{q}{1-q} + \alpha \int \frac{dt}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} \log \left\{ \int \frac{dh}{\sqrt{2\pi}} \mathcal{N}(h) \exp \left[-\frac{(h + \sqrt{q}t)^2}{2(1-q)} \right] \right\}. \quad (C28)$$

APPENDIX D: SADDLE-POINT EQUATIONS AND OPTIMAL CAPACITY

To evaluate the optimal capacity for the open quantum Hopfield model, we are interested in the limit where the saddle point corresponds to high replica correlation, $q \rightarrow 1$. This limit carries the notion that finding a suited coupling matrix encoding all patterns faithfully becomes hard. Indeed, the replica correlation, $q = \frac{1}{N} \sum_j \langle J_{ij}^\alpha J_{ij}^\beta \rangle$, is a measure of the degree of degeneracy of the suited coupling matrices. On increasing the capacity α , the task of finding a suited coupling matrix is supposed to become more difficult, therefore the degeneracy decreases, and q approaches 1, which is its maximum value. On enforcing the limit $q \rightarrow 1$ one therefore obtains the optimal load.

The first summand of the action diverges as $(1-q)^{-1}$, and we assume the same for the second one, such that a rescaling $\lambda \rightarrow \lambda/(1-q)$ can be introduced. The action takes the form

$$\frac{1-q}{nN} G = \frac{q}{2} + \alpha \int \frac{dt}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} (1-q) \log \left\{ \int \frac{dh}{\sqrt{2\pi}} \exp \left[\frac{1}{q-1} \left(-\frac{(h + \sqrt{q}t)^2}{2} + \lambda_1 [-M^z + a(M_z, h) + 2\Omega M^y] \right. \right. \right. \\ \left. \left. \left. + \lambda_2 \left[-\frac{1}{2} M^y b(M_z^\mu, h) - 2\Omega M^z \right] + |\lambda_\theta| \left[-\frac{1}{2} [a'(M_z, h) - 1] b(M_z, h) + \Omega b'(M_z, h) M^y + 4\Omega^2 \right] \right] \right\}. \quad (D1)$$

Performing a last saddle-point approximation for the h integral, the prefactor $(1-q)^{-1}$ serves as the large parameter that ensures the exactness of this treatment in the limit $q \rightarrow 1$. Note that we perform such a limit explicitly wherever no divergences are involved. Furthermore, we define a function $Y(h, t)$ that is to be maximized over h , given a fixed t , in order to obtain the

saddle-point value of h as

$$Y(h, t) := -\frac{(h+t)^2}{2} + \lambda_1 a(M_z, h) - \frac{\lambda_2}{2} M_y b(M_z, h) + |\lambda_\theta| \left\{ -\frac{1}{2} [a'(M_z, h) - 1] b(M_z, h) + \Omega b'(M_z, h) M_y \right\} \Big|_h \Rightarrow h(t). \quad (\text{D2})$$

On inserting the saddle-point solution, the action modulo constants can be written as

$$\frac{1-q}{nN} G \xrightarrow{q \rightarrow 1} \frac{1}{2} + \alpha \left[\lambda_1 (-M_z + 2\Omega M_y) - 2\lambda_2 \Omega M_z + |\lambda_\theta| 16\Omega^2 + \int Dt Y(t, h(t)) \right], \quad (\text{D3})$$

where the logarithm could be applied to the inner exponential and the normalized integral $\int \frac{dt}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} = \int Dt$ only resides with the t and $h(t)$ depend terms, which are summarized in Y . Now we consider solving for the saddle-point values $\{M_z, M_y, \lambda_1, \lambda_2, \lambda_\theta\}$ by imposing stationarity on G . We begin by discussing the stationarity equations for the Lagrange parameters λ , which should lead to a minimization of G as they stem from solving integrals over the imaginary axis. The equations regarding $\lambda_{1,2}$ and M_y read

