Symmetry of Skyrmions on the Sphere

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Abstract

We study Skyrmions on the Sphere which arise as critical points of a magnetic energy involving only exchange energy and easy – normal anisotropy. Due to the curved nature of the sphere, these two terms suffice to stabilize critical points against the scaling invariance of the Dirichlet term. Furthermore, the normal anisotropy breaks the invariance of this term under individual rotations of the domain and target sphere, leaving only invariance under joint rotations. The goal of this thesis is to understand the effect that this remaining invariance has on the symmetry of minimizers and critical points.

First, we focus on axisymmetric Skyrmions on the sphere, which are themselves invariant under joint rotations around a given axis. We use standard methods to show existence and regularity of minimizers in this symmetry class and then exploit the symmetry to study their shape in more detail. A fine analysis of the energy density and other energy arguments lead to the proof of several properties. We also give some estimates for the case of a high anisotropy parameter.

Secondly, we investigate the minimality of these Skyrmions in a broader class. We find that the Hessian associated to the magnetic energy is positive semidefinite and identify the elements of its kernel. Under the assumption of strict convexity within the axisymmetric class, we deduce local minimality of axisymmetric Skyrmions up to invariances of the energy.

Finally, we construct non-trivial periodic solutions for the Landau-Lifshitz equation associated to the magnetic energy functional. For this, we consider the minimization of the energy under a constraint on the angular momentum which enforces symmetry breaking. We show that constrained minimizers solve an Euler-Lagrange equation with Lagrange multiplier ω and employ a Łojasiewicz inequality for the magnetic energy to confirm $\omega \neq 0$.

Kurzzusammenfassung

Wir betrachten Skyrmionen auf der Sphäre, die als kritische Punkte eines magnetischen Energiefunktionals bestehend aus Dirichletterm und Anisotropie aufkommen. Durch die Krümmung der Sphäre sind diese beiden Terme ausreichend, um kritische Punkte gegen die Skalierungsinvarianz des Dirichletterms zu stabilisieren. Zudem wird durch die Anisotropie die Invarianz dieses Terms unter unabhängigen Rotationen der Definitions- und Bildsphäre gebrochen, sodass nur Invarianz unter gleichzeitigen Drehungen übrig bleibt. Das Ziel dieser Arbeit ist es, den Effekt dieser verbleibenden Invarianz auf Minimierer und kritische Punkte zu verstehen.

Zuerst konzentrieren wir uns auf achsensymmetrische Skyrmionen auf der Sphäre, die selbst invariant unter gleichzeitigen Drehungen um eine gegebene Drehachse sind. Wir verwenden Standardmethoden, um Existenz und Regularität vo Minimierern in dieser Symmetrieklasse zu zeigen und nutzen dann die Symmetrie, um ihre Form detaillierter zu untersuchen. Eine genaue Analyse der Energiedichte und andere Energieargumente führen zum Beweis diverser Eigenschaften. Zudem geben wir einige Abschätzungen für den Fall hoher Anisotropie an.

Als zweites untersuchen wir die Minimalität dieser Skyrmionen in einer größeren Klasse. Wir zeigen, dass die zur magnetischen Energie zugehörige Hessesche positiv semidefinit ist und bestimmen die Elemente ihres Kerns. Unter der Annahme strikter Konvexität innerhalb der achsensymmetrischen Klasse folgern wir die lokale Minimalität achsensymmetrischer Skyrmionen bis auf Invarianzen der Energie.

Schließlich konstruieren wir nichttriviale periodische Lösungen für die zugehörige Landau-Lifshitz Gleichung. Dafür betrachten wir die Minimierung der Energie unter einer Einschränkung für den Drehmoment, die Symmetriebrechung forciert. Wir zeigen, dass bedingte Minimierer eine Euler-Lagrange Gleichung mit Langrangemultiplikator ω lösen, und verwenden eine Łojasiewicz Ungleichung für die magnetische Energie um $\omega \neq 0$ zu bestätigen.

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Contents

1. Introduction

Skyrmions are topologically non-trivial critical points of nonlinear σ-models. They have originally been proposed by Tony Skyrme as a field theory for interacting particles [49], but are now discussed in the context of magnetism in condensed matter. Experimentally, they have first been observed in bulk materials [41] and subsequently on thin film [57].

Mathematically, the existence of Skyrmions is usually proven by minimizing a magnetic energy functional under a constraint on the mapping degree. Depending on the situation modeled, the energies under consideration are integral functionals over $\mathbb{R}^3$ for bulk material or, more commonly, over $\mathbb{R}^2$ for the thin film limit, which can be confirmed by proving Gamma convergence [13, 9]. They are applied to magnetizations $\mathbf{u}\colon \mathbb{R}^n \to \mathbb{S}^n$ for $n \in \{2, 3\}$ which converge to a fixed value as $|x| \to \infty$. Thus, identifying $\mathbb{R}^n \cup \{\infty\}$ with $\mathbb{S}^n$ via the stereographic projection π, the map $\tilde{\mathbf{u}} = \mathbf{u} \circ \pi$ is well defined and its mapping degree $\deg(\tilde{\mathbf{u}})$ is integer valued whenever $\tilde{\mathbf{u}}$ is sufficiently regular [8]. Focusing on the $\mathbb{R}^2$ model, it can be computed via the standard volume form $\omega_{\mathbb{S}^2}$ on $\mathbb{S}^2$:

$$Q(\mathbf{u}) := \deg(\tilde{\mathbf{u}}) = \int_{\mathbb{S}^2} \tilde{\mathbf{u}}^* \omega_{\mathbb{S}^2} = \frac{1}{4\pi} \int_{\mathbb{R}^2} \mathbf{u} \cdot (\partial_1 \mathbf{u} \times \partial_2 \mathbf{u}) \, \mathrm{d}x$$

By approximation with smooth functions, this integral expression of Q implies that the topological charge $Q(\mathbf{u})$ is well defined and integer valued even for fields $\mathbf{u} \in H^1(\mathbb{R}^2; \mathbb{S}^2)$. It separates $H^1(\mathbb{R}^2; \mathbb{S}^2)$ into topological sectors

$$\{\mathbf{u} \in H^1(\mathbb{R}^2; \mathbb{S}^2) \ : \ Q(\mathbf{u}) = k\}$$

which are preserved under continuous deformations of $\mathbf{u}$. Hence, minimizing under a constraint on Q yields critical points of the energy, which are locally minimizing and have higher energy than the ground state. Indeed, the classical topological bound for the exchange energy yields

$$\frac{1}{2} \int_{\mathbb{R}^2} |\nabla \mathbf{u}|^2 \, \mathrm{d}x \geq \int_{\mathbb{R}^2} |\partial_1 \mathbf{u}| |\partial_2 \mathbf{u}| \, \mathrm{d}x \geq 4\pi Q(\mathbf{u})$$

by a simple application of Young's inequality.

However, invariance of $\|\nabla \mathbf{u}\|^2_{L^2(\mathbb{R}^2)}$ under rescaling $\mathbf{u} \mapsto \mathbf{u}_\lambda = \mathbf{u}(\lambda \cdot)$ entails that minimizers are not localized and can vary in size. In order to overcome this problem and to obtain stable Skyrmions on the plane, other terms of opposite scaling behavior must be included, each carrying a different meaning in the context of micromagnetics. This results in energy functionals consisting of at least three terms. In particular, a frequent subject of study is the combination of Dzyaloshinski-Moriya interaction (DMI) and different anisotropy terms forcing $\mathbf{u}$ to assume fixed values at ∞ [32, 55, 12]. While the DMI is given by

$$\int_{\mathbb{R}^2} \mathbf{u} \cdot (\nabla \times \mathbf{u}) \, \mathrm{d}x$$

and results in chirality, anisotropy relates to the fact that aligning all spins in a certain direction is energetically favorable. The specific choice of anisotropy depends on the modeled material. However, easy plane anisotropy $(\mathbf{u} \cdot \hat{\mathbf{e}}_3)^2$ and easy axis anisotropy $1 - (\mathbf{u} \cdot \hat{\mathbf{e}}_3)^2$, along with the Zeeman term $1 - (\mathbf{u} \cdot \hat{\mathbf{e}}_3) = \frac{1}{2}|\hat{\mathbf{e}}_3 - \mathbf{u}|^2$ are most common. In practice, the conditions under which DMI effects and anisotropy lead to the observation of Skyrmions are hard to realize [50].

Stabilizing Effects of Curvature

This difficulty has led researchers to investigate the appearance of complex magnetic structures on curved thin films [14, 50] in order to reduce the requirements on the material in which Skyrmions can be stabilized. In the Γ-limit and under conditions on the surface, that are usually satisfied in applications, these curved thin films are well modeled by magnetization fields $\mathbf{m}\colon \mathcal{M} \to \mathbb{S}^2$ where $\mathcal{M}$ is a two dimensional manifold embedded in $\mathbb{R}^3$ [10].

In the case of a spherical shell $\partial B_R(0) \subset \mathbb{R}^3$, this manifests in the fact that a Skyrmion can already be induced by a uniform external magnetic field [26]. Mathematically, on the other hand, the reduced requirements are reflected by an energy

$$\mathcal{E}(\mathbf{m}) = \frac{1}{2} \int_{\mathbb{S}^2} |\nabla \mathbf{m}|^2 + \kappa(1 - (\mathbf{m} \cdot \nu)^2) \, \mathrm{d}\sigma$$

that only consists of two terms. Here, $\nu(\mathbf{x}) = \mathbf{x}$ is the outer unit normal and $\kappa > 0$ is the anisotropy parameter. Apart from the exchange energy, the easy-normal anisotropy is sufficient to stabilize minimizers against scaling, which for $\mathbf{m}\colon \mathbb{S}^2 \to \mathbb{S}^2$ is done on the level of the pullback via the inverse stereographic projection

$$\mathbf{m}_\lambda = \mathbf{m} \circ \pi^{-1}(\lambda \cdot) \circ \pi.$$

The stability is due to the fact that the anisotropy on a curved surface results in a curvature induced DMI [26]. This is easily shown for axisymmetric fields, see section 2.2.

For sufficiently regular magnetizations on the sphere, the topological charge can easily be defined as

$$Q(\mathbf{m}) = \deg(\mathbf{m}) = \int_{\mathbb{S}^2} \mathbf{m}^* \omega_{\mathbb{S}^2}.$$

It is identical to the topological charge of the pullback via the inverse stereographic projection, $\tilde{\mathbf{m}} = \mathbf{m} \circ \pi^{-1} \colon \mathbb{R}^2 \to \mathbb{S}^2$, which was defined above. Hence it is integer valued for $\mathbf{m} \in H^1(\mathbb{S}^2; \mathbb{S}^2)$. Interestingly, the hedgehog configurations $\mathbf{h}_\pm(x) = \pm\nu(x)$ satisfy $Q(\mathbf{h}_\pm) = \pm 1$ while also being the global minimizers [11] for $\kappa > 4$. In the planar case, the topological lower bound for the exchange energy usually implies that ground states are topologically trivial with $Q(u) = 0$. This shift in the topological charge results from the inverse stereographic projection $\mathbf{u} = \pi^{-1} \colon \mathbb{R}^2 \to \mathbb{S}^2$, which has charge $Q(\mathbf{u}) = 1$.

Apart from degree considerations, it is particularly interesting that the hedgehog configurations are the global minimizers because this shows that the global minimizers display the same symmetry as the energy functional, namely invariance under joint rotations and inversions

$$\mathbf{m} \mapsto \mathbf{m}_O = O\mathbf{m}(O^{-1}\cdot) \quad \text{for } O \in O(3).$$

Invariance of the energy under individual inversions, i.e. $\mathbf{m} \mapsto -\mathbf{m}$ or $\mathbf{m} \mapsto \mathbf{m}(-\cdot)$, would change the topological charge and cannot be expected to hold for minimizers within a fixed topological sector. Moreover, invariance under any individual inversion combined with invariance under joint rotations is only satisfied by $\mathbf{m} \equiv 0$ and is thus impossible for a unit vector field.

The question immediately arises whether the symmetry property of the ground states extends to minimizers in different topological sectors, i.e. Skyrmions. However, the only maps that are invariant under all joint rotations are the outer and inner hedgehog $h_\pm$ which are the ground states and therefore no candidates for Skyrmions . Instead, the maximal possible symmetry is invariance under joint rotations and inversions which preserve a chosen symmetry axis, meaning

$$\mathbf{m} = \mathbf{m}_O \quad \text{for } O \in O(3)_{\hat{\mathbf{e}}},$$

where $O(3)_{\hat{\mathbf{e}}}$ is the stabilizer subgroup of $O(3)$ with respect to $\hat{\mathbf{e}}$. This group is isomorphic to $O(2)$. Maps that exhibit this symmetry shall be called $\hat{\mathbf{e}}$-axisymmetric.

In general, the question whether minimizers are symmetric is very hard to answer [25]. As a slightly easier question, one may ask if minimizers within a given symmetry class are locally minimizing in a larger class. Examples in which minimality could be proven include the melting hedgehog in liquid crystals [23] and axisymmetric chiral Skyrmions in the plane [32]. On the other hand, radially symmetric solutions of the Ginzburg-Landau equation are only minimizing if they are of degree 0 or 1[39], [43].

In [26], the authors use numerical simulations to obtain axisymmetric critical points of the energy at different radii of the shell $\mathbb{S} = \partial B_R(0)$. Furthermore, in the appendix, they briefly touch on the question of local minimality by considering a Fourier decomposition of an operator yielding the second variation and numerically computing the eigenvalues of the first ten modes. However, a rigorous mathematical proof of existence and local minimality of these critical points has yet to be given. Doing so is the first aim of this work and shall be discussed in chapter 3 and 4.

In particular, existence, regularity, and several qualitative properties of axisymmetric minimizers are proven in chapter 3, using standard PDE tools and a fine analysis of the energy. Local minimality, on the other hand, is obtained by first proving that the Hessian is positive semi-definite and then expressing the energy difference compared to a critical point in terms of the Hessian. Treatment of the Hessian follows a method from [23], which has been employed in several other problems, e.g. [28], [32]: The functional is decomposed into several Fourier modes which are then analyzed individually. Due to a monotonicity result, it is sufficient to consider the first two.

Dynamics of Skyrmions on the Sphere

To model the evolution of a magnetization vector under the influence of an external field over time, Landau and Lifshitz [29] have proposed the Landau-Lifshitz (LL) equation

$$\partial_t \mathbf{u} = \mathbf{u} \times h_{\text{eff}},$$

where the effective field h_{eff} is given by the negative L^2-gradient of the modeling energy. It was originally stated for the bulk case of multi layered crystals and has been widely investigated for magnetizations of flat space

$$\mathbf{u} \colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{S}^2.$$

In the original work by Landau and Lifshitz [29], the authors proposed a second equation accounting for the effects of what they called 'relativistic interaction', often referred to as damping. For an equivalent choice of damping involving the time derivative of $\mathbf{u}$, one arrives at the Landau-Lifshitz-Gilbert (LLG) equation. It was recently shown in [10] for the (LLG) equation that in the limit for thin curved films, the limiting function of solutions solves the (LLG) equation on the manifold describing the film. Setting the Gilbert damping constant to zero, we therefore consider the equation

$$\partial_t \mathbf{m} = \mathbf{m} \times (-\nabla_{L^2} \mathcal{E}(\mathbf{m})) = \mathbf{m} \times (\Delta \mathbf{m} + \kappa (\mathbf{m} \cdot \nu)\nu) \qquad \text{on } \mathbb{S}^2 \tag{1.1}$$

for a magnetization of the sphere $\mathbf{m} \colon \mathbb{S}^2 \to \mathbb{S}^2$ and the same energy as above. Details on its L^2-gradient are given in section 2.1. For simplicity, we will denote it by $\nabla \mathcal{E}$ for the remainder of this introduction.

The geometric structure of the (LL)-equation implies that the energy and the absolute value of a solution are formally conserved over time:

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}(\mathbf{m}(t)) = \langle \partial_t \mathbf{m}, \nabla \mathcal{E}(\mathbf{m}) \rangle_{L^2} = \langle \mathbf{m} \times \nabla \mathcal{E}(\mathbf{m}), \nabla \mathcal{E}(\mathbf{m}) \rangle_{L^2} = 0$$

and
$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{m}|^2 = 2\partial_t \mathbf{m} \cdot \mathbf{m} = 2\left(\mathbf{m} \times \nabla \mathcal{E}(\mathbf{m})\right) \cdot \mathbf{m} = 0.$$
In particular, this means that $\|\nabla \mathbf{m}(t)\|_{L^2} + \|\mathbf{m}(t)\|_{L^2}$ are uniformly bounded in time for solutions of the equation. A more rigorous proof of the conservation can be given in a Hamiltonian framework for the equation. This is done in section 2.1.

However, existence of solutions and a-priori bounds in H^2, which would be needed for a stability analysis, are more subtle. In many cases, such well-posedness results are only availiable for small initial data, due to the relation to the Schrödinger equation which is well-posed for small initial data [2] but can be ill-posed for large data [38]. Unfortunately, due to the topological lower bound, this is not applicable to Skyrmions and the question becomes even more challenging. In the following, we give two examples of well-posedness results for specific Landau-Lifshitz equations.

In the planar case, well-posedness of the (LL)-equation involving only exchange energy and anisotropy has been proven for initial values close to certain critical points of the energy [17]. This has been done by a transformation into a nonlinear Schrödinger equation. However, when considering (1.1) for the pullback via the inverse stereographic projection, $\tilde{\mathbf{m}} = \pi^* \mathbf{m} \colon \mathbb{R}^2 \to \mathbb{S}^2$, the curvature induced DMI mentioned above introduces additional terms. These are order one in the derivatives of $\mathbf{m}$ and are not covered by the framework of [17].

Another approach to prove well-posedness of the planar (LL)-equation for arbitrary initial data involves the construction of high-order energy functionals [7, 52]. This is done for an energy functional consisting of the exchange energy and anisotropy with respect to a fixed axis. Since the method is tailored for a specific energy functional, it is unclear whether this method could be adapted to the normal anisotropy we consider.

Instead of dealing with the full Cauchy problem, we construct specific solutions of (1.1) that follow the orbit of a static magnetization under joint rotations. In order for the thus obtained family of vector fields $\mathbf{m}_t = \mathbf{m}_{R(t)}$to solve the (LL)-equation, $\mathbf{m}$ has to satisfy $\mathbf{m} \times \nabla \mathcal{E}(\mathbf{m}) = F(\mathbf{m})$, where
$$F(\mathbf{m}) = \frac{\mathrm{d}}{\mathrm{d}t} \mathbf{m}_{R(t)}|_{t=0}$$
is the generator of a family of joint rotations. Such static magnetizations are obtained by minimizing the energy under a constraint on the total angular momentum, as has already been observed in [37].

It is of course a common concept to obtain specific solutions to difficult PDE by applying motion to an initial field that solves a matching static equation. For the (LL)-equation and the related Schrödinger equation, examples include precessing bubbles [17] and traveling wave solutions [34]. In the presence of symmetry, the static equation is typically the Euler-Lagrange equation of a certain constrained minimization problem related to the symmetry by Noether's law: Every invariance of the energy under the action of a continuous symmetry group is related to a conserved quantity of the system. The constraint is then put on the quantity that is conserved under the action of the symmetry group. The general framework of this construction method is given in [15], where the authors also develop tools for the stability analysis of the constructed solutions.

Here, the conserved quantity related to invariance under joint rotations with $R \in SO(3)$ is the total angular momentum. Control over the angular momentum also ensures that the constrained minimizers from [37] are non-symmetric. Hence, jointly rotating them at a non-zero frequency yields non-trivial solutions. In chapter 5, we show that this frequency arises as a Lagrange multiplier ω from the constrained minimization problem. Furthermore, we prove a Łojasiewicz-Simon type inequality for $\mathcal{E}$ to show that under some assumptions, the minimality results of chapters 3 and 4 ensure $\omega \neq 0$. In total, we thus give an analytical proof of the existence of non-symmetric, rotating solutions of the Landau-Lifshitz equation. Such solutions have been observed numerically in [47].

In the vicinity of critical points, a Łojasiewiz inequality provides an estimate for the energy difference in terms of the L^2-norm of the gradient. In the present case, however, since the energy takes $\mathbb{S}^2$-valued fields, some more work is necessary. This results in an an estimate in terms of the projection of the L^2-gradient gradient onto the tangent space.

Originally proven for real analytic functions by Łojasiewicz [27], the inequality was extended to the infinite dimensional setting by Simon [48] in order to show that solutions to certain non-linear evolution equations converge asymptotically to solutions of a stationary equation. Since then, the proof has been simplified [24], generalised [19] and adapted, in particular to settings with no analyticity [4], [18] and to the case of Banach manifolds[46].