$$\begin{aligned} 0 &= \frac{\partial G}{\partial \lambda_1} = -M_z + 2\Omega M_y + \int Dt \tanh(\beta M_z h(t)), \\ 0 &= \frac{\partial G}{\partial \lambda_2} = -2\Omega M_z - \frac{M_y}{2} \left(1 + \frac{\beta^2}{2} \int Dt \{1 + M_z \tanh[\beta M_z h(t)]\} \{1 - \tanh^2[\beta M_z h(t)]\} \right), \\ 0 &= \frac{\partial G}{\partial M_y} = 2\Omega \lambda_1 - \frac{\lambda_2}{2} \left(1 + \frac{\beta^2}{2} \int Dt \{1 + M_z \tanh[\beta M_z h(t)]\} \{1 - \tanh^2[\beta M_z h(t)]\} \right) \\ &\quad + 2|\lambda_\theta| \beta \Omega \int Dt \{1 - \tanh^2[\beta M_z h(t)]\} [(1 - 2\beta h(t)) \tanh[\beta M_z h(t)] + \beta h(t) M_z \{1 - 3 \tanh[\beta M_z h(t)]\}]. \end{aligned} \quad (\text{D4})$$

As for λ_θ , it can take two different values to minimize the action depending on D

$$|\lambda_\theta| = \begin{cases} 0 & D \geq 0 \\ \infty & \text{else} \end{cases}, \quad (\text{D5})$$

corresponding to fulfilling and violating the stability constraint, where

$$\begin{aligned} D &= 16\Omega^2 + \int Dt \left[-2(\beta h(t) \{1 - \tanh^2[\beta M_z h(t)]\} - 1) \left(1 + \frac{\beta^2}{2} \{1 + M_z \tanh[\beta M_z h(t)]\} \{1 - \tanh^2[\beta M_z h(t)]\} \right) \right. \\ &\quad \left. + 2\beta \Omega M_y \{1 - \tanh^2[\beta M_z h(t)]\} [(1 - 2\beta h(t)) \tanh[\beta M_z h(t)] + \beta h(t) M_z \{1 - 3 \tanh[\beta M_z h(t)]\}] \right]. \end{aligned} \quad (\text{D6})$$

The second case leads directly to a vanishing volume, i.e., zero storage capacity. Therefore, we enforce $D \geq 0$ while solving the equations. Inserting back the saddle-point equations, the replicated averaged volume becomes

$$\langle V^n \rangle_\xi \xrightarrow{n \rightarrow 0} = \frac{nNC}{2(1-q)} \left\{ 1 + \alpha \int Dt [h(t) + t]^2 \right\}, \quad (\text{D7})$$

where $h(t)$ depends on the saddle-point solutions. Before addressing the saddle-point solutions, the replica calculation is concluded as $n^{-1} \langle V^n \rangle_\xi \xrightarrow{n \rightarrow 0} \langle \log(V) \rangle_\xi$, and we can state the condition for which the averaged volume is finite as follows:

$$\exp(\langle \log(V) \rangle_\xi) = \exp \left(\frac{NC}{2(1-q)} \left\{ 1 - \alpha \int \frac{dt}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} [h(t) + t]^2 \right\} \right) \neq 0 \iff \alpha < \alpha_c := \left\{ \int \frac{dt}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} [h(t) + t]^2 \right\}^{-1}, \quad (\text{D8})$$

where we defined the critical load α_c . From Eq. (D8), we thus recover Eq. (11) of the manuscript, i.e.,

$$\bar{V} = \exp \left\{ \frac{NC}{2(1-q)} \left[1 - \frac{\alpha}{\alpha_c(m, T, \Omega)} \right] \right\}. \quad (\text{D9})$$

The maximal capacity α_c can be calculated numerically and depends only on the external parameter $\{T, \Omega, m\}$. Its behavior in certain parameter regimes with respect to $\{T, \Omega, m\}$ will be shown and commented on in the next section. Before doing this, we give more details on how the maximal capacity can be actually computed.

For evaluating α_c , the saddle-point equations, $\frac{d}{do} S(o)|_{o=o^*} = 0$, have to be solved. These are a self-consistent set of equations, of two types: algebraic equations and integral equations. While the former can be solved analytically, the latter can be solved numerically. Indeed, in order to numerically compute the integrals over t , which depend on the functions $h(t)$, $Y(h, t)$, it must

be maximized for every t that is sampled, and simultaneously the saddle-points of λ and M must be determined. The latter is achieved by noting the following algebraic relation for λ_2 :

$$\lambda_2 = -\lambda_1 \frac{M_y}{M_z}, \quad (\text{D10})$$

that is the solution of one of the coupled saddle-point equations. The remaining equations read