The inequality is usually applied in the study of gradient flows to prove, for example, asymptotic stability, decay rates, or uniqueness of blowups [33], [5], [21]. To our knowledge, it has not yet been used to estimate a Lagrange multiplier.

1.1. Statement of Main Results

We study the energy

$$\mathcal{E}(\mathbf{m}) = \int_{\mathbb{S}^2} |\nabla \mathbf{m}|^2 + \kappa \left(1 - (\mathbf{m} \cdot \nu)^2\right) \, \mathrm{d}\sigma$$

for magnetizations $\mathbf{m}\colon \mathbb{S}^2 \to \mathbb{S}^2$ and under the restriction $Q = 0$. In particular, we are interested in axisymmetric fields of the form

$$\mathbf{m}(\Psi(x,\varphi)) = \begin{pmatrix} \sin\theta(x)\cos\varphi \\ \sin\theta(x)\sin\varphi \\ \cos\theta(x) \end{pmatrix},$$

where $\theta\colon [0,\pi] \to \mathbb{R}$ is the polar profile of $\mathbf{m}$ and $(x,\varphi) \in (0,\pi)\times(0,2\pi)$ are spherical coordinates. In this case, $Q(\mathbf{m}) = 0$ means that the profile θ satisfies

$$\cos(\theta(\pi)) - \cos(\theta(0)) = 0.$$

Axisymmetric fields are invariant under joint rotations and reflections

$$\mathbf{m} \mapsto \mathbf{m}_O = O^{-1}\mathbf{m}(O\cdot), \qquad \text{for } O \in SO(3)_{\hat{\mathbf{e}}_3}.$$

Under the assumption of symmetry, the energy of $\mathbf{m}$ only depends on the profile θ with

$$\mathcal{E}(\mathbf{m}) = 2\pi E(\theta) = 2\pi \int_0^\pi (\theta')^2 \sin x + \frac{\sin^2\theta}{\sin x} + \kappa \sin^2(\theta - x)\sin x \,\mathrm{d}x.$$

In Theorems 1 and 2, we show that minimizers in the class of axisymmetric fields with degree $Q(\mathbf{m}) = 0$ exist and are smooth. Furthermore, we give a qualitative analysis of these minimizers by investigating the polar profile θ. This is necessary to prove the following theorem about the minimality of axisymmetric minimizers beyond the class of symmetric magnetizations.

Theorem 3. *Given $\kappa > 24$, let $\mathbf{m}_0 = \mathbf{m}_\theta$ be minimizing among all axisymmetric fields of degree 0. Then the Hessian of $\mathcal{E}$ at $\mathbf{m}_0$ is positive semidefinite. Furthermore, if the reduced energy is strictly convex at θ in the sense that $\frac{d^2}{dt^2}E(\theta+t\beta) > 0$ for all variations $\beta \in C_0^\infty((0,\pi)) \setminus \{0\}|t=0$, then $\mathbf{m}_0$ is a local minimizer among all fields of degree 0 and there exist ε_0, $c > 0$ such that*

$$\mathcal{E}(\mathbf{m}) - \mathcal{E}(\mathbf{m}_0) \geq c \inf_{R \in SO(3)/SO(3)_{\hat{\mathbf{e}}_3}} \|\mathbf{m}_R - \mathbf{m}_0\|_{H^1}$$

for all $\|\mathbf{m} - \mathbf{m}_0\|_{H^1} < \varepsilon_0$.

After minimality, we turn to the dynamical problem and construct a class of jointly rotating solutions to the corresponding Landau Lifshitz equation by adding a constraint on the total angular momentum $\mathbf{J}$. In theorem 4, we prove that minimizers of the constrained minimization problem exist and solve the equation

$$\mathbf{m} \times \nabla\mathcal{E}(\mathbf{m}) = \omega\mathbf{m} \times \nabla\mathbf{J}_3(\mathbf{m}).$$

These constrained minimizers are then jointly rotated at frequency ω to obtain periodic solutions of the (LL)-equation. To prove $\omega \neq 0$, i.e. that these solutions are non-trivial, we establish a Łojasiewicz inequatliy for $\mathcal{E}$. Overall, and under some assumptions (A1)-(A3) which are discussed in chapter 5, this yields the following theorem:

Theorem 5. *Assume that for* $\kappa > 0$ *large enough, either (A1) and (A2) or (A3) hold. Then there exists* $\varepsilon_0 > 0$ *such that for all* $0 < \varepsilon < \varepsilon_0$ *there exist* $\omega \neq 0$ *and* $\mathbf{m} \in C^\infty(\mathbb{S}^2; \mathbb{S}^2)$ *such that* $\mathbf{J}(\mathbf{m}) = -(4\pi + \varepsilon)\hat{\mathbf{e}}_3$ *and*

$$\mathbf{m}(x,t) := \mathbf{m}_{R(\omega t)}(x)$$

is a nontrivial periodic solution of the Landau Lifshitz equation.

2. Preliminary Considerations

In this chapter, we will gather some preliminary results needed for the analysis in later chapters. After introducing the underlying function spaces, we will also investigate the role of symmetry.

2.1. Function Spaces and Admissible Variations

To start with, we will develop the framework of all computations by defining the relevant function spaces and parametrizations of the sphere.

For most computations, we will use spherical coordinates

$$\Psi\colon (0,\pi)\times(0,2\pi)\to \mathbb{S}^2\setminus\{(y_1,y_2,y_3)\in\mathbb{R}^3\ : y_2=0,\ y_1\geq 0\}$$

on the sphere and denote them by (x,φ). Recall that Ψ is given by

$$\Psi(x,\varphi)=\begin{pmatrix}\sin x\cos\varphi\\ \sin x\sin\varphi\\ \cos x\end{pmatrix}.$$

However, these coordinates are not defined at the poles $\pm\hat{\mathbf{e}}_3$, which are of particular interest in the symmetric case. Setting $\Psi(0,\varphi)=\hat{\mathbf{e}}_3$ and $\Psi(\pi,\varphi)=-\hat{\mathbf{e}}_3$, we can continuously extend the parametrization and overcome the polar gap in most cases. Still, we will need to be particularly careful when dealing with limits for $x\to\pi$ or $x\to 0$. A more detailed discussion of the extension at the poles and of the existence of some limits can be found in the appendix.

In other cases, the stereographic projections will be employed. For these we use the notation $\pi_\pm\colon \mathbb{S}^2\setminus\{\pm\hat{\mathbf{e}}_3\}\to\mathbb{R}^2$ where

$$\pi_\pm(\mathbf{x})=\frac{1}{1\mp x_3}\begin{pmatrix}x_1\\ x_2\end{pmatrix}\qquad\text{and}\qquad \pi_\pm(\Psi(x,\varphi))=\frac{\sin x}{1\mp\cos x}\begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}.$$

Furthermore, for $p\in\mathbb{S}^2$, π_p shall denote the stereographic projection centered at p such that for any $R\in SO(3)$ with $Rp=p$, it holds that $\pi_p(\mathbf{x})=\pi_-(R\mathbf{x})$. In particular, $\pi_p(p)=0=\pi_-(\hat{\mathbf{e}}_3)=\pi_+(-\hat{\mathbf{e}}_3)$.

The energy

$$\mathcal{E}(\mathbf{m}) = \int\limits_{\mathbb{S}^2} |\nabla_{\mathbb{S}^2}\mathbf{m}|^2 + \kappa\left(1 - (\mathbf{m}\cdot\nu)^2\right)\,\mathrm{d}\sigma$$

is finite for $\mathbf{m}\colon \mathbb{S}^2 \to \mathbb{S}^2$ such that $|\nabla\mathbf{m}|^2$ is integrable over $\mathbb{S}^2$. Due to the conformal invariance of the Dirichlet term, this is equivalent to square integrability of $|\nabla(\mathbf{m}\circ\pi_\pm^{-1})|$ over $\mathbb{R}^2$. For higher order derivatives, however, the Riemannian metric on $\mathbb{S}^2$ introduces additional factors of $(1+|x|^2)^{2k}$.

By Aubin [1], Sobolev spaces on Riemannian manifolds are independent of the choice of the metric. We make use of this fact by defining Sobolev spaces on the sphere via the stereographic projection.

Definition 1. *Let (η_1, η_2) be a partition of unity on $\mathbb{S}^2$ such that $\eta_1(\hat{\mathbf{e}}_3) = 0 = \eta_2(-\hat{\mathbf{e}}_3)$. For a C^k– function $\mathbf{m}\colon \mathbb{S}^2 \to \mathbb{R}$, we define*

$$\|\mathbf{m}\|^2_{\dot{H}^\ell(\mathbb{S}^2)} := \int\limits_{\mathbb{R}^2} \left(\left|\nabla^\ell\left(\mathbf{m}\circ\pi_+\right)\right|^2 \eta_1\circ\pi_+ + \left|\nabla^\ell(\mathbf{m}\circ\pi_-)\right|^2 \eta_2\circ\pi_-\right)(1+|x|^2)^{2\ell-2}\,\mathrm{d}x.$$

Let $\mathcal{C}^k(\mathbb{S}^2)$ be the space of C^∞ functions $\mathbf{m}$ on $\mathbb{S}^2$ such that $\|\mathbf{m}\|_{\dot{H}^\ell(\mathbb{S}^2)} < \infty$ for all $0 \le \ell \le k$. Then we define $H^k(\mathbb{S}^2)$ to be the completion of $\mathcal{C}^k(\mathbb{S}^2)$ with respect to the $\|\cdot\|_{H^k(\mathbb{S}^2)}$-norm, where

$$\|\mathbf{m}\|^2_{H^k(\mathbb{S}^2)} = \sum_{0=\ell}^{k} \|\mathbf{m}\|^2_{\dot{H}^\ell(\mathbb{S}^2)}.$$

Sobolev spaces over n-dimensional compact manifolds behave similarly to regular Sobolev spaces over $\Omega \subset \mathbb{R}^n$ open such that $\bar{\Omega}$ is compact. Most notably, the Sobolev imbedding and the Rellich-Kondrakov theorem hold [1, Theorem 2.20 and Theorem 2.34]. This ensures

$$H^k(\mathbb{S}^2) \subset C^0(\mathbb{S}^2)$$

for $k \ge 2$, thus justifying pointwise investigations for fields in $H^2(\mathbb{S}^2)$. Furthermore, the validity of the Sobolev imbedding is sufficient to make $H^k(M_n)$ an algebra for $k \ge \frac{n}{2}$. A proof of the statement in the flat case can be found in [53], an adaptation to the case of manifolds requires no changes.

Arguing componentwise, it easily follows that the Sobolev spaces $H^k(\mathbb{S}^2;\mathbb{R}^3)$ are well defined and that the imbeddings still hold. However, we are interested in $\mathbb{S}^2$-valued fields $\mathbf{m}\colon \mathbb{S}^2 \to \mathbb{S}^2$ which do no not form a linear space anymore. Pointwise tangential fields, on the other hand, do.

Definition 2. *Given $\mathbf{m}\colon \mathbb{S}^2 \to \mathbb{S}^2$ and a field $v\colon \mathbb{S}^2 \to \mathbb{R}^3$, we write $v\colon \mathbb{S}^2 \to T_{\mathbf{m}}\mathbb{S}^2$ if v satisfies*

$$v(x) \in T_{\mathbf{m}(x)}\mathbb{S}^2 \quad \textit{for almost every } x \in \mathbb{S}^2.$$

Moreover,

$$H^k(\mathbb{S}^2; T_{\mathbf{m}}\mathbb{S}^2) := \{v \in H^k(\mathbb{S}^2; \mathbb{R}^3) \;:\; v(x) \in T_{\mathbf{m}(x)}\mathbb{S}^2 \text{ a.e.}\}.$$

Lemma 2.1. *For every $\mathbf{m}\colon \mathbb{S}^2 \to \mathbb{S}^2$ and $k \geq 0$, the space $H^k(\mathbb{S}^2; T_{\mathbf{m}}\mathbb{S}^2)$ is a Hilbert space enjoying the same imbedding properties as $H^k(\mathbb{S}^2; \mathbb{R}^3)$.*

Proof. This follows from the fact that $H^k(\mathbb{S}^2; T_{\mathbf{m}}\mathbb{S}^2)$ is a closed subspace of $H^k(\mathbb{S}^2; \mathbb{R}^3)$. Indeed, it is a linear space due to the bilinearity of the scalar product in $\mathbb{R}^3$. Moreover, the tangential property is characterized by $v(x) \cdot \mathbf{m}(x) = 0$ which is conserved by pointwise convergence almost everywhere. This, on the other hand, follows from strong L^2-convergence and thus from H^k-convergence. □

To abbreviate notation, we will sometimes also write $v \in T_{\mathbf{m}}\mathbb{S}^2$ for fields $v\colon \mathbb{S}^2 \to T_{\mathbf{m}}\mathbb{S}^2$.

2.1.1. Manifold Structure

While the spaces $H^k(\mathbb{S}^2; \mathbb{S}^2)$ are not closed under summation and can therefore not be Sobolev spaces, they are Banach manifolds for $k \geq 2$ due to Sobolev estimates. We use the following definitions by Zeidler.

Definition
– from [58], p. 533, definition 73.2
Let M be a topological space. A chart (U, φ) is a pair where the set U is open in M and $\varphi\colon U \to U_\varphi$ is a homeomorphism onto an open subset U_φ of a Banach-space X_φ. We call φ a chart map.

Definition
– from [58], p. 535, definition 73.4
Let M be a topological space. A C^k*-atlas for* M, $0 \leq k \leq \infty$ is a collection of charts $(U_\alpha, \varphi_\alpha)$ (α ranging in some indexing set), which satisfies the following conditions:

(i) The U_α cover M.

(ii) Any two charts are C^k-compatible.

(iii) All chart spaces X_α are Banach spaces over $\mathbb{K}$.

M is said to be a C^k-Banach manifold if and only if there exists a C^k-atlas for M.

Note: In [58], two charts (U_i, φ_i) are called C^k-compatible if their domains are disjoint or if both $\varphi_1 \circ \varphi_2^{-1}$ and $\varphi_2 \circ \varphi_1^{-1}$ are C^k.

Following this definition, we will show that control of the C^0 or L^∞ norm together with H^1 is sufficient for a Banach manifold structure for a set of maps $\mathbf{m}\colon \mathbb{S}^2 \to \mathbb{S}^2$. To that end, for every $k \in \mathbb{N}_0$, let $X^k(\mathbb{S}^2;\mathbb{S}^2)$ be the set of $\mathbb{S}^2$-valued maps in $H^k(\mathbb{S}^2;\mathbb{R}^3)$ from above, equipped with the norm $\|\cdot\|_{X^k}$ where

$$\|\mathbf{m}\|_{X^k} = \|\mathbf{m}\|_{H^k(\mathbb{S}^2;\mathbb{R}^3)} + \|\mathbf{m}\|_{L^\infty(\mathbb{S}^2;\mathbb{R}^3)}.$$

We show that $X^k(\mathbb{S}^2;\mathbb{S}^2)$ is a Banach manifold for $k \in \mathbb{N}_0$ by defining local charts over a suitable set of maps $v\colon \mathbb{S}^2 \to T_{\mathbf{m}}\mathbb{S}^2$. The L^∞ norm is needed to ensure that the images of the charts remain bounded and to show compatibility of charts. While $\|\mathbf{m}\|_{L^\infty} = 1$ for all $\mathbf{m}\colon \mathbb{S}^2 \to \mathbb{S}^2$, the norm $\|\cdot\|_X$ introduces additional control over the difference of two such functions. For $k \geq 2$, the norm $\|\cdot\|_X$ is equivalent to the $H^k(\mathbb{S}^2;\mathbb{R}^3)$ norm due to the embedding of H^k into $C^0(\mathbb{S}^2;\mathbb{R}^3)$.

Lemma 2.2. *The space $X^k(\mathbb{S}^2;\mathbb{S}^2)$ defined above is a smooth Banach manifold.*

Proof. Fix $k \geq 0$. In the following, we will mostly omit the specification of differentiability and write $X = X^k(\mathbb{S}^2;\mathbb{S}^2) = (H^k(\mathbb{S}^2;\mathbb{S}^2), \|\cdot\|_X)$. Given $\mathbf{m}_0 \in H^k(\mathbb{S}^2;\mathbb{S}^2)$, consider the set

$$X^k(\mathbb{S}^2;T_{\mathbf{m}}\mathbb{S}^2) := \{v\colon \mathbb{S}^2 \to T_{\mathbf{m}}\mathbb{S}^2 \;:\; \|v\|_{X^k} < \infty\}.$$

Equipped with the norm $\|\cdot\|_X$, the spaces $X^k(\mathbb{S}^2;T_{\mathbf{m}}\mathbb{S}^2) = H^k(\mathbb{S}^2;T_{\mathbf{m}}\mathbb{S}^2) \cap L^\infty(\mathbb{S}^2;T_{\mathbf{m}}\mathbb{S}^2)$ are Banach spaces. Henceforth, we will also leave out the specification of k for the spaces $X^k(\mathbb{S}^2;T_{\mathbf{m}}\mathbb{S}^2)$. We will construct a chart around $\mathbf{m}$ by considering

$$\psi\colon X(\mathbb{S}^2;T_{\mathbf{m}}\mathbb{S}^2) \to X,\ \psi(v) = \frac{\mathbf{m}+v}{|\mathbf{m}+v|}.$$

First, note that

$$|\mathbf{m}+v|^2 = |\mathbf{m}|^2 + 2\mathbf{m}\cdot v + |v|^2 = 1 + |v|^2$$

so that ψ is well-defined. Furthermore, the L^∞ bounds for $\mathbf{m}$ and v imply that

$$\left|\frac{\mathbf{m}+v}{|\mathbf{m}+v|}\right| = 1, \qquad \left|\frac{1}{|\mathbf{m}+v|}\right| \leq 1 + \|v\|_{L^\infty}$$

and

$$\|\nabla\psi(v)\|_{L^2(\mathbb{S}^2)} \lesssim \left(\|\nabla\mathbf{m}\|_{L^2(\mathbb{S}^2)} + \|\nabla v\|_{L^2(\mathbb{S}^2)}\right)(1 + \|v\|_{L^\infty}).$$

For $k \geq 2$, the bounds for higher order derivatives follow componentwise from $H^k(\mathbb{S}^2;\mathbb{R})$ being an algebra. Thus, we find $\|\psi(v)\|_X \leq c\|v\|_X$ for all v in some neighborhood of $0 \in X(\mathbb{S}^2;T_{\mathbf{m}}\mathbb{S}^2)$ and some $c > 0$. To compute the range of ψ, note that

$$\mathbf{m}\cdot\psi(v) = \frac{1+(\mathbf{m}\cdot v)}{|\mathbf{m}+v|} = (|\mathbf{m}+v|)^{-1} = \left(1+|v|^2\right)^{-\frac{1}{2}} > 0.$$

On the other hand, given $\tilde{\mathbf{m}}$ with $\mathbf{m} \cdot \tilde{\mathbf{m}} > 0$, the tangent field $v = \frac{\tilde{\mathbf{m}}}{\mathbf{m} \cdot \tilde{\mathbf{m}}} - \mathbf{m}$ satisfies

$$\mathbf{m} + v = \frac{\tilde{\mathbf{m}}}{\tilde{\mathbf{m}} \cdot \mathbf{m}}$$

such that $\psi(v) = \tilde{\mathbf{m}}$ and the range of ψ is

$$\{\tilde{\mathbf{m}} \in X \ : \ \tilde{\mathbf{m}} \cdot \mathbf{m} > 0\}.$$

To ensure compatibility of the charts, we set

$$U := \left\{ \tilde{\mathbf{m}} \in X \ : \ \tilde{\mathbf{m}} \cdot \mathbf{m} > 0 \text{ and } \ \|\tilde{\mathbf{m}} - \mathbf{m}\|_X < \frac{1}{2} \right\}.$$

By above computations, $\psi^{-1} \colon U \to U_\varphi$ with

$$\psi^{-1}(\tilde{\mathbf{m}}) = \frac{\tilde{\mathbf{m}}}{\mathbf{m} \cdot \tilde{\mathbf{m}}} - \mathbf{m}$$

is the inverse of ψ. Continuity follows from the bounds for $\|\mathbf{m}\|_X$ in a similar fashion as continuity of ψ. Thus, the pair (U, ψ^{-1}) defines a local chart around $\mathbf{m}$. It remains to be shown that any two charts are compatible. For that, consider charts (U_1, ψ_1), (U_2, ψ_2) around $\mathbf{m}_1$, $\mathbf{m}_2 \in X$. If $\mathbf{m}_1(x) \cdot \mathbf{m}_2(x) < \frac{1}{2}$ for some $x \in \mathbb{S}^2$, then

$$|\mathbf{m}_1(x) - \mathbf{m}_2(x)|^2 = 1 - 2\mathbf{m}_1(x) \cdot \mathbf{m}_2(x) + 1 > 1.$$

Hence, if the inequality holds on a set of positive measure, then $\|\mathbf{m}_1 - \mathbf{m}_2\|_{L^\infty} > 1$ and $U_1 \cap U_2 = \emptyset$. On the other hand, if the intersection is non-empty, then $\mathbf{m}_1 \cdot \mathbf{m}_2 \geq \frac{1}{2}$ almost everywhere and the composition of the two charts is given by

$$\psi_1^{-1} \circ \psi_2(v) = \psi_1^{-1}\left(\frac{\mathbf{m}_2 + v}{|\mathbf{m}_2 + v|}\right) = \frac{\mathbf{m}_2 + v}{\mathbf{m}_1 \cdot (\mathbf{m}_2 + v)} - \mathbf{m}_1 = \frac{\mathbf{m}_2 + v}{\mathbf{m}_1 \cdot \mathbf{m}_2} - \mathbf{m}_1,$$

which is a smooth mapping from $X(\mathbb{S}^2; T_{\mathbf{m}}\mathbb{S}^2)$ to $X(\mathbb{S}^2; T_{\mathbf{m}}\mathbb{S}^2)$ due to $\mathbf{m}_1 \cdot \mathbf{m}_2 > \frac{1}{2}$ almost everywhere. In the last step it has been used that $v \in \psi_2^{-1}(U_2) \subset X(\mathbb{S}^2; T_{\mathbf{m}}\mathbb{S}^2)$. □

Corollary 2.1. *Given $\mathbf{m} \in X^k(\mathbb{S}^2; \mathbb{S}^2)$, a representation of the tangent space $T_{\mathbf{m}}X^k(\mathbb{S}^2; \mathbb{S}^2)$ is given by $X^k(\mathbb{S}^2; T_{\mathbf{m}}\mathbb{S}^2)$.*

Proof. Since the spaces $X^k(\mathbb{S}^2; T_{\mathbf{m}}\mathbb{S}^2)$ are the chart spaces for $X^k(\mathbb{S}^2; \mathbb{S}^2)$, this follows directly from proposition 73.12 in [58]. □

2.1.2. First and Second Variation of the Energy

Considering $\mathcal{E}$ as a functional on $X^1(\mathbb{S}^2; \mathbb{S}^2)$, it follows that $\mathrm{d}\mathcal{E}$ is given by $\mathrm{d}(\mathcal{E} \circ \psi)$. Fixing $\mathbf{m} \in X^1(\mathbb{S}^2; \mathbb{S}^2)$, we thus define

$$\mathbf{m}_t := \frac{\mathbf{m} + tv}{|\mathbf{m} + tv|} = \psi(tv) \quad v \in X^1(\mathbb{S}^2; T_{\mathbf{m}}\mathbb{S}^2), \ t \in \mathbb{R}.$$

To simplify computations, consider the nearest point projection

$$\Pi\colon \mathbb{R}^3 \setminus \{0\} \to \mathbb{S}^2;\ x \mapsto \frac{x}{|x|}.$$

For $v \in T_{\mathbf{m}}\mathbb{S}^2$, $\Pi(\mathbf{m}+v) = \psi(v)$ is well defined and smooth with

$$D\Pi(\mathbf{m}+v)\langle w\rangle = \frac{1}{|\mathbf{m}+v|}\left(\mathbb{1} - \frac{1}{|\mathbf{m}+v|^2}(\mathbf{m}+v)\otimes(\mathbf{m}+v)\right) w,$$
$$\frac{d}{dt}\Pi(\mathbf{m}+tv)\Big|_{t=0} = D\Pi(\mathbf{m})\langle v\rangle = v - (\mathbf{m}\otimes\mathbf{m})v = v$$

since $\mathbf{m}\cdot v = 0$. Furthermore, the second derivative is given by the second fundamental form A on the sphere:

$$\frac{d^2}{dsdt}\Pi(\mathbf{m}+tv+sw)\Big|_{s=t=0} = D^2\Pi(\mathbf{m})\langle v,w\rangle = -A(\mathbf{m})\langle v,w\rangle = (v\cdot w)\mathbf{m}.$$

Furthermore, we introduce $\tilde{\mathcal{E}}$, the extension of $\mathcal{E}$ to $H^1\cap L^\infty(\mathbb{S}^2;\mathbb{R}^3)$. For $v \in X^1(\mathbb{S}^2;T_{\mathbf{m}}\mathbb{S}^2)$, we now employ Π and $\tilde{\mathcal{E}}$ to compute the Gateaux derivative of $\mathcal{E}\circ\psi$ at $0 = \psi^{-1}(\mathbf{m})$ in direction v.

$$\begin{aligned}\delta\mathcal{E}(\mathbf{m})\langle v\rangle &= \delta\left(\mathcal{E}\circ\psi\right)(0)\langle v\rangle = \frac{d}{dt}\mathcal{E}(\mathbf{m}_t)\Big|_{t=0}\\ &= \delta\tilde{E}(\mathbf{m})\langle D\Pi(\mathbf{m})\langle v\rangle\rangle = \delta\tilde{\mathcal{E}}(\mathbf{m})\langle v\rangle\\ &= \int_{\mathbb{S}^2} \nabla\mathbf{m}\cdot\nabla v - \kappa(\mathbf{m}\cdot\nu)\nu\cdot v\,\mathrm{d}\sigma\\ &= \int_{\mathbb{S}^2} \left(-\Delta_{\mathbb{S}^2}\mathbf{m} - \kappa\left(\mathbf{m}\cdot\nu\right)\nu\right)\cdot v\,\mathrm{d}\sigma,\end{aligned}$$

Consequently, the $L^2(\mathbb{S}^2;T_{\mathbf{m}}\mathbb{S}^2)$-gradient of $\mathcal{E}$ is given by $-\Delta\mathbf{m}-\kappa(\mathbf{m}\cdot\nu)\nu$. For a general testfunction $\phi \in C^\infty(\mathbb{S}^2;\mathbb{R}^3)$, we may set $v = \mathbf{m}\times\phi$ such that

$$\delta\mathcal{E}(\mathbf{m})\langle\phi\rangle = \int_{\mathbb{S}^2}\left(-\Delta_{\mathbb{S}^2}\mathbf{m} - \kappa\left(\mathbf{m}\cdot\nu\right)\nu\right)\cdot(\mathbf{m}\times\phi)\,\mathrm{d}\sigma = \int_{\mathbb{S}^2}\left(\mathbf{m}\times\left(-\Delta_{\mathbb{S}^2}\mathbf{m} - \kappa\left(\mathbf{m}\cdot\nu\right)\nu\right)\right)\cdot\phi\,\mathrm{d}\sigma.$$

It follows that critical points of the energy are weak solutions of the equation

$$0 = \mathbf{m}\times\left(-\Delta_{\mathbb{S}^2}\mathbf{m} - \kappa(\mathbf{m}\cdot\nu)\nu\right) =: \left(\nabla\mathcal{E}(\mathbf{m})\right)^{\tan}.$$

For the second variation and setting $\mathbf{m}_{st} := \Pi(\mathbf{m}+sv+tw)$ where $v,w \in X^1(\mathbb{S}^2;T_{\mathbf{m}}\mathbb{S}^2)$, the product rule gives

$$\begin{aligned}\delta^2\mathcal{E}(\mathbf{m})\langle v,w\rangle &= \frac{d^2}{dsdt}\mathcal{E}(\mathbf{m}_{st}) = \frac{d}{ds}\left(\delta\tilde{\mathcal{E}}(\mathbf{m}+sv)\left\langle D\Pi(\mathbf{m}+sv)\langle w\rangle\right\rangle\right)\Big|_{s=0}\\ &= \delta^2\tilde{\mathcal{E}}(\mathbf{m})\langle D\Pi(\mathbf{m})\langle v\rangle, D\Pi(\mathbf{m})\langle w\rangle\rangle + \delta\tilde{\mathcal{E}}(\mathbf{m})\langle D^2\Pi(\mathbf{m})\langle v,w\rangle\rangle\\ &= \delta^2\tilde{\mathcal{E}}(\mathbf{m})\langle v,w\rangle - \delta\tilde{\mathcal{E}}(\mathbf{m})\langle(v\cdot w)\mathbf{m}\rangle.\\ &= \int_{\mathbb{S}^2}\nabla v\cdot\nabla w - \kappa(v\cdot\nu)(w\cdot\nu)\,\mathrm{d}\sigma - \int_{\mathbb{S}^2}\left(|\nabla\mathbf{m}|^2 - \kappa\left(\mathbf{m}\cdot\nu\right)^2\right)(v\cdot w)\,\mathrm{d}\sigma.\end{aligned}$$

2.1.3. Hamiltonian Framework

We now return to the Landau-Lifshitz equation (1.1). In order to give a more detailed reasoning for the energy conservation, we will establish a triplet (P, ω, H) with $H = \mathcal{E}$ such that the flow associated with the Hamiltonian H describes the solutions of (1.1). Assuming the existence of solutions, conservation of energy then follows from a result for infinite dimensional Hamiltonian systems.

All definitions and the theorem cited below are taken from the first lecture of a set of notes of Marsden [36]. However, some details have been omitted.

Definition
– from [36], p. 5
A *symplectic manifold* is a pair (P, ω) where P is a C^∞ Banach manifold and ω is a C^∞ two-form on P, such that

(i) $\mathrm{d}\omega = 0$;

and

(ii) ω is (weakly) nondegenerate: for all $x \in P$ and $v_x \in T_x P$,

$$\omega_x(v_x, w_x) = 0$$

for all $w_x \in T_x P$ implies $v_x = 0$.

Note: The exterior derivative for differential forms on Banach manifolds that was used in (i) is defined in a similar manner as the finite dimensional one, see [30, Proposition 3.2]. However, it is often simpler to define ω via an almost complex structure $\mathcal{J}$, see [36, Remark 6].

Setting $P = X^0(\mathbb{S}^2; \mathbb{S}^2)$ from above, we define ω by

$$\omega_{\mathbf{m}}(v_{\mathbf{m}}, w_{\mathbf{m}}) = \langle v_{\mathbf{m}}, \mathbf{m} \times w_{\mathbf{m}} \rangle_{L^2(\mathbb{S}^2)} \quad \text{for } v_{\mathbf{m}},\ w_{\mathbf{m}} \in T_{\mathbf{m}} P = X^0(\mathbb{S}^2; T_{\mathbf{m}}\mathbb{S}^2).$$

We will omit the subscripts $\mathbf{m}$ whenever it is unambiguous.

The form ω is thus defined via the Riemannian metric on $X^0(\mathbb{S}^2; \mathbb{S}^2)$,

$$g(\mathbf{m})\langle v_m, w_m \rangle = \langle v_m, w_m \rangle_{L^2(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2)} \quad v_m,\ w_m \in X^0(\mathbb{S}^2; T_{\mathbf{m}}\mathbb{S}^2),$$

and the almost complex structure $\mathcal{J}$ that is given by

$$\mathcal{J} \colon TX^2(\mathbb{S}^2; \mathbb{S}^2) \to TX(\mathbb{S}^2; \mathbb{S}^2),\ \mathcal{J}_{\mathbf{m}} v_m = \mathbf{m} \times v_m.$$

It satisfies $\mathcal{J}^2 v = \mathbf{m} \times (\mathbf{m} \times v) = (\mathbf{m} \cdot \mathbf{m})v - (\mathbf{m} \cdot v)\mathbf{m} = v$. Not only does this definition of ω imply that $\omega\langle v, w\rangle = g\langle v, \mathcal{J} w\rangle$ is a closed and smooth two-form on $X^0(\mathbb{S}^2; \mathbb{S}^2)$ but it also follows that ω is weakly nondegenerate, as we shall briefly show. Given $\mathbf{m} \in P$, the identity $\omega(v, w) = 0$ for all $w \in X^0(\mathbb{S}^2; T_{\mathbf{m}}\mathbb{S}^2)$ implies

$$0 = \omega(v, \mathbf{m} \times v) = -\omega(\mathbf{m} \times v, v) = -\langle \mathbf{m} \times v, \mathbf{m} \times v\rangle_{L^2(\mathbb{S}^2;\mathbb{R}^3)} = -\|\mathbf{m} \times v\|^2_{L^2(\mathbb{S}^2)}.$$

But, since $\mathbf{m} \times v \in L^2(\mathbb{S}^2; \mathbb{R}^3)$, it follows from $\|\mathbf{m} \times v\|_{L^2} = 0$ that $\mathbf{m} \times v = 0$ almost everywhere on $\mathbb{S}^2$. On the other hand, $(\mathbf{m} \times v)(x) = 0$ implies that $v(x)$ is a multiple of $\mathbf{m}(x)$ and thus

$$v(x) = ((v \cdot \mathbf{m})\mathbf{m})(x) = 0$$

since $v \colon \mathbb{S}^2 \to T_{\mathbf{m}}\mathbb{S}^2$ means that $v(x) \in T_{\mathbf{m}(x)}\mathbb{S}^2$ for almost every $x \in \mathbb{S}^2$. Therefore, (P, ω) is a weak symplectic manifold.

Definition
– from [36], p. 11
Let (P, ω) be a (weak) symplectic manifold and $H \colon D_H \to \mathbb{R}$ a C^1 function where D_H is a manifold domain in P. We call the triple (P, ω, H) a Hamiltonian system. Set

$$D_{X_H} = \{x \in D_H \;: \exists v \in T_x P,\; \mathrm{d}H(x)\langle w\rangle = \omega(v, w) \text{ for all } w \in T_x D_H\}$$

and call X_H with $X_H(x) = v_x$ the Hamiltonian vector field of H.

Note: Here, a manifold domain is a subset $D_H \subset P$ for which the inclusion of the tangent spaces is dense, see [36, Remark 5]. It generalizes the notion of domains to non-linear spaces. Introducing domains allows for H to be defined on a set of functions with higher regularity than those of the underlying manifold. In particular, $X^k(\mathbb{S}^2; \mathbb{S}^2)$ is a manifold domain of $X^0(\mathbb{S}^2; \mathbb{S}^2)$ for all $k \geq 1$.

As announced in the introduction of the section, we choose $H = \mathcal{E} \colon P \to \mathbb{R}$ with domain $D_H = X^1(\mathbb{S}^2; \mathbb{S}^2)$. Then, given $\mathbf{m} \in P$ and $w \in X^1(\mathbb{S}^2; T_{\mathbf{m}}\mathbb{S}^2) = T_{\mathbf{m}} D_H$, we have

$$\begin{aligned}
\mathrm{d}H(\mathbf{m})\langle w\rangle &= \langle(-\Delta\mathbf{m} - \kappa(\mathbf{m} \cdot \nu)\nu), w\rangle_{L^2} = \langle(-\Delta\mathbf{m} - \kappa(\mathbf{m} \cdot \nu)\nu), -\mathbf{m} \times (\mathbf{m} \times w)\rangle_{L^2} \\
&= \langle \mathbf{m} \times (\Delta\mathbf{m} + \kappa(\mathbf{m} \cdot \nu)\nu)\,, \mathbf{m} \times w\rangle \\
&= \omega(P_{\mathbf{m}}\left(\nabla\mathcal{E}\right), w)
\end{aligned}$$

and thus

$$X_H(\mathbf{m}) = \mathbf{m} \times (\Delta\mathbf{m} + \kappa(\mathbf{m} \cdot \nu)\nu)\,.$$

To ensure $\mathbf{m} \times (\Delta\mathbf{m} + \kappa(\mathbf{m} \cdot \nu)\nu) \in T_{\mathbf{m}} P$ it would be sufficient to choose $\mathbf{m} \in X^2(\mathbb{S}^2; \mathbb{S}^2)$. Thus, $X^2(\mathbb{S}^2; \mathbb{S}^2) \subset D_{X_H}$. However, D_{X_H} might be larger due to regularity considerations.

Note: In general, given a function $f \colon X^k(\mathbb{S}^2; \mathbb{S}^2) \to \mathbb{R}$, the vector field X_f satisfying $df(x)\langle w\rangle = \omega(v, w)$ is given by $\mathbf{m} \times (-\nabla_{L^2(\mathbb{S}^2; T_{\mathbf{m}}\mathbb{S}^2)} f)$ whenever this L^2-gradient exists.

Definition
– from [36], p. 12
Let P be a Banach manifold and $D \subset P$ be a manifold domain. Let $G\colon D \to TP$ be a vector field with domain D. By a semiflow for G we mean a map $F\colon R \subset (D \times [0,\infty)) \to D$ where $R \subset D \times [0,\infty)$ is open, with the following properties:

(i) F is continuous.

(ii) $D \times \{0\} \subset R$ and $F(x,0) = x$ for all $x \in D$.

(iii) Let $t, s \geq 0$ and $x \in D$. Then
$$(x, t+s) \in R \Leftrightarrow (x,s) \in R \text{ and } (F(x,s),t) \in R.$$
In this case, $F(x,t+s) = F(F(x,s),t)$.

(iv) For $t \geq 0$,
$$\frac{d}{dt}F(x,t) = G(F(x,t)).$$

In contrast to finite dimensional Hamiltonian systems, even local existence and uniqueness of a semiflow for a vectorfield are not guaranteed. However, if the flow for $G = X_H$ exists on a set $R \subset D_{D_H}$, then for $\mathbf{m}_0 \in R$, setting $\mathbf{m}(t) = F(\mathbf{m}_0, t)$ gives a solution to the equations
$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{m}(t) &= X_H(\mathbf{m}(t)) = \mathbf{m} \times (\Delta\mathbf{m} + \kappa(\mathbf{m}\cdot\nu)\nu) \\ \mathbf{m}(0) &= \mathbf{m}_0. \end{cases}$$
Conservation of the energy follows from the following theorem with $K = H$.

Theorem
– from [36], p. 12
Let (P, ω, H) be a Hamiltonian system and let $K\colon D_K \to \mathbb{R}$ be a C^1 function. Assume:

(i) X_H has a semiflow F.

(ii) $D_{X_H} \subset P$ is a manifold domain.

(iii) $D_{X_K} \supset D_{X_H}$ and $X_K\colon D_{X_H} \to TP$ is continuous. Then, for each $x_0 \in D_{X_H}$ and $t > 0$, $(x_0, t) \in R$,
$$\frac{d}{dt}K(F(x_0,t)) = \{K,H\}(F(x_0,t))$$
where the derivative is from the right for $t = 0$.

Note: The induced Poisson bracket is defined via the symplectic form by
$$\{f,g\}(x) = \omega_x(X_f(x), X_g(x))$$

for functions $f, g\colon P \to \mathbb{R}$. If the $L^2(\mathbb{S}^2; T_{\mathbf{m}}\mathbb{S}^2)$-gradients of f, g exist and are given by $\nabla_{L^2} f$ and $\nabla_{L^2} g$, respectively, then the poisson bracket takes the form

$$\{f, g\}(\mathbf{m}) = \omega_{\mathbf{m}}(X_f(\mathbf{m}), X_g(\mathbf{m})) = \langle \mathbf{m} \times \nabla_{L^2} f, \mathbf{m} \times (\mathbf{m} \times \nabla_{L^2} g) \rangle_{L^2} = \omega_{\mathbf{m}}(\nabla_{L^2} f, \nabla_{L^2} g).$$

For $K = H$, it immediately follows from the alternating property of two-forms that $\omega_{\mathbf{m}}(X_H, X_H) = 0$. Furthermore, given $x \in \mathbb{S}^2$, the function $K_x\colon X^2(\mathbb{S}^2; \mathbb{S}^2) \to \mathbb{R}$ with $K_x(\mathbf{m}) = |\mathbf{m}(x)|^2$ is well defined due to the embedding $H^2(\mathbb{S}^2; \mathbb{R}^3) \hookrightarrow C^0(\mathbb{S}^2; \mathbb{S}^2)$. It satisfies

$$dK_x(\mathbf{m})\langle v \rangle = 2\mathbf{m}(x) \cdot v(x) = 0$$

for all $v \in X^2(\mathbb{S}^2; T_{\mathbf{m}}\mathbb{S}^2)$ and therefore $X_{K_x} = 0$. Hence, $\{K_x, H\} = 0$ for every $x \in \mathbb{S}^2$ and $|\mathbf{m}|$ is pointwisely conserved by the flow of X_H.