$$M_z = 2\Omega M_y + \int Dt \tanh[\beta M_z h(t)], \quad M_y = -4\Omega M_z \left(1 + \frac{\beta^2}{2} \int Dt \{1 + M_z \tanh[\beta M_z h(t)]\} \{1 - \tanh^2[\beta M_z h(t)]\} \right)^{-1}, \quad (\text{D11})$$

where $h(t)$ depends on the solution of λ_1 and M_z . Under the assumption that the stability condition is fulfilled, i.e., $\lambda_\theta = 0$, the stationarity condition $\partial_h Y = 0$ for the maximization of Y can be expressed as

$$h(t) = -t + \frac{\lambda_1 \beta M_z}{\{1 - \tanh^2[\beta M_z h(t)]\}^{-1}} \left[1 + 4\beta^2 \Omega^2 M_z \frac{M_z(1 - 2 \tanh[\beta M_z h(t)] - \tanh^2[\beta M_z h(t)] - 2 \tanh[\beta M_z h(t)])}{\left(1 + \frac{\beta^2}{2} \int Dt \{1 + M_z \tanh[\beta M_z h(t)]\} \{1 - \tanh^2[\beta M_z h(t)]\}\right)^2} \right]. \quad (\text{D12})$$

We will find that $\alpha_c(M_z)$ and therefore the action is strictly decreasing with M_z , and therefore the saddle-point value for M_z is given by the lower bound m . In other words, we replace M_z by m , and we just need to numerically solve for M_y and λ_1 numerically. This is done by a modified Newton method, where the first equation is solved by optimizing λ_1 , while new values for M_y are given by the fixed-point value of the second equation for M_y . Such a fixed-point value is obtained by reinserting the previous value for M_y multiple times, given an iteration, in the λ_1 optimization. On convergence of this solving algorithm, the function $h(t)$ is fully determined, and, by maximizing Y , the maximal capacity is calculated as given by Eq. (D8).

APPENDIX E: LIMITING CASES

1. Classical Gardner limit

Based on the results of the previous calculation, we now focus on the case $\Omega = 0$. By means of the saddle-point equations (D10) and (D11) we find that $M^y = \lambda_2 = 0$ holds, as expected. Hence, the numerical computation reduces to solving the equations

$$M_z = \int Dt \tanh[\beta M_z h(t)], \quad h(t) = -t + \lambda_1 \beta \{1 - \tanh^2[\beta M_z h(t)]\}. \quad (\text{E1})$$

The last line yields $h(t)$ by finding the maximum of Y , defined by Eq. (D2), which is done by setting $0 = \partial_h Y(h, t)$. This set of equations corresponds to the known classical results [35] and can be solved numerically. We focus first on the limit of zero temperature, $\beta \rightarrow \infty$, where the equations take the form

$$M_z = \int Dt \operatorname{sgn}[h(t)], \quad (\text{E2})$$

$$h(t) = -t + 2\lambda_1 \beta M_z \delta_T[h(t)]. \quad (\text{E3})$$

Here δ_T denotes a Dirac sequence for small T . For $t < 0$ this equation leads to the solution $h(t) = -t$. For $t \geq 0$ there are two possible solutions, $h(t) = -t$, and $h(t) = 0^+$, for which

λ_1 can be chosen accordingly. Further, the solution $h(t) = 0^+$ must be chosen in a certain regime $t \in [0, a]$ in order to fulfill Eq. (E2). Thus, the solution for $h(t)$ reads

$$h(t) = \begin{cases} 0^+, & 0 < t < a \\ -t, & \text{else} \end{cases}. \quad (\text{E4})$$

The constant a is determined by solving Eq. (E2) as $a = \sqrt{2} \operatorname{erf}^{-1}(M_z)$. As a result, the capacity at zero temperature is given by

$$\alpha_c(T = \Omega = 0) = \left[\int_0^{\sqrt{2} \operatorname{erf}^{-1}(M_z)} \frac{dt}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} \right]^{-1}. \quad (\text{E5})$$

By setting $M_z = m$, which corresponds to the maximal value of the action, we derive the limit obtained by Gardner [30], $\alpha_c(T = \Omega = 0) \xrightarrow{m \rightarrow 1} 2$.

2. Large temperature behavior

We now proceed to consider the large-temperature case, while keeping $\Omega = 0$. It is to be expected that for large temperature the capacity vanishes on demanding a large minimal overlap m . We intend to use the following calculation in a temperature regime where an expansion in $\beta \ll 1$ is reasonable and at the same time the capacity is finite. We resort to the saddle-point equations (D11) and expand them to first order in β ,

$$M_z = 2\Omega M_y + \beta M_z \int Dt h(t) + \mathcal{O}(\beta^2), \quad M_y = -4\Omega M_z + \mathcal{O}(\beta^2). \quad (\text{E6})$$

On combining these equations, and further setting $M_z = m$, we obtain

$$m(1 + 8\Omega^2) = m\beta \int Dt h(t) + \mathcal{O}(\beta^2) < 1, \quad (\text{E7})$$

which sets a maximal value for the Hamiltonian drive up to which a solution can still be found. This critical value is given

by

$$\Omega_c(m) = \frac{1}{2} \sqrt{\frac{1}{2} \left(\frac{1}{m} - 1 \right)}. \quad (\text{E8})$$

In order to proceed and derive the maximal storage capacity, we employ a large temperature expansion of the equation $0 = \partial_h Y(h, t)$. At the first order we obtain

$$h(t) = -t + \beta \lambda_1 m + \mathcal{O}(\beta^2). \quad (\text{E9})$$

Combining the latter with Eq. (E7), one finds $\lambda_1 = \frac{1+8\Omega^2}{m\beta^2}$, and therewith

$$h(t) = -t + \frac{1+8\Omega^2}{\beta}, \quad \alpha_c = \left(\frac{\beta}{1+8\Omega^2} \right)^2, \quad (\text{E10})$$

which shows a quadratic decay of the maximal capacity at large temperature.

3. Behavior for weak Hamiltonian drive

In order to investigate the dependency of the maximal capacity regarding a perturbative Hamiltonian drive, $\Omega \ll 1$, we separate $h(t)$ and λ_1 into a classical contribution and a quantum contribution as follows:

$$h(t) = h_c(t) + \Omega h_q(t) + \mathcal{O}(\Omega^2),$$

$$\lambda_1 = \lambda_{1,c} + \Omega \lambda_{1,q} + \mathcal{O}(\Omega^2). \quad (\text{E11})$$

Here the functions $h_c(t)$, $h_q(t)$ and the constants $\lambda_{1,c}$, $\lambda_{1,q}$ are independent of the Hamiltonian drive Ω , and thus only the term $h_q(t)$, $\lambda_{1,q}$ contributes at first order in Ω . The classical parts, denoted by the label $()_c$, should thereby correspond to the solution $\Omega = 0$.

The saddle-point equations can be written as

$$\begin{aligned} M_z &= \int Dt \tanh[\beta M_z h_c(t)] + \Omega \beta M_z \int Dt h_q(t) \{1 - \tanh^2[\beta M_z h_c(t)]\} + \mathcal{O}(\Omega^2), \\ M_y &= -\frac{4\Omega M_z}{1 + \frac{\beta^2}{2} \int Dt \{1 + M_z \tanh[\beta M_z h_c(t)]\} \{1 - \tanh^2[\beta M_z h_c(t)]\}} + \mathcal{O}(\Omega^2), \end{aligned} \quad (\text{E12})$$

implying that

$$\int Dt h_q(t) \{1 - \tanh^2[\beta M_z h_c(t)]\} = 0. \quad (\text{E13})$$

Further, on expanding the equation $0 = \partial_h Y(h, t)$ to first order in Ω , it is

$$0 = -h_c - t + \beta M_z \lambda_{1,c} \{1 - \tanh^2[\beta M_z h_c(t)]\} + \Omega \{-h_q + \beta M_z \lambda_{1,q} (1 - \tanh^2[\beta M_z h_c(t)])\} + \mathcal{O}(\Omega^2), \quad (\text{E14})$$

from which we derive

$$h_q = \beta M_z \lambda_{1,q} (1 - \tanh^2[\beta M_z h_c(t)]). \quad (\text{E15})$$