2.2. Symmetry

In this section, we take a closer look at the invariance of the energy under joint rotations and inversions and at the corresponding symmetries. Since Emmi Noether famously discovered a connection between one-parameter symmetries and invariants of a system [42], researchers have been interested in finding solutions that display as many symmetries as the system they solve. On the other hand, symmetry can also reduce equations to more accessible ones, allowing for better results than in the general case. In particular, radial solutions exploiting the invariance of $\|\nabla \mathbf{m}\|_{L^2}^2$ under rotations on either domain or target space gave rise to many discoveries for wave maps , see e.g. [51]. On the other hand, this symmetry is broken in many Skyrmion models, for example due to chiral terms like DMI [17, 32] or, in the present case, anisotropy on a curved surface. In this scenario, individual rotational symmetry is reduced to equivariance under the operation of a rotation group.

Definition
– see [54], p. 4
Let X, Y be sets and G a group acting on X, Y. A map $f\colon X \to Y$ is called G-equivariant if f commutes with the action of G i.e. $f(g.x) = g.f(x)$ for all $x \in X$, $g \in G$.

In mathematical physics, G is usually chosen to be a suitable rotation group in matrix representation acting on X and Y via simple multiplication. In contrast, the term k-equivariance is used when the action of X is simple multiplication while the action of G on Y is multiplication by g^k:

$$g.x = R_g x \text{ and } g.y = R_g^k y.$$

2.2.1. Symmetry for Curvature Stabilized Skyrmions

The energy functional for curvature stabilized Skyrmions is not invariant under individual rotations of the domain or target sphere. This is due to the anisotropy term $(1-(\mathbf{m}\cdot\nu)^2)$, which relates $\mathbf{m}(x)$ to the outer unit normal $\nu(x)$. However, ν satisfies $\nu(Rx) = R\nu(x)$ such that the anisotropy term is invariant under joint rotations $\mathbf{m} \mapsto \mathbf{m}_R$ where

$$\mathbf{m}_R(x) = R^{-1}\mathbf{m}(Rx).$$

Maps that are invariant under all joint rotations $R \in G \subset SO(3)$ are exactly the G-equivariant maps. For $G = SO(3)$, this is a very small class containing only the hedgehog $h(x) \equiv x$ and $-h$. If $\kappa \geq 4$, this is the ground state [11] with topological degree $Q(\pm h) = \pm 1$. In the search for nontrivial energy minimizers with $Q(\mathbf{m}) = 0$, we therefore have to choose a suitable subgroup of $SO(3)$, for example by restricting the invariance of maps to rotations around a fixed axis $\hat{\mathbf{e}}$. This corresponds to $G = SO(3)_{\hat{\mathbf{e}}}$, the stabilizer subgroup of $SO(3)$ with respect to $\hat{\mathbf{e}}_3$. We will use the term $\hat{\mathbf{e}}$-equivariant for $SO(3)_{\hat{\mathbf{e}}}$-equivariant maps and drop the specification of the axis for $\hat{\mathbf{e}} = \hat{\mathbf{e}}_3$. Most of the following properties hold for maps that are $\hat{\mathbf{e}}$-equivariant almost everywhere on $\mathbb{S}^2$ and all other cases will be specifically marked.

We observe some properties of $\hat{\mathbf{e}}_3$-equivariant maps.

Lemma 2.3. *For $\hat{\mathbf{e}} \in \mathbb{S}^2$, let $\mathbf{m}\colon \mathbb{S}^2 \to \mathbb{S}^2$ be $\hat{\mathbf{e}}$-equivariant. Then the following statements hold true:*

(1) For $Q \in O(3)$, $\mathbf{m}_Q$ is $(Q^{-1}\hat{\mathbf{e}})$-equivariant.

(2) For $u\colon \mathbb{S}^2 \to \mathbb{S}^2$, we have

$$\langle \mathbf{m}, u_R\rangle_{L^2(\mathbb{S}^2)} = \langle \mathbf{m}, u\rangle_{L^2(\mathbb{S}^2)} \text{ for all } R \in SO(3)_{\hat{\mathbf{e}}}.$$

(3) If $\mathbf{m}_R(\pm\hat{\mathbf{e}}) = \mathbf{m}(\pm\hat{\mathbf{e}})$ for all $R \in SO(3)_{\hat{\mathbf{e}}}$ then $\mathbf{m}(\hat{\mathbf{e}}) = \pm\hat{\mathbf{e}}$ and $\mathbf{m}(-\hat{\mathbf{e}}) = \pm\hat{\mathbf{e}}$.

Proof. (1) For $R \in SO(3)_{Q^{-1}\hat{\mathbf{e}}}$ it holds that

$$QRQ^{-1}\hat{\mathbf{e}} = Q\left(R\left(Q^{-1}\hat{\mathbf{e}}\right)\right) = QQ^{-1}\hat{\mathbf{e}} = \hat{\mathbf{e}}$$

and therefore $QRQ^{-1} \in SO(3)_{\hat{\mathbf{e}}}$. But then

$$\begin{aligned}(\mathbf{m}_Q)_R &= R^{-1}Q^{-1}\mathbf{m}(QRx)\\ &= Q^{-1}\left(QR^{-1}Q^{-1}\right)\mathbf{m}((QRQ^{-1})Qx) = \left(\mathbf{m}_{QRQ^{-1}}\right)_Q = \mathbf{m}_Q.\end{aligned}$$

(2) Orthogonality of R and $\det R = 1$ imply $\langle a, b_R\rangle = \langle a_{R^{-1}}, b\rangle$ and the statement follows by the equivariance of $\mathbf{m}$.

(3) For $R \in SO(3)_{\hat{\mathbf{e}}}$ it holds that $\mathbf{m}(\hat{\mathbf{e}}) = \mathbf{m}_R(\hat{\mathbf{e}}) = R^{-1}\mathbf{m}(\hat{\mathbf{e}})$. Therefore $SO(3)_{\hat{\mathbf{e}}} \subset SO(3)_{m(\hat{\mathbf{e}})}$. But since the two groups are isomorphic, the inclusion implies equality which only holds for $\mathbf{m}(\hat{\mathbf{e}}) = \pm\hat{\mathbf{e}}$.

□

Furthermore, in the special case of $\hat{\mathbf{e}} = \hat{\mathbf{e}}_3$ we have the following representation of equivariant maps in spherical coordinates $\Psi\colon (0,\pi) \times (0, 2\pi) \to \mathbb{S}^2 \setminus \{\mathbf{x}_1 \geq 0,\ \mathbf{x}_2 = 0\}$ on the domain sphere.

Lemma 2.4. *Consider a continuous map* $\mathbf{m}\colon \mathbb{S}^2 \to \mathbb{S}^2$. $\mathbf{m}$ *is equivariant iff*

$$\mathbf{m}(\Psi(x,\varphi)) = \begin{pmatrix} \sin\theta(x)\cos(\chi(x)+\varphi) \\ \sin\theta(x)\cos(\chi(x)+\varphi) \\ \cos\theta(x) \end{pmatrix}$$

for some functions θ, $\chi\colon (0,\pi) \to \mathbb{R}$.

Proof. In spherical coordinates on the domain and target sphere, any map can be expressed as

$$\mathbf{m}(\Psi(x,\varphi)) = \Psi(\theta(x,\varphi), \chi(x,\varphi)) = \begin{pmatrix} \sin\theta(x,\varphi)\cos(\chi(x,\varphi)) \\ \sin\theta(x,\varphi)\sin(\chi(x,\varphi)) \\ \cos\theta(x,\varphi) \end{pmatrix}$$

for $x \in (0,\pi)$, $\varphi \in [0, 2\pi)$ and $\mathbf{m}(\Psi(x,\varphi)) \in \mathbb{S}^2 \setminus \{\hat{\mathbf{e}}_3\}$. A more detailed discussion of this representation can be found in A.1.

Consider a rotation around $\hat{\mathbf{e}}_3$ by the angle α, i.e.

$$R_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} = R_{-\alpha}^{-1}.$$

Then $R(\Psi(x,\varphi)) = \Psi(x, \varphi+\alpha)$ if Ψ is 2π-periodically extended in the second component. Therefore,

$$\begin{aligned} R_\alpha \mathbf{m}(R_{-\alpha}\Psi(x,\varphi)) &= R_\alpha \mathbf{m}(\Psi(x,\varphi-\alpha)) \\ &= \begin{pmatrix} \sin\theta(x,\varphi-\alpha)\cos(\chi(x,\varphi-\alpha)+\alpha) \\ \sin\theta(x,\varphi-\alpha)\sin(\chi(x,\varphi-\alpha)+\alpha) \\ \cos\theta(x,\varphi-\alpha) \end{pmatrix} \end{aligned}$$

while

$$\mathbf{m}(\Psi(x,\varphi)) = \begin{pmatrix} \sin\theta(x,\varphi)\cos\chi(x,\varphi) \\ \sin\theta(x,\varphi)\sin\chi(x,\varphi) \\ \cos\theta(x,\varphi) \end{pmatrix}.$$

First, consider the third component:

$$(\mathbf{m}_R)_3 = (\mathbf{m})_3 \text{ iff } \theta(x, \varphi - \alpha) = \theta(x, \varphi)$$

for all $x \in (0, \pi)$ and $\alpha \in \mathbb{R}$. Thus, θ must be independent of φ for $\mathbf{m}$ to be equivariant. On the other hand, comparing the first two components, one can conclude that $\mathbf{m}$ is equivariant iff $\chi(x, \varphi) - \chi(x, \varphi - \alpha) = \alpha + 2\pi k$ for some $k \in \mathbb{Z}$ and all $x \in (0, \pi)$. For small α, we may assume $k = 0$ and by taking the limit

$$\lim_{\alpha \searrow 0} \frac{\chi(x, \varphi) - \chi(x, \varphi - \alpha)}{\alpha} = 1$$

we conclude that χ grows linearly in φ, independently of x. Therefore, $\chi(x, \varphi) = \tilde{\chi}(x) + \varphi$ for some function $\tilde{\chi} \colon (0, \pi) \to \mathbb{R}$ which in the following we will again denote by χ. $\square$

On the other hand, the hedgehog in spherical coordinates is given by

$$\mathbf{m}(\Psi(x, \varphi)) = \begin{pmatrix} \sin x \cos \varphi \\ \sin x \sin \varphi \\ \cos x \end{pmatrix}$$

and is therefore equivariant with $\theta(x) = x$, $\chi(x) \equiv 0$. Interested in as much symmetry as possible, we will investigate equivariant maps with $\chi(x) = 0$, leaving θ to be varied. In fact, these are exactly the $O(3)_{\hat{\mathbf{e}}_3}$-equivariant maps, i.e. those that are invariant under joint rotations and joint reflections leaving $\hat{\mathbf{e}}_3$ unchanged. While the invariance has been observed in [16], we are not aware of a proof of the characterization and provide it here.

Lemma 2.5. *Consider a continuous map $\mathbf{m} \colon \mathbb{S}^2 \to \mathbb{S}^2$. $\mathbf{m}$ is $O(3)_{\hat{\mathbf{e}}_3}$-equivariant iff*

$$\mathbf{m}(\Psi(x, \varphi)) = \begin{pmatrix} \sin \theta(x) \cos(\varphi) \\ \sin \theta(x) \sin(\varphi) \\ \cos \theta \end{pmatrix}$$

for a function $\theta \colon (0, \pi) \to \mathbb{R}$.

Due to this characterization in terms of the $\hat{\mathbf{e}}_3$-axis, we will refer to equivariant maps with $\chi = 0$ as axisymmetric maps.

Proof. Starting from the previous characterization of equivariant maps, we need to show that the additional invariance under joint reflections implies the existence of a function θ such that $\chi(x) = 0$. For this purpose, we additionally consider the reflection at the $\hat{\mathbf{e}}_1$-$\hat{\mathbf{e}}_3$-plane, i.e.

$$F_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with $F\hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_3$. Then, again extending Ψ periodically in the second component,

$$F_2\Psi(x,\varphi) = \begin{pmatrix} \sin x\cos\varphi \\ -\sin x\sin\varphi \\ \cos x \end{pmatrix} = \begin{pmatrix} \sin x\cos(-\varphi) \\ \sin x\sin(-\varphi) \\ \cos x \end{pmatrix} = \Psi(x,-\varphi)$$

such that

$$\begin{aligned} F_2\mathbf{m}(F_2^{-1}\Psi(x,\varphi)) &= F_2\mathbf{m}(\Psi(x,-\varphi)) \\ &= \begin{pmatrix} \sin\theta(x)\cos(\chi(x)-\varphi) \\ -\sin\theta(x)\sin(\chi(x)-\varphi) \\ \cos\theta(x) \end{pmatrix} \\ &= \begin{pmatrix} \sin\theta(x)\cos(-\chi(x)+\varphi) \\ \sin\theta(x)\sin(-\chi(x)+\varphi) \\ \cos\theta(x) \end{pmatrix}. \end{aligned}$$

Again, it follows from comparison with $\mathbf{m}(x,\varphi)$ in the first and second component that $\chi(x)+\varphi = -\chi(x)+\varphi+2\pi k$ and therefore $\chi(x) = \pi k$ for some $k \in \mathbb{Z}$. If k is even, χ can be dropped due to the periodicity of the trigonometric functions. For odd k, we have

$$\mathbf{m}(\Psi(x,\varphi)) = \begin{pmatrix} \sin\theta(x)\cos(\varphi+k\pi) \\ \sin\theta(x)\sin(\varphi+k\pi) \\ \cos\theta(x) \end{pmatrix} = \begin{pmatrix} -\sin\theta(x)\cos\varphi \\ -\sin\theta(x)\sin\varphi \\ \cos\theta(x) \end{pmatrix} = \begin{pmatrix} \sin(-\theta(x))\cos\varphi \\ \sin(-\theta(x))\sin\varphi \\ \cos(-\theta(x)) \end{pmatrix}$$

which also is of the proposed form.

To confirm invariance for all elements of $O(3)_{\hat{\mathbf{e}}_3}$, note that any $O \in O(3)_{\hat{\mathbf{e}}_3}$ is either a rotation or can be decomposed into a rotation $R \in SO(3)_{\hat{\mathbf{e}}_3}$ applied to F_2: If $\det(O) = 1$ then $O = R \in SO(3)_{\hat{\mathbf{e}}_3}$. If $\det(O) = -1$ then $O = (OF_2)F_2$ where $R = OF_2 \in SO(3)_{\hat{\mathbf{e}}_3}$. In this decomposition, we have

$$\mathbf{m}_O = \mathbf{m}_{(RF_2)} = (\mathbf{m}_{F_2})_R = \mathbf{m}_R = \mathbf{m}.$$

□

2.2.2. The Energy Functional for Axisymmetric Maps

We will now further investigate the relevant functionals for equivariant and axisymmetric maps. In spherical coordinates, we have

$$\begin{aligned} |\nabla\mathbf{m}| &= \left(\frac{d}{dx}\mathbf{m}(\Psi(x,\varphi))\right)^2 + \frac{1}{\sin^2 x}\left(\frac{d}{d\varphi}\mathbf{m}(\Psi(x,\varphi)\right) \\ &= \left|\theta'\begin{pmatrix} \cos\theta\cos(\chi+\varphi) \\ \cos\theta\sin(\chi+\varphi) \\ -\sin\theta \end{pmatrix} + \chi'\begin{pmatrix} -\sin\theta\sin(\chi+\varphi) \\ \sin\theta\cos(\chi+\varphi) \\ 0 \end{pmatrix}\right|^2 + \left|\begin{pmatrix} -\sin\theta\sin(\chi+\varphi) \\ \sin\theta\cos(\chi+\varphi) \\ 0 \end{pmatrix}\right|^2 \\ &= (\theta')^2 + (\chi')^2\sin^2\theta + \frac{\sin^2\theta}{\sin^2 x} \end{aligned}$$

and

$$
\begin{aligned}
1-(\mathbf{m}\cdot\nu)^2 &= 1-(\sin\theta\sin x\,(\cos(\chi+\varphi)\cos\varphi+\sin(\chi+\varphi)\sin\varphi)+\cos\theta\cos x)^2\\
&= 1-(\cos(\theta-x)+\sin\theta\sin x\,(\cos\chi-1))^2\\
&= \sin^2(\theta-x)-2\cos(\theta-x)\sin\theta\sin x\,(\cos\chi-1)-\sin^2\theta\sin^2 x\,(\cos\chi-1)^2\,.
\end{aligned}
$$

Thus, the energy is reduced to

$$
\begin{aligned}
\mathcal{E}(\mathbf{m}) = 2\pi\int_0^\pi &\left((\theta')^2+(\chi')^2\right)\sin x+\frac{\sin^2\theta}{\sin x}+\kappa\Big(\sin^2(\theta-x)\\
&+2\cos(\theta-x)\sin\theta\sin x(1-\cos\chi)-\sin^2\theta\sin^2 x\,(1-\cos\chi)^2\Big)\sin x\,\mathrm{d}x
\end{aligned}
$$

and in particular, for axisymmetric $\mathbf{m}$,

$$
\mathcal{E}(\mathbf{m}) = 2\pi\int_0^\pi(\theta')^2\sin x+\frac{\sin^2\theta}{\sin x}+\kappa\sin^2(\theta-x)\sin x\,\mathrm{d}x =: 2\pi E(\theta).
$$

For the topological degree Q, we compute the pullback $\mathbf{m}^*\omega_{\mathbb{S}^2}=\omega(\mathbf{m})\,\mathrm{d}x_1\wedge\mathrm{d}x_2$ in spherical coordinates $x_1=x\in(0,\pi)$, $x_2=\varphi\in(0,2\pi)$, and $\mathbf{x}=\Psi(x_1,x_2)\in\mathbb{S}^2$. Note that they are orientation preserving due to

$$
\det(D(\pi_-\circ\Psi)) = \frac{\sin x}{(1+\cos x)^2},
$$

where $\sin x>0$ for $x\in(0,\pi)$. Starting out with the vector product, we have

$$
\begin{aligned}
&\frac{\partial\mathbf{m}}{\partial x}\times\frac{\partial\mathbf{m}}{\partial\varphi}\\
&= \left(\theta'\begin{pmatrix}\cos\theta\cos(\chi+\varphi)\\ \cos\theta\sin(\chi+\varphi)\\ -\sin\theta\end{pmatrix}+\chi'\begin{pmatrix}-\sin\theta\sin(\chi+\varphi)\\ \sin\theta\cos(\chi+\varphi)\\ 0\end{pmatrix}\right)\times\begin{pmatrix}-\sin\theta\cos(\chi+\varphi)\\ \sin\theta\sin(\chi+\varphi)\\ 0\end{pmatrix}\\
&= (\theta'\sin\theta)\mathbf{m}.
\end{aligned}
$$

Therefore, the topological charge for equivariant fields is given by

$$
\begin{aligned}
Q(\mathbf{m}) &= \frac{1}{4\pi}\int_{\mathbb{S}^2}\mathbf{m}^*\omega_{\mathbb{S}^2}\\
&= \frac{1}{4\pi}\int_0^{2\pi}\int_0^\pi\mathbf{m}\cdot\left(\frac{\partial\mathbf{m}}{\partial x}\times\frac{\partial\mathbf{m}}{\partial\varphi}\right)\mathrm{d}x\,\mathrm{d}\varphi\\
&= \frac{1}{2}\int_0^\pi\theta'\sin\theta\,\mathrm{d}x = \frac{1}{2}\left(\cos(\theta(0))-\cos(\theta(\pi)\right).
\end{aligned}
$$

Recall that equivariance implies $\mathbf{m}(\pm\hat{\mathbf{e}}_3) \in \{\hat{\mathbf{e}}_3, -\hat{\mathbf{e}}_3\}$, and therefore $Q(\mathbf{m}) \in \{-1, 0, 1\}$. For higher degrees, k-equivariant fields would have to be considered.

The group $SO(3)_{\hat{\mathbf{e}}_3}$ is a compact Lie group as a subgroup of $SO(3)$. It acts on $X^1(\mathbb{S}^2;\mathbb{S}^2)$ via the operation $\mathbf{m} \mapsto \mathbf{m}_R$ and $\mathcal{E}\colon X^1 \to \mathbb{R}$ is $SO(3)_{\hat{\mathbf{e}}_3}$ - invariant. By the principle of symmetric criticality [44], critical points of $\mathcal{E}$ within the set of equivariant maps are critical for $\mathcal{E}$. Thus, we now express the terms in the Euler-Lagrange equations for equivariant fields in spherical coordinates to obtain equations that are solved by minimizers within the set of axisymmetric fields. Note that minimality is not covered by Palais' principle.

$$\begin{aligned}
\Delta\mathbf{m} \times \mathbf{m} &= \left(\frac{\theta'' \sin x + \theta' \cos x}{\sin x} + \frac{\sin(2\theta)}{2\sin x}\left((\chi')^2 - \frac{1}{\sin x}\right)\right)\begin{pmatrix}\sin(\chi+\varphi)\\ -\cos(\chi+\varphi)\\ 0\end{pmatrix}\\
&\quad + \left(\sin\theta\frac{\chi'\cos x + \chi''\sin x}{\sin x} + 2\theta'\chi'\cos\theta\right)\begin{pmatrix}\cos\theta\cos(\chi+\varphi)\\ \cos\theta\sin(\chi+\varphi)\\ -\sin\theta\end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
(\mathbf{m}\cdot\nu)\nu\times\mathbf{m} &= (\sin\theta\sin x\cos\chi + \cos\theta\cos x)\,\nu\times\mathbf{m}\\
&= (\sin\theta\sin x\cos\chi + \cos\theta\cos x)\,(\cos\theta\sin x\cos\chi - \sin\theta\cos x)\begin{pmatrix}\sin(\chi+\varphi)\\ -\cos(\chi+\varphi)\\ 0\end{pmatrix}\\
&\quad + (\sin\theta\sin x\cos\chi + \cos\theta\cos x)\,(-\sin x\sin\chi)\begin{pmatrix}\cos\theta\cos(\chi+\varphi)\\ \cos\theta\sin(\chi+\varphi)\\ -\sin\theta\end{pmatrix}.
\end{aligned}$$

Combining these two terms, the equation $0 = \mathbf{m}\times(\Delta\mathbf{m} + \kappa(\mathbf{m}\cdot\nu)\nu)$ is reduced to two coupled ordinary differential equations for θ and χ.

$$\begin{aligned}
&\frac{\sin\theta}{\sin x}\Big(\chi'\cos x + \chi''\sin x\Big) + \frac{1}{\sin x}\Big(2\theta'\chi'\cos\theta\Big) \\
&\qquad = \kappa\,(\sin\theta\sin x\cos\chi + \cos\theta\cos x)\sin x\sin\chi
\end{aligned} \tag{2.1}$$

$$\begin{aligned}
&\theta''\sin x + \theta'\cos x + \sin(2\theta)\frac{(\chi')^2}{2} - \frac{\sin(2\theta)}{2\sin x} \\
&\qquad = \kappa\,(\sin\theta\sin x\cos\chi + \cos\theta\cos x)\sin x\,(\cos\theta\sin x\cos\chi - \sin\theta\cos x)
\end{aligned} \tag{2.2}$$

In particular, for axisymmetric solutions with $\chi \equiv 0$, the profile θ solves the equation

$$\sin x\theta'' + \cos x\theta' = \frac{\sin(2\theta)}{2\sin x} + \kappa\sin(2\theta - 2x)\sin x. \tag{2.3}$$

Note that for constant χ, the left-hand side of the first equation vanishes. If $\sin\chi \neq 0$ i.e. $\chi \notin \{0, \pi\}$, the reappearance of the right-hand side of (2.1) in the right-hand side of (2.2) implies that θ has to solve $(\sin x\theta')' = \frac{\sin(2\theta)}{2\sin x}$, independently of κ. This is the Euler-Lagrange equation for axisymmetric critical points of

$$\int_{\mathbb{S}^2} |\nabla\mathbf{m}|^2 \,\mathrm{d}\sigma$$

The equivariant $\mathbf{m}$ with profile θ and azimuthal shift χ would thus solve the harmonic map equation. However, that is not what we're interested in.

In the following, we will restrict our analysis to $\chi \in \{0, \pi\}$, i.e. axisymmetric fields, and the corresponding functionals. Recall that the change $\chi = 0$ to $\chi = \pi$ corresponds to the change $\theta \mapsto -\theta$ on the level of profiles. Apart from the considerations about harmonic maps, the restriction to axisymmetric fields is further justified by the fact that for any given profile θ, the angles $\chi \equiv 0$ and $\chi \equiv \pi$ are critical among all χ for which the expressions are defined. To see this set $\mathbf{m}_t := (\sin\theta\cos((\chi+t\phi)+\varphi), \sin\theta\sin((\chi+t\phi)+\varphi), \cos\theta)^T$ for a test function $\phi \in C_0^\infty(0,\pi)$. Then,

$$\frac{d}{dt}\mathcal{E}(\mathbf{m}_t)|_{t=0} = 2\pi \int_0^\pi \phi'\chi' \sin x \\ + \kappa\left(2\cos(\theta - x)\sin\theta\sin x\sin\chi - 2\sin^2\theta\sin^2 x\sin\chi(1-\cos\chi)\right)\phi \,\mathrm{d}x$$

which vanishes for $\chi \equiv 0$ and $\chi \equiv \pi$ due to the $\sin\chi$-terms. Note that the anisotropy term is $\sin^2(\theta - x)$ for $\chi \equiv 0$ and $\sin^2(\theta + x)$ for $\chi \equiv \pi$ which corresponds to the shift from θ to $-\theta$ that we have seen in the proof of lemma 2.5.

2.2.3. Analysis of the Energy Density

Recall that in the planar case, Dzyaloshinski-Moriya interaction (DMI) in combination with a potential term stabilizes Skyrmions $\mathbf{u}\colon \mathbb{R}^2 \to \mathbb{S}^2$ against the scaling invariance of $\|\nabla\mathbf{u}\|_{L^2}^2$. It is given by

$$\int_{\mathbb{R}^2} \mathbf{u}\cdot(\nabla\times\mathbf{u}) \,\mathrm{d}x.$$

In the case of the sphere, it has been observed in [26] and [37] that the stabilizing effect of curvature can be traced to a so-called curvature-induced DMI. By expressing $\mathbf{m}$ in curvilinear coordinates $(\nu, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_\varphi)$ on the sphere [26] or curvilinear stereographic coordinates (τ_1, τ_2, ν) on $\mathbb{R}^2$ [37], a DMI-term emerges from the exchange energy. Both these observations require extensive calculations. Under the assumption of symmetry, however, the curvilinear coordinates are simply induced by considering the difference profile $\Theta := \theta - x$:

$$\begin{pmatrix}\sin\theta\cos\varphi\\ \sin\theta\sin\varphi\\ \cos\theta\end{pmatrix} = \begin{pmatrix}\sin\Theta\cos x\cos\varphi\\ \sin\Theta\cos x\sin\varphi\\ -\sin\Theta\sin x\end{pmatrix} + \begin{pmatrix}\cos\Theta\sin x\cos\varphi\\ \cos\Theta\sin x\sin\varphi\\ \cos\Theta\cos x\end{pmatrix} = \sin\Theta\hat{\mathbf{e}}_1 + \cos\Theta\hat{\mathbf{e}}_n$$

in the notation of [26]. The energy, expressed in terms of Θ, is given by

$$E(\theta) = \int_0^\pi (\Theta')^2\sin x + 2\Theta'\sin x + \sin x + \frac{\sin^2\Theta\cos^2 x}{\sin x} + \frac{\sin(2\Theta)\cos x}{\sin x} + \cos^2\Theta\sin x \\ + \kappa\sin^2(\Theta)\sin x \,\mathrm{d}x.$$

On $\mathbb{R}^2$, the properties of axisymmetric Skyrmions have been investigated in [32] for an energy functional involving exchange energy, DMI and easy-axis anisotropy with a large anisotropy parameter h. For axisymmetric Skyrmions, this results in

$$E_{[32]}(\theta) = 2\pi \int_0^\infty \left(\frac{(\theta')^2}{2} + \frac{\sin^2\theta}{2r^2} + \theta' + \frac{\sin\theta\cos\theta}{r} + h(1-\cos\theta) \right) r\,\mathrm{d}r.$$

This can be compared with the case on the sphere by considering the pullback of $\mathbf{m}\colon \mathbb{S}^2 \to \mathbb{S}^2$ via the inverse stereographic projection. On the level of profiles, this corresponds to setting

$$\theta_p\colon (0,\infty) \to \mathbb{R},\ \theta_p(r) = \Theta(2\arctan(r))$$

when identifying $-\hat{\mathbf{e}}_3$ with ∞. For the projection from $\hat{\mathbf{e}}_+$, π_+, the same would be achieved by $\Theta(2\arctan(\frac{1}{r}))$. Thus, using the identities $\sin(2\arctan(r)) = \frac{2r}{1+r^2}$ and $\cos(2\arctan(r)) = \frac{1-r^2}{1+r^2}$, the energy of θ, expressed via θ_p, is given by

$$\begin{aligned}
\mathcal{E}(\theta) &= \int_0^\infty \Bigg((\theta_p')^2 \frac{(1+r^2)^2}{4} \frac{2r}{1+r^2} + 2\theta_p' \frac{1+r^2}{2} \frac{2r}{1+r^2} + \sin^2\theta_p \frac{(1-r^2)^2}{(1+r^2)^2} \frac{1+r^2}{2r} \\
&\qquad + \sin(2\theta_p) \frac{1-r^2}{1+r^2} + (1+\cos^2\theta_p + \kappa \sin^2\theta_p) \frac{2r}{1+r^2} \Bigg) \frac{2}{1+r^2}\,\mathrm{d}r \\
&= \int_0^\infty \Bigg((\theta_p')^2 + 2\theta_p' \frac{2}{1+r^2} + \frac{\sin^2\theta_p}{r^2} \frac{1-4r^2+r^4}{1+4r^2+r^4} \\
&\qquad + \frac{\sin(2\theta_p)}{r} \frac{1-r^2}{1+2r^2+r^4} + (2+(\kappa-1)\sin^2\theta_p) \frac{4}{(1+r^2)^2} \Bigg) r\,\mathrm{d}r \\
&= 2\int_0^\infty \left(\frac{(\theta_p')^2}{2} + \theta_p' + \frac{\sin^2\theta_p}{2r^2} + \frac{\sin(2\theta_p)}{2r} + 4(\kappa-1)\sin^2\theta_p \right) r\,\mathrm{d}r \\
&\quad + \int_0^\infty \theta_p' \frac{(1-r^2)r}{1+r^2} - \frac{8r\sin^2\theta_p}{1+4r^2+r^4} - \sin(2\theta_p) \frac{3r^3+r^4}{1+2r+r^4} \\
&\qquad + \frac{8r-(8r^2+4r^4)\sin^2\theta_p}{(1+r^2)^2}\,\mathrm{d}r.
\end{aligned}$$

Up to the change in the potential from a Zeeman term to anisotropy, the first integral reproduces the energy for axisymmetric chiral Skyrmions on $\mathbb{R}^2$. The integrand of the second integral converges to 0 as $r \to 0$ where, in the planar case, the Skyrmion forms. For large r, on the other hand, the expansion is not meaningful because the second integral yields a significant contribution to the energy.

On the sphere, chapter two will reveal that the Skyrmion in the axisymmetric case may form at either of the poles. The computations above have shown that similarities to the planar case are to be expected if the Skyrmion forms at the north pole $\hat{\mathbf{e}}_3$ with $\theta(\hat{\mathbf{e}}_3) = \theta_p(0)$. For a Skyrmion at $-\hat{\mathbf{e}}_3$, one may choose the stereographic projection accordingly and the same expansion of the energy is achieved for

$$\tilde{\theta}_p = \Theta\left(2\arctan\left(\frac{1}{r}\right)\right),$$

thus allowing the comparison of Skyrmions at the south pole.

In preparation for Chapter 2 on axisymmetric minimizers, we end this section with a detailed analysis of the terms in the reduced energy for axisymmetric fields. For that purpose, let

$$e_I(\theta, x) = (\theta'(x))^2 \sin x \quad \text{and} \quad e_{II}(\theta, x) = \frac{\sin^2\theta(x)}{\sin x} + \kappa\sin^2(\theta(x) - x)\sin x$$

as well as

$$E_I(\theta) = \int_0^\pi e_I(\theta, x)\,\mathrm{d}x \quad \text{and} \quad E_{II}(\theta, x) = \int_0^\pi e_{II}(\theta)\,\mathrm{d}x.$$

Furthermore, given $0 \le a < b \le \pi$, we define

$$E_{[a,b]}(\theta) = \int_a^b e_I(\theta, x) + e_{II}(\theta, x)\,\mathrm{d}x.$$

This choice of grouping terms seems to be non-intuitive because

$$\int_{\mathbb{S}^2} |\nabla \mathbf{m}_\theta|^2\,\mathrm{d}\sigma = 2\pi\int_0^\pi (\theta')^2\sin x + \frac{\sin^2\theta}{\sin x}\,\mathrm{d}x,$$

which is split among e_I and e_{II}. However, on the level of profiles, minimality has different effects on the two terms stemming from $|\nabla \mathbf{m}_\theta|^2$.

The first part of the energy, e_I, is obviously minimized by constant profiles. However, it is not relevant whether this constant is π, $\frac{\pi}{2}$, or any other value.

For e_{II}, on the other hand, the pointwise value of θ is relevant. The energy density is a superposition of the trigonometric terms $\sin^2\theta$ and $\sin^2(\theta - x)$, weighted by $1/\sin x$ and $\kappa\sin x$. For very small x, namely $x < \arcsin(\frac{1}{\kappa})^2$, minimizing the first term by having θ close to 0 or π therefore has a larger effect on minimizing e_{II} then minimizing the second term which enforces $\theta(x)$ to be close to x. For energy arguments, it will be helpful to find the balance and to know the pointwise optimal value of θ with respect to e_{II}.

For any $x \in (0, \pi)$, the value of θ is critical with respect to e_{II} if

$$\frac{d}{d\delta}e_{II}(\theta + \delta, x)|_{\delta=0} = \frac{\sin(2\theta)}{2\sin x} + \frac{\kappa}{2}\sin(2\theta - 2x)\sin x = 0.$$

In order to solve this equation for θ when x is fixed, define $d_\kappa\colon (0,\pi) \to \mathbb{R}$ by

$$d_\kappa(x) = \begin{cases} 0 & x = \frac{\pi}{2} \\ \frac{1}{2}\operatorname{arccot}\left(\frac{\kappa}{2}\tan x + \cot(2x)\right) & \text{otherwise} \end{cases}$$

where $\operatorname{arccot}\colon \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ is chosen to be discontinuous at 0.

```
Plot[{ 1 / 2 ArcCot[(100 / 2) Tan[x] + Cot[2 x]],
   1 / 2 ArcCot[(50 / 2) Tan[x] + Cot[2 x]],  1 / 2 ArcCot[(20 / 2) Tan[x] + Cot[2 x]],
   1 / 2 ArcCot[(10 / 2) Tan[x] + Cot[2 x]],  1 / 2 ArcCot[(2 / 2) Tan[x] + Cot[2 x]], Pi / 8},
  {x, 0, Pi}, PlotLegends → {k = 100 , k = 50 , k = 20 , k = 10 , k = 2,  Pi / 8}]
```

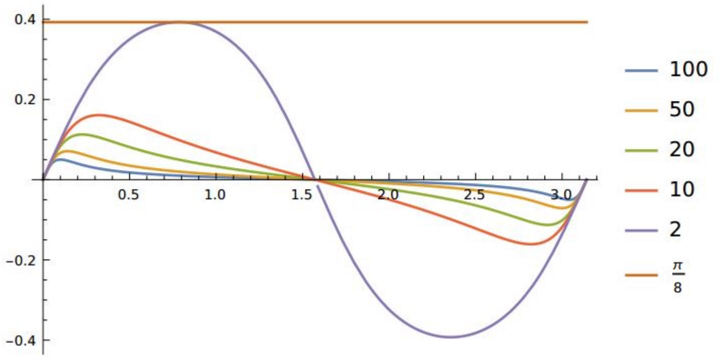

Figure 2.1.: Plots of d_κ over $[0,\pi]$ for different values of κ and compared to $\frac{\pi}{8}$.

Lemma 2.6 (Properties of d_κ)**.** *The function $d_\kappa\colon (0,\pi) \to \mathbb{R}$ given as above has the following properties:*

(1) For every $\kappa > 0$, d_κ is well-defined and continuous.

(2) For every $n \in \mathbb{N}$ and every $x \in (0,\pi)$, the value $\theta(x) = x + \frac{n\pi}{2} - d_\kappa(x)$ is a critical point of $y \mapsto e_{II}(y,x)$.

(3) $\kappa \mapsto d_\kappa(x)$ is strictly monotonously increasing on $(0,\frac{\pi}{2})$ and strictly monotonously decreasing on $(\frac{\pi}{2},\pi)$. Moreover, $\lim_{\kappa\to\infty} d_\kappa(x) = 0$ for all $x \in (0,\pi)$.

(4) Let $\kappa > 2$. Then, $d_\kappa\left((0,\frac{\pi}{2})\right) \subset (0,\frac{\pi}{8})$ and $d_\kappa\left((\frac{\pi}{2},\pi)\right) \subset (-\frac{\pi}{8},0)$.

(5) Define

$$\eta(x) := x + \frac{\pi}{2} - d_\kappa(x) \qquad \text{and} \qquad \xi(x) := x - d_\kappa(x).$$

Then for every $n \in \mathbb{Z}$ and $x \in (0,\pi)$, the density $e_{II}(\theta,x)$ is pointwise maximal if $\theta(x) = \eta(x) + \pi n$ and minimal if $\theta(x) = \xi(x) + \pi n$. The energy density is strictly monotonous in θ between these extrema.

(6) e_{II} is symmetric around $x - d_\kappa + \frac{n\pi}{2}$ for every $n \in \mathbb{Z}$.

The pointwise analysis of e_{II} directly implies the following functional statement for E_{II}:

Corollary 2.2. *For every $n \in \mathbb{N}$, the continuous function $\mathrm{id} + \frac{n\pi}{2} - d_\kappa$ is a critical point of E_{II}.*

Moreover, the functions η and ξ as defined above are a maximizing and minimizing function of E_{II}, respectively.

Proof. Recall that by arccot, we refer to the discontinuous function which satisfies $\operatorname{arccot}(-\infty) = \operatorname{arccot}(\infty) = 0$.

(1) On $(0, \frac{\pi}{2})$ and $(\frac{\pi}{2}, \pi)$, d_κ is continuous as a combination of continuous functions due to the choice of arccot. For continuity in $x = \frac{\pi}{2}$, note that

$$\lim_{x \nearrow \frac{\pi}{2}} \frac{\kappa}{2} \tan(x) + \cot(2x) = \infty \text{ and } \lim_{x \searrow \frac{\pi}{2}} \frac{\kappa}{2} \tan(x) + \cot(2x) = -\infty$$

with $\lim_{y\to\infty} \operatorname{arccot}(y) = \lim_{y\to-\infty} \operatorname{arccot}(y) = 0 = d_\kappa(\frac{\pi}{2})$ for this choice of arccot.

(2) For $\theta(x) = x + \frac{n\pi}{2} - d_\kappa(x)$, trigonometric identities give

$$\begin{aligned}
\cot(2\theta) &= \cot(2x + n\pi - 2d_\kappa(x)) = \cot(2x - 2d_\kappa(x)) \\
&= \frac{\cot(2x)\cot(2d_\kappa(x)) + 1}{\cot(2d_\kappa(x)) - \cot(2x)} \\
&= \frac{\cot(2x)\left(\frac{\kappa}{2}\tan(x) + \cot(2x)\right)}{\frac{\kappa}{2}\tan(x)} + \frac{1}{\frac{\kappa}{2}\tan(x)} \\
&= \cot(2x) + \frac{2}{\kappa}\frac{\cos x}{\sin x}\left(\cot^2(2x) + 1\right) \\
&= \cot(2x) + \frac{1}{\kappa}\frac{1}{\sin^2 x \sin(2x)}.
\end{aligned}$$

Next, multiplication by $\kappa \sin(2x)\sin x$ and $\sin(2\theta)$ give

$$\kappa \sin x\,(\cos(2\theta)\sin(2x) - \sin(2\theta)\cos(2x)) = \frac{\sin(2\theta)}{\sin x},$$

and therefore $\frac{d}{d\delta} e_{II}(\theta + \delta, x)|_{\delta=0} = 0$.

(3) Fix $x \in (0, \pi) \setminus \{\frac{\pi}{2}\}$. As a function in κ, $d_\kappa(x)$ is differentiable with derivative

$$\frac{d}{d\kappa} d_\kappa(x) = \frac{\tan(x)}{4} \frac{-1}{1 + \left(\frac{\kappa}{2}\tan(x) + \cot(2x)\right)^2}.$$

This expression has the opposite signum of $\tan x$ and is therefore negative for $x < \frac{\pi}{2}$ and positive for $x > \frac{\pi}{2}$.