This last equation means that h_q has a fixed sign on varying t , and therefore $\lambda_{1,q} = 0$ must hold in order to fulfill Eq. (E13). As a consequence, the first-order contribution in Ω to $h(t)$ vanishes, i.e., it is $h_q = 0$, and the lowest-order contribution to the maximal capacity can only scale as Ω^2 . Indeed, the expansion of $h(t)$ to second order reads

$$h(t) = h_c(t) + \Omega h_{q_1}(t) + \Omega^2 h_{q_2}(t) + \mathcal{O}(\Omega^3), \quad (\text{E16})$$

and we can insert it to calculate the maximal storage capacity as

$$\begin{aligned} \alpha_c(m, T, \Omega) &= \left\{ \int Dt [h_c(t) + t + \Omega h_{q_1}(t) + \Omega^2 h_{q_2}(t) + \mathcal{O}(\Omega^3)]^2 \right\}^{-1} \\ &= \left\{ \int Dt [h_c(t) + t]^2 \right\}^{-1} - 2\Omega \frac{\int Dt [h_c(t) + t] h_{q_1}(t)}{\left\{ \int Dt [h_c(t) + t]^2 \right\}^2} \\ &\quad - \frac{\Omega^2}{\left\{ \int Dt [h_c(t) + t]^2 \right\}^2} \left(\int Dt \{h_{q_1}^2(t) + 2h_{q_2}(t)[h_c(t) + t]\} - 8 \frac{\int Dt (h_c + t)^2 h_{q_1}^2(t)}{\left\{ \int Dt [h_c(t) + t]^2 \right\}} \right) + \mathcal{O}(\Omega^3). \end{aligned} \quad (\text{E17})$$

Employing $h_{q_1} = 0$ that expression of the maximal storage capacity at small Hamiltonian drive reads

$$\alpha_c(m, T, \Omega) = \alpha_c(m, T, \Omega = 0) - 2\Omega^2 \alpha_c(m, T, \Omega = 0)^2 \int Dt h_{q_2}(t) [h_c(t) + t] + \mathcal{O}(\Omega^3), \quad (\text{E18})$$

for which we still need to determine h_{q_2} . Similarly to before, we expand the relevant saddle-point equations,

$$M_z = \int Dt \tanh[\beta M_z h_c(t)] + \Omega^2 \beta M_z \int Dt h_{q_2}(t) \{1 - \tanh^2[\beta M_z h_c(t)]\} + \mathcal{O}(\Omega^3), \quad (\text{E19})$$

which yield $\int Dt h_{q_2}(t) \{1 - \tanh^2[\beta M_z h_c(t)]\} = 0$. Further, the equation $0 = \partial_h Y(h, t)$ is expanded as

$$0 = -h_c - t + \beta M_z \lambda_{1,c} [1 - \tanh^2(\beta M_z h_c)] + \Omega^2 \left(-h_{q_2} + \beta M_z \lambda_{1,q_2} [1 - \tanh^2(\beta M_z h_c)] \right. \\ \left. + 4\beta^3 (M_z)^2 \lambda_{1,c} \frac{[1 - \tanh^2(\beta M_z h_c)] \{M_z [1 - 2 \tanh(\beta M_z h_c) - \tanh^2(\beta M_z h_c)] - 2 \tanh(\beta M_z h_c)\}}{\left\{1 + \frac{\beta^2}{2} \int Dt [1 + M_z \tanh(\beta M_z h_c)] [1 - \tanh^2(\beta M_z h_c)]\right\}^2} \right). \quad (\text{E20})$$

This implies that the second-order contribution in Ω , given by the last angular bracket, has to vanish. Together with the previous condition, one can determine λ_{1,q_2} formally as

$$\lambda_{1,q_2} = -4\lambda_{1,c} \beta^2 M_z \frac{\int Dt [1 - \tanh^2(\beta M_z h_c)]^2 \{M_z [1 - 2 \tanh(\beta M_z h_c) - \tanh^2(\beta M_z h_c)] - 2 \tanh(\beta M_z h_c)\}}{\left\{ \int Dt [1 - \tanh^2(\beta M_z h_c)]^2 \right\} \left\{ 1 + \frac{\beta^2}{2} \int Dt [1 + M_z \tanh(\beta M_z h_c)] [1 - \tanh^2(\beta M_z h_c)] \right\}^2}. \quad (\text{E21})$$

Thereby the function $h_{q_2}(t)$ is fully determined by the solution of the classical problem, and furthermore it is in general nonvanishing, i.e., $h_{q_2}(t) \neq 0$. In conclusion, this shows that the effect of the Hamiltonian drive to the maximal storage capacity is quadratic in Ω . This behavior is confirmed by the results we find from numerically solving the system dynamics.

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