For the asymptotic behavior as $\kappa \to \infty$, recall the asymptotic behavior of cot. For $x = \frac{\pi}{2}$, we have $d_\kappa(x) = 0$ for all κ. Otherwise, the factor $\tan(x)$ is non-zero and $\lim_{\kappa\to\infty} \left|\frac{\kappa}{2}\tan x + \cot(2x)\right| = \infty$.

(4) Due to the monotonicity of d_κ in κ and symmetry of the trigonometric functions, it suffices to prove the lower bound for $\kappa = 2$ and $x \in (0, \frac{\pi}{2})$ and the upper bound of 0 for $x \in (0, \frac{\pi}{2})$. Figure 2.1 depicts the graphs of d_κ for different κ and is topped by an auxiliary line at $\frac{\pi}{8}$.

Both bounds are visible in the plot but also easily shown analytically. For the maximum of d_2 on $(0, \frac{\pi}{2})$, consider the derivative with respect to x,

$$\frac{\mathrm{d}}{\mathrm{d}x} d_2(x) = \frac{1}{2}\frac{-2\cos(2x)}{\sin(2x)}\frac{-1}{1+\left(\frac{1}{\sin(2x)}\right)^2} = \frac{\cos(2x)}{(\sin(2x))^2+1}.$$

The only root of this in $(0, \frac{\pi}{2})$ is $x = \frac{\pi}{4}$ and $d_2(\frac{\pi}{4}) = \frac{1}{2}\operatorname{arccot}(1) = \frac{\pi}{8}$. On $(\frac{\pi}{2}, \pi)$, the only root is $x = \frac{3\pi}{4}$ with $d_2(\frac{3\pi}{4}) = \frac{1}{2}\operatorname{arccot}(-1) = -\frac{\pi}{8}$.

For the lower bound, trigonometric identities imply

$$\begin{aligned}\frac{\kappa}{2}\tan x + \cot(2x) &= \frac{\kappa-1}{2}\tan x + \frac{1}{2}\cot x \geq \frac{1}{2}(\tan x + \cot x)\\ &= \frac{1}{2\sin x\cos x} = \frac{1}{\sin(2x)} > 0\end{aligned}$$

since $\kappa \geq 2$ and $\tan x, \sin(2x) > 0$ for $x \in (0, \frac{\pi}{2})$. Since arccot is positive on $(0, \infty)$,

$$d_\kappa = \frac{1}{2}\operatorname{arccot}\left(\frac{\kappa}{2}\tan x + \cot(2x)\right) > 0 \qquad \text{for all } x \in \left(0, \frac{\pi}{2}\right).$$

On $(\frac{\pi}{2}, \pi)$, both $\tan x$ and $\sin(2x)$ are negative, resulting in the opposite inequality and negativity of $d_\kappa(x)$.

(5) Consider the second derivative of e_{II} with respect to θ,

$$\frac{\cos(2\theta)}{\sin x} + \kappa\cos(2\theta - 2x)\sin x.$$

This can be rewritten as

$$\begin{aligned}&\frac{\cos(2\theta)}{\sin x} + \kappa\cos(2\theta-2x)\sin x\\ &\quad= \cos(2\theta-2x)\left(\frac{\cos(2x)}{\sin x} + \kappa\sin x\right) - \frac{\sin(2\theta-2x)\sin(2x)}{\sin x}\\ &\quad= \cos(2\theta-2x)\left(\frac{\cos^2 x + (\kappa-1)\sin^2 x}{\sin x}\right) - 2\sin(2\theta-2x)\cos x.\end{aligned}$$

For $\kappa > 2$, the expression in parentheses is always larger than $\frac{1}{\sin x} > 0$.

If $x \in (0, \frac{\pi}{2})$ and $\theta(x) - x \in (\frac{\pi}{4}, \frac{\pi}{2})$ then $\cos(2(\theta - x)) < 0$, $\sin(2(\theta - x)) \cos x > 0$ and the whole expression is negative. Similarly, it is negative if $x \in (\frac{\pi}{2}, \pi)$ and $\theta - x \in (\frac{\pi}{2}, \frac{3\pi}{4})$. By (4), this is true for $\eta(x) - d_\kappa(x) < \theta(x) < \eta(x) + d_\kappa(x)$ and in particular for $\theta(x) = \eta(x)$.

On the other hand, the whole expression is positive for $x \in (0, \frac{\pi}{2})$ if $\theta - x \in (-\frac{\pi}{4}, 0)$ and for $x \in (\frac{\pi}{2}, \pi)$ if $\theta - x \in (0, \frac{\pi}{4})$. By (4), this is true for θ between $\xi(x) + d_\kappa(x)$ and $\xi(x) - d_\kappa(x)$ and in particular for $\theta(x) = \xi(x)$.

(6) This follows from the criticality of e_{II} at these points. By periodicity, the argument is the same in both cases and we will proceed to show $e_{II}(\eta + \delta) = e_{II}(\eta - \delta)$ for $\delta > 0$.

$$
\begin{aligned}
e_{II}(\eta(x) + \delta) - e_{II}(\eta(x)) &= \int_0^\delta \frac{d}{dt} e_{II}(\eta(x) + t)\, dt \\
&= \int_0^\delta \frac{\sin(2\eta(x) + 2t)}{\sin x} + \kappa \sin(2\eta(x) - 2x + 2t)\, dt \\
&= \int_0^\delta \cos(-2t) \left(\frac{\sin(2\eta(x))}{\sin x} + \kappa \sin(2\eta - 2x) \sin x \right) \\
&\quad + \sin(2t) \left(\frac{\cos(2\eta)}{\sin x} + \kappa \cos(2\eta - 2x) \sin x \right)\, dt \\
&\overset{(a)}{=} - \int_0^\delta \cos(-2t) \left(\frac{\sin(2\eta(x))}{\sin x} + \kappa \sin(2\eta - 2x) \sin x \right) \\
&\quad + \sin(-2t) \left(\frac{\cos(2\eta)}{\sin x} + \kappa \cos(2\eta - 2x) \sin x \right)\, dt \\
&= \int_0^\delta \frac{d}{dt} e_{II}(\eta(x) + t)\, dt = e_{II}(\eta(x) - \delta) - e_{II}(\eta(x))
\end{aligned}
$$

Note that $\cos(2t) \neq -\cos(2t)$ in (a) is irrelevant because the expression in parenthesis identically vanishes identically.

□

3. Symmetric Minimizers

In this chapter, we consider the minimization problem

$$E(\mathbf{m}) \to \min \ \text{ for } \{\mathbf{m} \in H^1(\mathbb{S}^2;\mathbb{S}^2) : Q(\mathbf{m}) = 0, \ \mathbf{m} \text{ axisymmetric}\}.$$

Since the energy and the topological degree are invariant under joint rotations and reflections, fixing the symmetry axis to be $\hat{\mathbf{e}}_3$ was only a matter of convenience because axisymmetric fields can be then expressed as

$$\mathbf{m} = \mathbf{m}_\theta = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$$

with a profile θ fulfilling $\theta(0) = \pi$, $\theta(\pi) = (2k+1)\pi$ for some $k \in \mathbb{Z}$. More details are given in the introduction, section 2.2. However, changing the symmetry axis would result in an equivalent problem.

For axisymmetric fields, the energy can be rewritten as

$$\mathcal{E}(\mathbf{m}_\theta) = 2\pi E(\theta) = 2\pi \int_0^\pi (\theta')^2 \sin x + \frac{\sin^2\theta}{\sin x} + \kappa \sin^2(\theta - x)\sin x \,\mathrm{d}x.$$

The Euler-Lagrange equation takes the special form of

$$0 = \mathbf{m}_\theta \times \nabla\mathcal{E}(\mathbf{m}_\theta) = f(\theta) \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}$$

where

$$f(\theta) = -\theta'' - \theta'\frac{\cos x}{\sin x} + \frac{\sin(2\theta)}{2\sin^2 x} + \frac{\kappa}{2}\sin(2\theta - 2x).$$

The chapter is divided into two parts. First, we prove existence of symmetric minimizers and discuss their non-uniqueness due to further invariances of $\mathcal{E}$. We then derive several properties of minimizing profiles which will be useful in the next chapters.

3.1. Existence, Regularity, and Non-Uniqueness

3.1.1. Existence

While the Euler-Lagrange equation of the reduced energy E, equation (2.3), is irregular at both 0 and π, making a boundary value problem hard to solve, the existence of symmetric minimizers for the functional $\mathcal{E}$ in the case $\kappa > 0$ is easily proven via the direct method of the calculus of variations.

Theorem 1. *For $\kappa > 0$, the minimum of $\mathcal{E}$ in*

$$\{\mathbf{m} \in H^1(\mathbb{S}^2; \mathbb{S}^2) \ : \ Q(\mathbf{m}) = 0, \ \mathbf{m} \ \textit{axisymmetric}\}$$

is attained.

Note: For $\kappa = 0$, constant fields are energy minimizing with zero energy. Thus, the global energy minimum is attained in the topological sector of $Q = 0$. For $\kappa > 0$, on the other hand, constant fields are not even critical points anymore and for $\kappa > 4$, the ground state satisfies $Q = 1$ [10]. Therefore, while Theorem 1 holds for all $\kappa > 0$, we are mostly interested in the case $\kappa > 4$.

Proof of Theorem 1. We apply the direct method of the calculus of variations exactly as in [37] and add an extra argument to ensure that axial symmetry is preserved for the minimizer.

The energy is bounded from below by 0 and the set

$$\mathcal{C} = \{\mathbf{m} \in H^1(\mathbb{S}^2; \mathbb{S}^2) : Q(\mathbf{m}) = 0, \ \mathbf{m} \text{ axisymmetric}\}$$

is non-empty due to $\mathbf{m} \equiv \hat{\mathbf{e}} \in \mathcal{C}$.

Let $(\mathbf{m}_k)_{k\in\mathbb{N}}$ be a minimizing sequence in $\mathcal{C}$. Then $\|\mathbf{m}_k\|^2_{H^1(\mathbb{S}^2;\mathbb{R}^3)} \leq \mathcal{E}(\mathbf{m}_k) + 4\pi$ so the sequence is uniformly bounded in $H^1(\mathbb{S}^2; \mathbb{R}^3)$. Passing to a subsequence, we have weak $H^1(\mathbb{S}^2; \mathbb{R}^3)$ convergence $\mathbf{m}_k \rightharpoonup \mathbf{m}$ for some $\mathbf{m} \in H^1$ with $\mathbf{m}_k \to \mathbf{m} \in L^2(\mathbb{S}^2; \mathbb{R}^3)$ strongly and $|\mathbf{m}(x)| = \lim_{k\to\infty} |\mathbf{m}_k(x)| = 1$ due to pointwise convergence almost everywhere. Hence, $\mathbf{m} \in H^1(\mathbb{S}^2; \mathbb{S}^2)$.

The energy is composed of a norm and the integral over a polynomial in $\mathbf{m}$ which is nonnegative due to $(\mathbf{m} \cdot \nu)^2 \leq 1$. Thus, pointwise convergence of $(\mathbf{m}_k)_{k\in\mathbb{N}}$ almost everywhere and the lemma of Fatou imply

$$\int_{\mathbb{S}^2} 1 - (\mathbf{m} \cdot \nu)^2 \,\mathrm{d}\sigma \leq \liminf_{k\to\infty} \int_{\mathbb{S}^2} 1 - (\mathbf{m}_k \cdot \nu)^2 \,\mathrm{d}\sigma$$

and in total we find $\mathcal{E}(\mathbf{m}) \leq \liminf_{k\to\infty} \mathcal{E}(\mathbf{m}_k)$ because norms are weakly lower semi-continuous.

To prove axisymmetry of $\mathbf{m}$, recall that it is characterized by invariance under joint rotations and joint reflections. Due to the pointwise convergence almost everywhere, we find

$$R^{-1}\mathbf{m}(R\mathbf{x})e = \lim_{k\to\infty} R^{-1}\mathbf{m}_k(R\mathbf{x}) = \lim_{k\to\infty} \mathbf{m}_k(\mathbf{x}) = \mathbf{m}(\mathbf{x}) \qquad \text{a.e. and for } R \in O(3)_{\hat{\mathbf{e}}_3}.$$

Since critical points are smooth (see Theorem 2 below), these identities extend to all $x \in \mathbb{S}^2$ and $\mathbf{m}$ is axisymmetric. It follows that the class of axisymmetric functions is weakly closed.

Finally, in order to study the topological degree, consider the topological density measure

$$\omega(\mathbf{m}) = \tilde{\mathbf{m}} \cdot (\partial_1 \tilde{\mathbf{m}} \times \partial_2 \tilde{\mathbf{m}})\, \mathrm{d}x_1\, \mathrm{d}x_2$$

where, once more, $\tilde{\mathbf{m}} = \mathbf{m} \circ \pi \colon \mathbb{R}^2 \to \mathbb{S}^2$. Due to [3, Theorem E1], boundedness of $\|\nabla \mathbf{m}_k\|_{L^2}^2$ and pointwise convergence almost everywhere imply that there exist integers $q_1, \ldots, q_p \in \mathbb{Z}$ and points $x_1, \ldots x_p \in \mathbb{R}^2$ such that

$$\omega(\mathbf{m}_k) \to \omega(\mathbf{m}) + 4\pi \sum_{i=1}^{N} q_i \delta_{x_i}$$

weakly in the sense of measures. This implies $0 = \lim_{k\to\infty} Q(\mathbf{m}_k) = Q(\mathbf{m}) + \sum_{i=1}^{N} q_i$. Combining this with [35, Lemma 4.3], we find that

$$\liminf_{k\to\infty} \int_{\mathbb{S}^2} |\nabla \mathbf{m}_k|^2 \,\mathrm{d}\sigma \geq \int_{\mathbb{S}^2} |\nabla \mathbf{m}|^2 \,\mathrm{d}\sigma + 4\pi \sum_{i=1}^{N} |q_i|.$$

Due to the upper bound in [37], $\liminf_{k\to\infty} \int_{\mathbb{S}^2} |\nabla \mathbf{m}_k|^2 \,\mathrm{d}\sigma \leq \liminf_{k\to\infty} E(\mathbf{m}_k) < 8\pi$ and N may be at most 1 with $|q_1| = 1$. But even then $Q(\mathbf{m}) = \pm 1$ and $\mathcal{E}(\mathbf{m}) \geq \|\nabla \mathbf{m}\|_{L^2}^2 \geq 4\pi$ due to the topological lower bound.Then,

$$8\pi > \liminf_{k\to\infty} \mathcal{E}(\mathbf{m}_k) \geq \liminf_{k\to\infty} \|\nabla \mathbf{m}_k\|_{L^2}^2 \geq \|\nabla \mathbf{m}\|_{L^2}^2 + 4\pi \geq 8\pi,$$

a contradiction. □

Note: If the direct method was applied on the level of profiles $\theta \colon [0, \pi] \to \mathbb{R}$ with fixed boundary values, axisymmetry would automatically be preserved and conservation of Q would simply depend on the conservation of boundary values. However, the reduced energy E does not provide a bound for $\|\theta_k\|_{H^1(0,\pi)}$ due to the trigonometric terms.

3.1.2. Regularity

Proving regularity of critical points again follows standard methods for harmonic maps, see e.g. [20] or[40]. The details are given below.

Theorem 2. *Let* $\mathbf{m}$ *be a critical point of* $\mathcal{E}$*. Then* $\mathbf{m}$ *is smooth.*

Proof. We show smoothness of $\tilde{\mathbf{m}} := \mathbf{m} \circ \pi_{\pm}^{-1}$ where $\pi_{\pm}$ is the stereographic projection mapping $\pm\hat{\mathbf{e}}_3$ to ∞. If not specified otherwise, all norms are between $\mathbb{R}^2$ and $\mathbb{R}^3$. For convenience, write $\mathbf{m} = \tilde{\mathbf{m}}$.

The field $\mathbf{m}$ solves the equation

$$-\Delta\mathbf{m} = \Omega(\mathbf{m}) : \nabla\mathbf{m} + f(\mathbf{m})$$

where

$$\begin{aligned} \Omega(\mathbf{m}) &= \mathbf{m} \otimes \nabla\mathbf{m} - \nabla\mathbf{m} \otimes \mathbf{m} \\ f(\mathbf{m}) &= \frac{4\kappa}{(1+|x|^2)^2}\mathbf{m} \times ((\mathbf{m}\cdot\nu)\nu \times \mathbf{m})\,. \end{aligned}$$

Due to the special form of Ω, we have

$$|\Omega|^2 \le 2|\nabla\mathbf{m}|^2$$

and

$$\begin{aligned} \text{div }\Omega^{ij} &= \mathbf{m}^i\Delta\mathbf{m}^j - \mathbf{m}^j\Delta\mathbf{m}^i \\ &= \mathbf{m}^i f(\mathbf{m})^j - \mathbf{m}^j f(\mathbf{m})^i + \mathbf{m}^i(\Omega(\mathbf{m}):\nabla\mathbf{m})^j - \mathbf{m}^j\left(\Omega(\mathbf{m}):\nabla\mathbf{m}\right)^i \\ &= (\mathbf{m}\otimes f - f\otimes\mathbf{m})^{ij} \end{aligned}$$

so that

$$\|\text{div }\Omega\|_{L^2} \le \|f\|_{L^2} \le \left\|\frac{4\kappa}{(1+|x|^2)^2}\right\|_{L^2}.$$

Performing a Helmholtz decomposition for Ω, we have $\Omega = \Omega_0 + \Omega_1$ with div $(\Omega_0) = 0$, $\text{curl}(\Omega_1) = 0$ and the following estimates:

$$\|\nabla\Omega_1\|_{L^2} \le \|\text{div }\Omega\|_{L^2} \le \|f\|_{L^2}$$

and

$$\|\Omega_i\|_{L^2} \le \|\Omega\|_{L^2} \le 2\|\nabla\mathbf{m}\|_{L^2}.$$

Moreover,

$$\|\Omega_1\|_{L^r} \le c\|\Omega_1\|_{L^2}^{\frac{2}{r}}\|\nabla\Omega_1\|_{L^2}^{1-\frac{2}{r}}$$

for all $r \in (2, \infty)$ due to the endpoint Gagliardo Nierenberg inequality. Combining this with Hölder, we find for all $q \in (1, 2)$ and $r = \frac{2q}{2-q} \in (2, \infty)$:

$$\begin{aligned} \|\Omega_1 \nabla \mathbf{m}\|_{L^q} &= \||\Omega_1|^q |\nabla \mathbf{m}|^q\|_{L^1(\mathbb{R}^2)}^{\frac{1}{q}} \\ &\leq \|\Omega_1\|_{L^{\frac{2q}{2-q}}} \|\nabla \mathbf{m}\|_{L^2} \\ &\leq c \|\nabla \mathbf{m}\|_{L^2} \left(\|\Omega_1\|_{L^2}^{\frac{2-q}{q}} \|\nabla \Omega_1\|_{L^2}^{\frac{2(q-1)}{q}} \right), \end{aligned}$$

implying $\Omega_1 \nabla \mathbf{m} \in L^q$ for all $q \in (1, 2)$.

Now set $g = \Omega_1 \colon \nabla \mathbf{m}$, $R > 0$ and let u be a solution of

$$\begin{aligned} -\Delta u &= f + g && \text{on } B_R(0) \\ u &= \mathbf{m} && \text{on } \partial B_R(0). \end{aligned}$$

Since $L^2(B_R(0)) \hookrightarrow L^q(B_R(0))$ due to the boundedness of $B_R(0)$ and $1 < q < 2$, the Calderón-Zygmund inequality implies

$$\|D^2 u\|_{L^q(B_R)} \lesssim \|\Delta u\|_{L^q(B_R)} \leq \|f + g\|_{L^q(B_R)}$$

and hence $u \in W^{1,q}(B_R)$ for all $q \in (1, 2)$. By Morrey's inequality with $\alpha = \frac{2(q-1)}{q}$ and $\alpha \in (0, 1)$ for $q \in (1, 2)$ we deduce continuity of u on $B_R(0)$.

On the other hand,

$$\begin{aligned} -\Delta(\mathbf{m} - u) &= \Omega_0 : \nabla \mathbf{m} + \Omega_1 : \nabla \mathbf{m} + f - (f + g) \\ &= \Omega_0 : \nabla \mathbf{m} + g + f - g - f \\ &= \Omega_0 : \nabla \mathbf{m} && \text{on } B_R(0), \end{aligned}$$

extended by 0 on $\mathbb{R}^2 \setminus B_R$ belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^2; \mathbb{R}^3)$ because $\operatorname{div} \Omega_0 = 0$[6]. It follows via Lorentz space estimates [20] that $\mathbf{m} - u$ is continuous on $B_R(0)$. Thus, $\mathbf{m}$ is continuous on B_R and, by repetition of the local argument, on $\mathbb{R}^2$.

Smoothness follows from continuity by standard arguments for semilinear equations with quadratic growth [53]. Here, the right-hand side of the equation $-\Delta \mathbf{m} = b(x, \mathbf{m}, \nabla \mathbf{m})$ is given by

$$\begin{aligned} b(x, \mathbf{m}, \nabla \mathbf{m}) &= \Omega : \nabla \mathbf{m} + f(\mathbf{m}) \\ &= |\nabla \mathbf{m}|^2 \mathbf{m} + \frac{4\kappa}{(1 + |x|^2)^2} \left(\mathbf{m} \times (\mathbf{m} \cdot \nu) \, \nu \times \mathbf{m} \right), \end{aligned}$$

where $|b(x, z, p)| \leq |p|^2 |z| + \frac{4\kappa}{(1+|x|^2)^2} |z|^3 \leq c(1 + |p|^2)$ for some constant c depending only on κ and ε if $||z| - 1| < \varepsilon$. This is sufficient since we are only interested in solutions $\mathbf{m} \colon \mathbb{R}^2 \to \mathbb{S}^2$. □

3.1.3. Non-Uniqueness

Axisymmetric minimizers of $\mathcal{E}$ exist and are smooth. However, they are not unique due to further invariances of the energy. This is reflected in the fact that the solution of (2.3) is non-unique, even when fixing boundary values. In the following, we discuss the non-uniqueness of magnetizations $\mathbf{m}$ and profiles θ, starting from an axisymmetric minimizer with $\mathbf{m}(\pm\hat{\mathbf{e}}_3) = -\hat{\mathbf{e}}_3$.

Without loss of generality we may therefore assume that $\theta(0) = \pi$. However, while $Q(\mathbf{m}) = 0$ implies $\mathbf{m}(\hat{\mathbf{e}}_3) = \mathbf{m}(-\hat{\mathbf{e}}_3)$, this is not true on the level of profiles where $\theta(0)$ and $\theta(\pi)$ may vary by multiples of 2π. Later on we will see that minimality of profiles does indeed imply $\theta(0) = \theta(\pi)$ so the non-uniqueness of boundary values can be ruled out for minimizing profiles by fixing $\theta(0)$.

Apart from such effects that are due to the periodicity of trigonometric functions, non-uniqueness of symmetric minimizers and thus their profiles is also due to the symmetry of the energy functional. To account for the invariance of $\mathcal{E}$ under joint rotations and joint reflections we have already fixed the axis of symmetry to be $\hat{\mathbf{e}}_3$ and the values at the poles as $-\hat{\mathbf{e}}_3$. However, both the energy and the condition $Q = 0$ are also invariant under individual reflections on the domain or target manifold.

The first type, $\mathbf{m} \mapsto \mathbf{m}(-\cdot)$, can not be ruled out by fixing values at the poles. Such a single reflection on the domain changes whether a Skyrmion forms at the north pole $\hat{\mathbf{e}}_3$ or the south pole $-\hat{\mathbf{e}}_3$. For the polar profile, $\mathbf{m} \mapsto \mathbf{m}(-\mathbf{x})$ corresponds to a change in profile $\theta \mapsto \pi - \theta(\pi - x)$, which also preserves boundary values.

The second type, $\mathbf{m} \mapsto -\mathbf{m}(\cdot)$, is ruled out by fixing $\mathbf{m}(\hat{\mathbf{e}}_3) = -\hat{\mathbf{e}}_3$. However, since it does not affect the energy of the condition $Q = 0$, it yields additional minimizers. Such a single reflection on the target sphere affects whether the Skyrmion forms inside the sphere or outside.

Due to the invariance under reflections on the domain we can not expect the solution to the boundary value problem

$$\begin{aligned}(\theta' \sin x)' &= \frac{\sin(2\theta)}{\sin x} + \kappa \sin(2\theta - 2x) \sin x \\ \theta(0) = \theta(\pi) &= \pi\end{aligned}$$

to be unique, even when minimality is additionally requested. Instead, we will focus on profiles with $\theta(\frac{\pi}{2}) < \pi$. We will later see that this corresponds to a Skyrmion formation at the north pole.

Definition 3. *Let* $\mathbf{m} \in \{\mathbf{m} \in H^1(\mathbb{S}^2;\mathbb{S}^2) : Q(\mathbf{m}) = 0,\ \mathbf{m}$ *axisymmetric*$\}$ *be a minimizer and* $\theta\colon [0,\pi] \to \mathbb{R}$ *a profile such that* $\mathbf{m} = \mathbf{m}_\theta$ *and* $\theta(0) = \pi$.

(1) θ *is called a minimizing profile.*

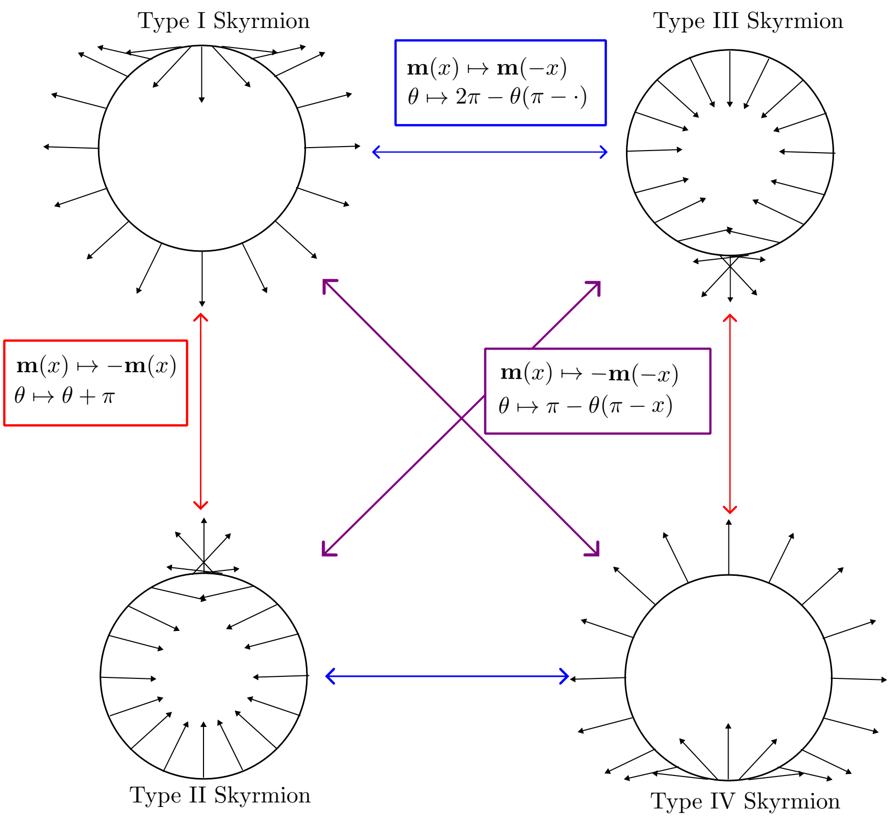

Figure 3.1.: Types of axisymmetric Skyrmions and the transformations by which they are related. On the left, the Skyrmion forms at the north pole and on the right, the Skyrmion forms at the south pole. The Skyrmions are related by the transformations indicated at the arrows and all have the same energy.

(2) *If $\theta(\frac{\pi}{2}) < \pi$ then θ is called a lower minimizing profile.*
If $\theta(\frac{\pi}{2}) > \pi$ then θ is called an upper minimizing profile.

(3) *Given a profile $\theta\colon [0,\pi] \to \mathbb{R}$, the profile $\bar{\theta} = \pi - \theta(\pi - x)$ is called the mirrored profile (associated to θ).*

Notes:

(1) Minimizing profiles are minimizers of the reduced energy E.

(2) We will later see that the case $\theta(\frac{\pi}{2}) = \pi$ does not occur for minimizing profiles.

(3) θ is a lower minimizing profile if and only if the mirrored profile $\bar{\theta}$ is an upper minimizing profile.

(4) If θ is a lower minimizing profile then $\mathbf{m}_0$ is a type I Skyrmion. If θ is an upper minimizing profile then $\mathbf{m}_0$ is a type III Skyrmion.

3.2. Properties of Minimizing Profiles

In this section we will analyze the properties of minimizing profiles. After establishing that minimality implies $\theta(0) = \theta(\pi)$ we will focus on lower minimizing profiles, investigating their range and shape as well as their behaviour near π. Overall, we prove the following:

Proposition 3.1. *Assume $\kappa > 24$ and let θ be a lower [upper] minimizing profile with $\theta(0) = \pi$. Then the following holds*

(1) $\theta(\pi) = \pi$ and for $x \in (0, \pi)$, we have $x < \theta(x) < \pi$ $[\pi < \theta(x) < x + \pi]$.

(2) θ has exactly one minimum x_m [maximum x^m]. As $\kappa \to \infty$, we have $x_m \to 0$ $[x^m \to \pi]$.

(3) $\theta - \mathrm{id}$ is monotonically decreasing on $(0, \pi)$.

(4) The limit of θ' at π [at 0] exists and is smaller than 1.

3.2.1. Range

The upper bound for the energy directly implies

Lemma 3.1. *Let θ be a minimizing profile with $\theta(0) = \pi$. Then $\theta(\pi) = \pi$.*

Proof. Assume $\theta(\pi) = (2k+1)\pi$ with $|k| \geq 1$. We show that the energy then exceeds the upper bound for the minimum in [37]. For that, write $x_a := \min\{x \in (0, \pi) \ : \ \theta(x) = a\}$ where $a \in \{2k\pi, (2k+1)\pi\}$. Then by Young's inequality,

$$\int_0^\pi (\theta')^2 \sin x + \frac{\sin^2\theta}{\sin x}\,\mathrm{d}x \geq \int_0^{x_{2k\pi}} -2\theta' \sin\theta\,\mathrm{d}x + \int_{x_{2k\pi}}^{x_{2k+1}\pi} 2\theta' \sin\theta\,\mathrm{d}x$$
$$= 2\left(\cos(2\pi) - \cos(\pi) + \cos(2\pi) - \cos(\pi)\right) = 8.$$

On the other hand, minimality implies $E(\theta) = \frac{1}{2\pi}\mathcal{E}(\mathbf{m}_\theta) < \frac{16\pi}{2\pi} = 8$, a contradiction. $\square$

From a simple energy argument, it follows that minimizing profiles are enclosed between id and id $+\pi$, implying $\sin(\theta - x) > 0$ for all $x \in (0, \pi)$.

Lemma 3.2. *Let θ be a minimizing profile with $\theta(0) = \theta(\pi) = \pi$. Then $x < \theta(x) < x+\pi$ for all $x \in (0, \pi)$.*

Proof. Assume that there exists $c \in \{x \in (0,\pi)|\ \theta(x) < x\}$. Then, by the intermediate value theorem and due to $\theta(0) > 0$ and $\theta(\pi) = \pi$, there are $0 < a < c < b \leq \pi$ with $\theta(a) = a$ and $\theta(b) = b$. The contribution of θ on $[a,b]$ can be estimated with Young's inequality:

$$\int_a^b \sin x(\theta')^2 + \frac{\sin^2\theta}{\sin x}\ dx \geq 2\int_a^b \theta'(x)\sin\theta\ dx = 2[-\cos\theta]_a^b$$
$$= 2\cos(\theta(a)) - 2\cos(\theta(a)) = 2\cos a - 2\cos b$$

On the other hand, consider the adapted profile

$$\tilde{\theta}\colon [0,\pi] \to \mathbb{R}, \quad \tilde{\theta}(x) = \begin{cases} x & a \leq x \leq b \\ \theta(x) & \text{else.} \end{cases}$$

Due to $\theta(a) = a = \tilde{\theta}(a)$ and $\theta(b) = b = \tilde{\theta}(b)$, $\tilde{\theta}$ is continuous and has the same boundary values as θ. Since

$$\int_a^b (\tilde{\theta}')^2 \sin x + \frac{\sin^2\tilde{\theta}}{\sin x}\,\mathrm{d}x = \int_a^b \sin x + \frac{\sin^2 x}{\sin x}\,\mathrm{d}x = 2\cos a - 2\cos b$$

and

$$\int_a^b \sin^2(\tilde{\theta} - x)\sin x\,\mathrm{d}x = 0 < \int_a^b \sin x\sin^2(\theta - x)\,\mathrm{d}x$$

while the contribution outside $[a,b]$ remains unchanged, we find $E(\tilde{\theta}) < E(\theta)$, which is a contradiction to θ being minimizing. Thus, the assumption of $\theta(c) < c$ has been wrong.

Furthermore, $\mathrm{id}_{[0,\pi]}$ is a solution of the differential equation apart from the initial value. Therefore, if $\theta = \mathrm{id}$ on some interval $[a,b]$, unique solvability of the ODE on $[\varepsilon, \pi-\varepsilon]$ for any $\varepsilon > 0$ would imply that $\theta = \mathrm{id}$ on $[\varepsilon, \pi-\varepsilon]$. As the profile of a smooth axisymmetric $\mathbf{m}$, the function θ is continuous, see appendix A.1. Thus, $\theta = \mathrm{id}$ on any closed subintervall of $[0,\pi]$ implies $\theta = \mathrm{id}$ on $[0,\pi]$ which contradicts $\theta(0) = \pi$.

In conclusion, if there was $x_0 \in (0,\pi)$ such that $\theta(x_0) = x_0$ then θ would not be minimizing or $\theta = \mathrm{id}$ on some interval $[x_0, b]$. As both options are impossible, there can be no such x_0 and $\theta > x$ on $(0,\pi)$.

By the same energy argument, $\theta(x) < x + \pi$ for all $x \in (0,\pi)$. □

For lower minimizing profiles, the upper bound can be refined. This will be done in two steps. First, we show that for minimizing profiles, the root of π has at most one element p in $(0,\pi)$. For lower minimizing profiles it follows that $p < \frac{\pi}{2}$. We then perform all further analysis under the assumption that $p > 0$. However, an argument in Chapter two will show that $p = 0$, completing the proof of the first statement in Proposition 3.1.

Lemma 3.3. *Assume that* $\theta\colon [0,\pi] \to \mathbb{R}$ *with* $\theta(0) = \theta(\pi) = \pi$ *minimizes* E *and that there exists* $x_0 \in (0,\pi)$ *with* $\theta(x_0) < \pi$. *Then either* $\theta < \pi$ *for all* $x \in (0,\pi)$ *or there exists a unique* $p \in (0,\pi)$ *such that* $\theta > \pi$ *on* $(0,p)$ *and* $\theta < \pi$ *on* (p,π).

The second case will be referred to as overshooting. As mentioned above, it can only be ruled out later on and thus has to be considered a possibility in this chapter.

Proof. There is nothing to prove in the first case. If it does not occur then there exist $0 \le p_1 < x_0 < p_2 \le \pi$ such that $\theta(x) < \pi$ on (p_1,p_2), $\theta(p_1) = \theta(p_2) = \pi$, and $p_1 > 0$ or $p_2 < \pi$. We will show $p_2 = \pi$ in two steps and then conclude by a symmetry argument, using $\bar{\theta}$. Note that the energy argument heavily relies on the analysis of e_{II} in Lemma 2.6. There, we have defined $\eta(x) = x + \frac{\pi}{2} + d_\kappa(x)$ as the value of $\theta(x)$ for which $e_{II}(\theta(x))$ is maximal.

Step 1 First, assume $p_2 \le \frac{\pi}{2}$ and let

$$x_m = \mathrm{argmin}_{[p_1,p_2]}\theta(x)$$

be the minimizer of θ in $[p_1,p_2]$. If $\theta(x_m) \ge \frac{\pi}{2}$ then $\theta > \frac{\pi}{2}$ on (p_1,p_2) and setting $\tilde{\theta}(x) = 2\pi - \theta(x)$ for $x \in (p_1,p_2)$ would reduce the energy since

$$E(\theta) - E(\tilde{\theta}) = \kappa \int_{p_1}^{p_2} \left(\sin^2(\theta - x) - \sin^2(\theta + x)\right) \sin x \,\mathrm{d}x = \kappa \int_{p_1}^{p_2} -\sin(2\theta)\sin(2x)\,\mathrm{d}x \quad \circledast$$

which is positive due to $\sin(2\theta)\sin(2x) < 0$ for all $(x,\theta) \in (0,\frac{\pi}{2}) \times (\frac{\pi}{2},\pi)$.

On the other hand, if $\theta(x_m) < \frac{\pi}{2} < \eta(x_m)$ and $\pi = \theta(p_2) > \eta(p_2)$ due to $p_2 < \frac{\pi}{2}$, then there must be intersection points a,b,c with $x_m \in (a,b) \subset (p_1,p_2)$ and $c \in (p_2,\pi)$ such that $\theta(x) = \eta(x)$ for $x \in \{a,b,c\}$, $\theta > \eta$ on $(0,a) \cup (b,c)$ and $\theta < \eta$ on (a,b). Note that oscillations of θ around η as opposed to $\theta < \eta$ on (a,b) and $\theta > \eta$ on (b,c) would result in an increase of both $E_I(\theta)$ and $E_{II}(\theta)$ and can therefore be excluded. Also, $c > p_2$ follows from the fact that $\eta(p_2) < \pi = \theta(p_2)$ due to $p_2 < \frac{\pi}{2}$.

Starting from this assumption on θ and a,b,c we perform the following two adaptations to construct $\tilde{\theta}$ with strictly smaller energy and $\tilde{\theta} > \eta > \frac{\pi}{2}$ for all $x \in (p_1,p_2)$, leading back to the first case.

Adaptation 1 Find $y = \mathrm{argmax}_{(a,b)}(\eta(y) - \theta(y))$ and set $m := \eta(y) - \theta(y)$ as well as $\theta_1 := \eta(y) + m$. If $\theta_1 \le \pi$ choose $\alpha \in (p_1,a)$ such that $\theta(\alpha) = \theta_1$ and set

$$\tilde{\theta}(x) = \begin{cases} \theta(x) & x \in (0,\alpha) \\ \theta_1 & x \in (\alpha,y) \end{cases}.$$

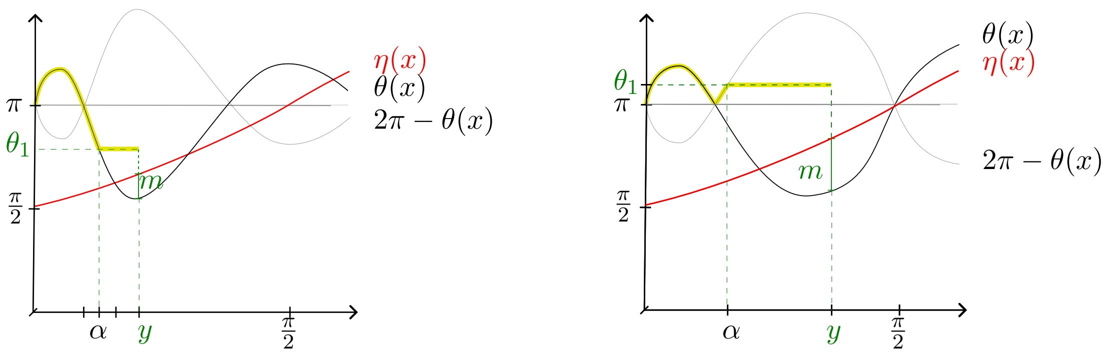

Figure 3.2.: Adaptation 1. $\tilde{\theta}$ is highlighted in yellow.

Otherwise, if $\theta_1 > \pi$, choose $\alpha \in (p_1, y)$ such that $2\pi - \theta(\alpha) = \theta_1$ and set

$$\tilde{\theta}(x) = \begin{cases} \theta(x) & x \in (0, p_1) \\ 2\pi - \theta(x) & x \in (p_1, \alpha) \\ \theta_1 & x \in (\alpha, y) \end{cases}.$$

Adaptation 2 Compute $\theta_2 = \eta(b) + m$. If $\theta_2 \leq \pi$, choose $\beta \in (b, p_2)$ minimal such that $\theta(\beta) = \theta_2$ and set

$$\tilde{\theta}(x) = \begin{cases} \eta(x) + m & x \in (y, b) \\ \theta_2 & x \in (b, \beta) \\ \theta(x) & x \geq \beta \end{cases}.$$

If $2\pi - \theta(b) > \theta_2 > \pi$, choose $\beta \in (b, p_2)$ minimal such that $2\pi - \theta(\beta) = \theta_2$ and set

$$\tilde{\theta}(x) = \begin{cases} \eta(x) + m & x \in (y, b) \\ \theta_2 & x \in (b, \beta) \\ 2\pi - \theta(x) & x \in (\beta, p_2) \\ \theta(x) & x > x_0 \end{cases}.$$

Finally, if $\theta_2 > 2\pi - \theta(b)$, choose $\beta \in (y, b)$ minimal such that $2\pi - \theta(\beta) = \eta(\beta) + m$ and set

$$\tilde{\theta}(x) = \begin{cases} \eta(x) + m & x \in (y, \beta) \\ 2\pi - \theta(x) & x \in (\beta, p_2) \\ \theta(x) & x > p_2 \end{cases}.$$

Consistency Before proving that the adaptations yield a reduction of the energy, we first confirm the existence of α, β in all cases and infer continuity of $\tilde{\theta}$. For the first adaptation, the inequalities

$$\pi = \theta(p_1) \geq \theta_1 = \eta(y) + m > \eta(a) = \theta(a)$$

$$\text{and} \qquad \pi = 2\pi - \theta(p_1) < \theta_1 = 2\eta(y) - \theta(y) < 2\pi - \theta(y),$$

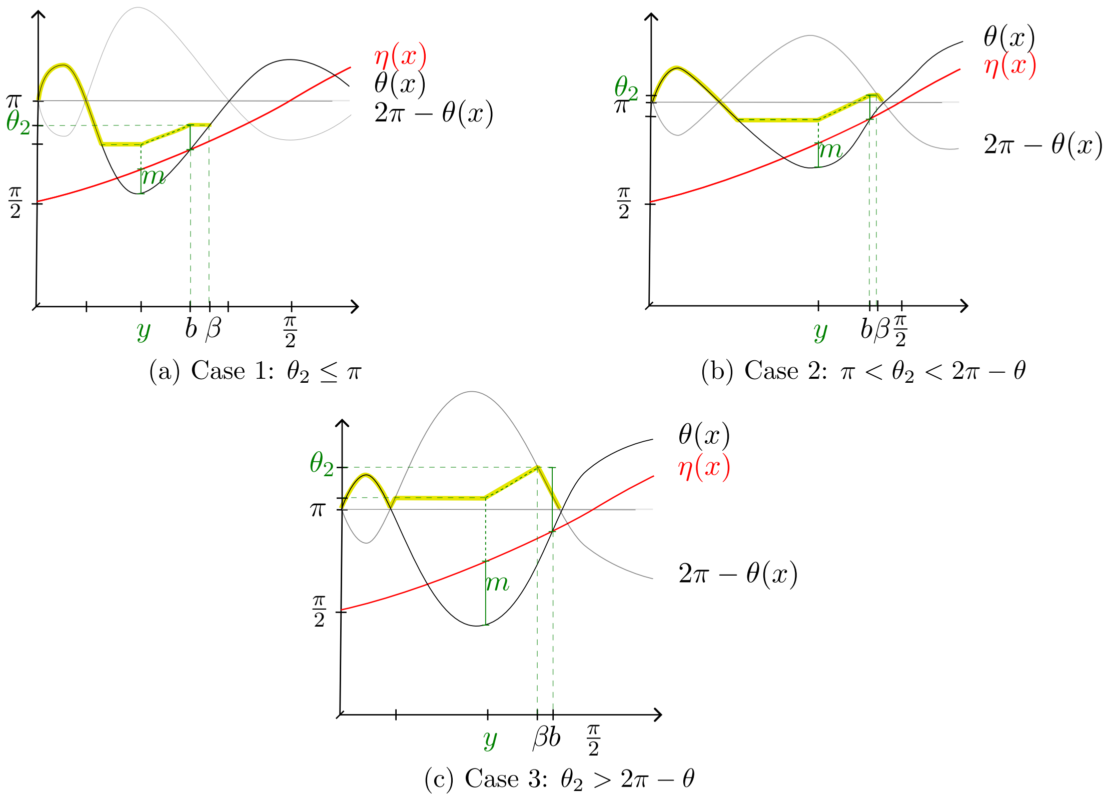

Figure 3.3.: Adaptation 2: positions of b and β

in the first and second case, respectively, imply the existence of α by the intermediate value theorem in each of the cases. Due to $\theta(p_1) = \pi = 2\pi - \theta(p_1)$ and $\theta(\alpha) = \theta_1$ or $2\pi - \theta(\alpha) = \theta_1$ in the respective cases, $\tilde{\theta}$ is continuous on $(0, y)$. For the second adaptation, existence again follows from the intermediate value theorem under consideration of the inequalities

$$\theta(b) = \eta(b) < \theta_2 \leq \pi = \theta(p_2),$$
$$2\pi - \theta(b) > \theta_2 > \pi = 2\pi - \theta(p_2),$$
$$\text{and} \quad \pi - \theta(y) > \eta(y) + m \quad \text{while} \quad 2\pi - \theta(b) < \eta(b) + m$$

in each of the respective cases. Continuity at b and β follows from the construction wile $\tilde{\theta}(y) = \eta(y) + m$ from Step 1 ensures continuity at y. Note that $\theta_1, \theta_2 > \eta$ and therefore $\tilde{\theta} > \eta > \frac{\pi}{2}$ on (p_1, p_2).

Energy reduction In general, e_{II} is reduced when $|\tilde{\theta}(x) - \eta(x)| > |\theta(x) - \eta(x)|$. For the first move we have $\tilde{\theta} \geq \theta > \eta$ on $(0, a)$ and $|\tilde{\theta}(x) - \eta(x)| \geq \theta_1 - \eta(y) = m \geq \eta(x) - \theta(x)$ on (α, y). In the second case, where $a < \alpha$ is possible, note that

$$\tilde{\theta}(x) - \eta(x) = 2\pi - \theta(x) - \eta(x) > 2\eta(x) - \theta(x) - \eta(x) = |\eta(x) - \theta(x)|$$

on (a, α).

For the second move there are three cases. Just as before, though, the reasoning is mostly the same with $\tilde{\theta} - \eta(x) = m \geq \theta(x) - \eta(x)$ on (y, b) (or on (y, β) in the third case) and $\tilde{\theta} > \pi > \theta(x) > \eta(x)$ on (b, p_2). For the third case, we again have the additional estimate $\tilde{\theta}(x) - \eta(x) = 2\pi - \theta(x) - \eta(x) > \eta(x) - \theta(x)$ on (β, b).

The first part of the energy is reduced pointwise on $(0, y)$ because $0^2 \leq (\theta')^2$. On (y, b), however, where $\tilde{\theta}' = \eta'$, we have to consider the possibility that $\eta - \theta$ admits local extrema, which result in a change in the sign of $(\eta - \theta)'$, thus preventing a pointwise comparison of $e_I(\theta)$ and $e_I(\tilde{\theta})$. Instead, we prove a reduction of E_I on certain intervals.

For that, assume that $\eta - \theta$ has a local maximum at z. Then there are $z_1 < z$ and $z_2 > z$ such that $(\eta - \theta)(z_1) = (\eta - \theta)(z_2)$, $(\eta - \theta)' > 0$ on (z_1, z) and $(\eta - \theta)' < 0$ on (z, z_2). We apply the integral version of the mean value theorem to obtain

$$
\begin{aligned}
&\int_{z_1}^{z_2} \left((\eta')^2 - (\theta')^2\right) \sin x \,\mathrm{d}x \\
&= \int_{z_1}^{z} \underbrace{(\eta' - \theta')}_{>0} \underbrace{(\eta' + \theta')}_{<2\theta'} \sin x \,\mathrm{d}x + \int_{z}^{z_2} \underbrace{(\eta' - \theta')}_{<0} \underbrace{(\eta' + \theta')}_{>2\theta'} \sin x \,\mathrm{d}x \\
&< \int_{z_1}^{z} (2\theta' \sin x)(\eta' - \theta') \,\mathrm{d}x + \int_{z}^{z_2} (2\theta' \sin x)(\eta' - \theta') \,\mathrm{d}x \\
&= 2\theta'(\xi_1)\sin(\xi_1)\Big(\eta(z) - \theta(z) - \eta(z_1) + \theta(z_1)\Big) \\
&\quad + 2\theta'(\xi_2)\sin(\xi_2)\Big(\eta(z_2) - \theta(z_2) - \eta(z) + \theta(z)\Big) \\
&= \Big((\eta(z) - \theta(z)) - (\eta(z_1) - \theta(z_1)) \Big)\Big(2\theta'(\xi_1)\sin(\xi_1) - 2\theta'(\xi_2)\sin(\xi_2)\Big) < 0,
\end{aligned}
$$

where the last expression is negative as the product of a positive and a negative term. Positivity of the first term follows from maximality of $\eta - \theta$ at z while the second term is negative due to the monotonicity of $\theta' \sin x$, which follows from differentiation. In fact,

$$
(\theta' \sin x)' = \theta'' \sin x + \theta' \cos x = \frac{\sin(2\theta)}{2 \sin x} + \frac{\kappa}{2} \sin(2\theta - 2x) \sin x = \frac{d}{d\theta} e_{II}(\theta, x)
$$

and the right and side is positive since e_{II} is monotonically increasing in θ due to $\theta < \eta$.

It follows that $E_{I,[z_1,z_2]}(\tilde{\theta}) < E_{I,[z_1,z_2]}(\theta)$ around a local maximum of $\eta - \theta$. By the choice of y, the global maximum of $\eta - \theta$ in $[y, b] \subset [a, b]$ is attained at y. Hence, $\eta - \theta$ is decreasing near y and the first extremum must be a minimum. Furthermore, $\eta - \theta > 0 = \eta(b) - \theta(b)$ implies that $\eta - \theta$ is decreasing near b. It follows that the last extremum must be a maximum. Since no two minima can occur without a maximum between them and vice versa, all local extrema on (y, b) appear in pairs of minima and maxima. In particular, every locally minimal value is assumed again at a later point, allowing a comparison as above. In this way, $E_{I,J}$ can be improved on a collection of intervals J such that $(\eta - \theta)' < 0$ on $(y, b) \setminus J$. On this remaining set, e_I is reduced pointwisely and overall, a reduction of $E_{I,[y,b]}$ follows. On (y, p_2), reduction is trivial with $\tilde{\theta}' \in \{0, -\theta'\}$.

Since the assumption $p_2 \leq \frac{\pi}{2}$ has allowed to construct $\tilde{\theta}$ with $E(\tilde{\theta}) < E(\theta)$, thus contradicting the minimality of θ, it must be wrong and we have shown that $\theta(x) < \pi$ on $(p_1, \frac{\pi}{2}]$ for any minimizing profile θ fulfilling $\theta(x_0) < \pi$ for some $x_0 < \frac{\pi}{2}$. Here, $p_1 = \inf\{x \in (0, x_0) \ : \ \theta(x) < \pi\}$.

Step 2 Next, consider the case where $\frac{\pi}{2} < p_2 < \pi$ so that $\theta(x_1) > \pi$ for some $x_1 \in (p_2, \pi)$. The mirrored profile $\bar{\theta}$ then fulfils $\bar{\theta}(\pi - x_1) < \pi$ and $\pi - x_1 \in (0, \frac{\pi}{2})$ and since it is also minimising, all arguments given above apply. From the first case it follows that $\bar{\theta} < \pi$ on $(p_1', \frac{\pi}{2})$ for some $p_1' \in (0, x_1)$, leading to a contradiction at $\pi - p_2 \in (p_1', \frac{\pi}{2})$ with $\bar{\theta}(\pi - p_2) = \pi$. In total, $p_2 = \pi$ is the only remaining option.

If $\theta > x$ on (p_0, p_1) for some $0 \leq p_0 < 1$ then performing the argument again for $\bar{\theta}$ with $\bar{\theta} < \pi(\pi - p_1, \pi - p_0)$ proves $p_0 = 0$. Therefore, in the case where θ overshoots, $p := p_1$ is the unique root of θ for π in $(0, \pi)$. □

3.2.2. Shape

First, we show that a minimizing profile has exactly one minimum. We then proceed to show an upper bound for the derivative of θ. Here and for the rest of the section we use that there exists $p \in [0, \pi)$ such that $\theta(x) > \pi$ on $(0, p)$ and $\theta(x) < \pi$ on (p, π). For lower minimizing profiles, $\theta(\frac{\pi}{2}) < \pi$ implies $p < \frac{\pi}{2}$.

Lemma 3.4. *Let θ be a minimizing profile. If $p < \pi$ then θ has exactly one minimum in (p, π).*

Proof. The proof again relies heavily on Lemma 2.6. In general, this implies that θ can not have a local minimum $x_{\min}$ with $\theta(x_{\min}) < \xi(x_{\min})$ or $\theta(x_{\min}) > \eta(x_{\min})$ nor a local maximum such that $\xi(x_{\max}) < \theta(x_{\max}) < \eta(x_{\max})$. Otherwise, constructing a new profile by setting $\tilde{\theta} \equiv \text{const} = \theta(a) = \theta(b)$ on a suitable interval $(a, b) \ni x_{\min}$ [or $(a, b) \ni x_{\max}$ at maxima] with $\xi(x) < \theta(x) < \theta(a)$ on (a, b) [or $\eta > \theta > \theta(a) > \xi$] would give a profile with strictly smaller energy.

Let $x_m \in (0, \pi)$ be the global minimum of θ. Since $\theta \equiv \pi$ does not solve the equation, $\theta_m := \theta(x_m) < \pi$. We now proceed separately on (p, x_m) and (x_m, π).

On (p, x_m), extrema appear in pairs $x_{\min} < x_{\max}$. Due to above results, it has to hold that $\xi(x_{\min}) < \theta(x_{\min}) < \eta(x_{\min})$ as well as $\theta(x_{\max}) < \xi(x_{\max})$ or $\eta(x_{\max}) < \theta(x_{\max})$. The maximal value can not be smaller than $\xi(x_{\max})$ since monotonicity of ξ would then imply the estimates $\theta_m < \theta(x_{\max}) < \xi(x_m)$, a contradiction to the first observation applied to x_m. Thus,

$$\xi(x_{\min}) < \theta(x_{\min}) < \eta(x_{\min}) < \eta(x_{\max}) < \theta(x_{\max})$$

is the only remaining option.

Assuming this setting, consider $x_1 < x_2 \in (x_{\min}, x_{\max})$ such that $|\theta - \eta|$ is maximal at x_1 and x_2. Furthermore, by the intermediate value theorem, there are points $y_1 < x_{\min}$ and $y_2 > x_{\max}$ such that $\theta(y_1) = \theta(x_{\max})$ and $\theta(y_2) = \theta(x_{\min})$. If $\eta(x_1) - \theta(x_1) < \theta(x_{\max}) - \eta(x_1)$ then

$$|\theta(x_{\max}) - \eta(x)| > |\theta(x) - \eta(x)| \qquad \text{for } x \in (y_1, x_{\max})$$

and setting $\tilde{\theta} = \theta(x_{\max})$ on $(y_1, x_{\max})$ results in $e_{II}(\tilde{\theta}) < e_{II}(\theta)$ on $(y_1, x_{\max})$. The derivative term is, as always, trivially reduced to 0 and so $\tilde{\theta}$ has smaller energy than the supposed minimizer θ, a contradiction.

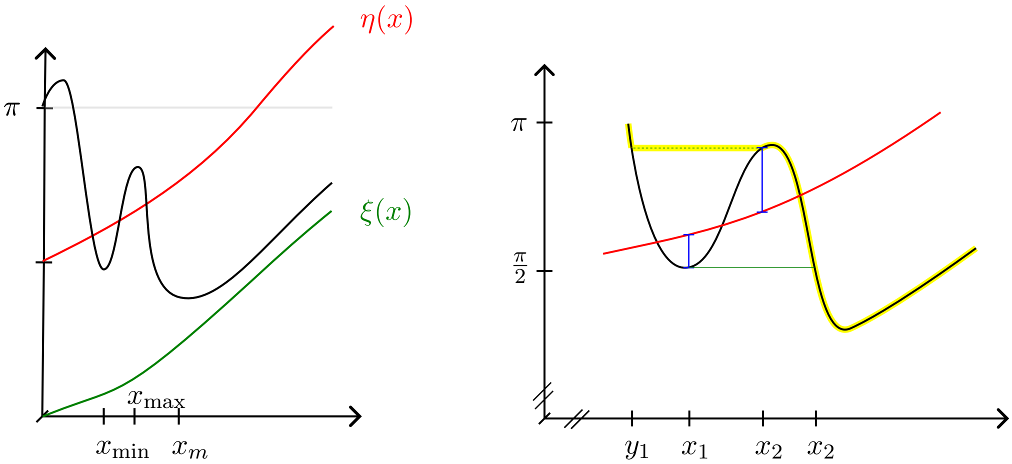

Figure 3.4.: Configuration of local extrema and the construction in the discussed case: $\eta(x_1) - \theta(x_1) < \theta(x_{\max}) - \eta(x_1)$

Similarly, if $\theta(x_2) - \eta(x_2) < \eta(x_2) - \theta(x_{\min})$, setting $\tilde{\theta}(x) = \theta(x_{\min})$ on $(x_{\min}, y_2)$ would yield an improvement in energy. Finally, since both θ and η are strictly increasing on $(x_{\min}, x_{\max})$, one of the two inequalities has to hold whenever there are minimum and maximum in the considered configuration. Since this was the last remaining case, there can not be any extrema in (p, x_m).

On (x_m, π) extrema appear in pairs $x_{\max} < x_{\min}$. Again, the restrictions from the first observations apply and $\theta(x_{\max}) < \xi(x_{\max})$ would imply $\theta(x_{\min}) < \xi(x_{\min})$ for the following minimum, a contradiction. The remaining option of $\theta(x_{\max}) > \eta(x_{\max})$ and $\eta(x_{\min}) > \theta(x_{\min}) > \xi(x_{\min})$ implies that θ intersects η at two points in $(x_m, x_{\max})$ and $(x_{\max}, x_{\min})$. We perform a construction similar to that in the proof of Lemma 3.3, which reduces the energy.

Under the assumption of two intersections, there are points a, b and $y \in (a, b)$ such that $\eta - \theta$ is maximal at y and $(\theta - \eta)(a) = (\theta - \eta)(b) = 0$. If $\theta_1 = 2\eta(y) - \theta(y) < \theta(x_{\max})$ then there exists $\alpha < a$ such that $\theta(\alpha) = \theta(x_{\max})$ and setting $\tilde{\theta} \equiv \theta(x_{\max})$ on $(\alpha, x_{\max})$ reduces the energy. Otherwise if $\theta_{\max} < \theta_1 < \pi$ there is $\beta \in (y, x_{\max})$ such that $2\theta(x_{\max}) - \theta(\beta) = 2\eta(y) - \theta(y)$ since

$$2\theta(x_{\max}) - \theta(y) > 2\eta(y) - \theta(y) > \theta(x_{\max}).$$

Setting $\tilde{\theta} \equiv \theta_1$ on (α, β) and $\tilde{\theta}(x) = 2\theta(x_{\max}) - \theta(x)$ on $(\beta, x_{\max})$ reduces the energy. Finally, it is possible that $\theta_1 > \pi$. In this case follow the corresponding construction in Lemma 3.3, Move 1 and again choose β such that $\theta_1 = 2\theta(x_{\max}) - \theta(\beta)$. The resulting $\tilde{\theta}$ has smaller energy. □

By considering $\bar{\theta} = 2\pi - \theta(\pi - \cdot)$, we can conclude

Corollary 3.1. *Let θ be a minimizing profile. If $p > 0$ then θ has exactly one maximum in $(0, p)$.*

We shall denote a maximum in $(0, p)$ by x^m and its value by $\theta^m = \theta(x^m)$. Recall from the proof of Lemma 3.4 that the global minimum is denoted by x_m and its value by $\theta_m = \theta(x_m)$.

Before we continue to prove monotonicity of $\theta - \mathrm{id}$, some more notation is introduced, regarding the root of $\frac{\pi}{2}$. Let θ be a minimizing profile.

Lemma 3.5. *The set $\{x \in (0, \pi)| \ \theta(x) = \frac{\pi}{2}\}$ contains at most 2 elements.*

Proof. Assume $|\theta^{-1}(\{\frac{\pi}{2}\})| > 2$. Then there would be points $a < b \in (p, \frac{\pi}{2})$ such that $\pi > \theta(x) > \frac{\pi}{2}$ on (a, b) (due to Lemma 3.3) and $\theta(a) = \theta(b) = \frac{\pi}{2}$ and the energy could be reduced by setting $\tilde{\theta}(x) = \pi - \theta(x) < \frac{\pi}{2}$ on $[a, b]$. □

As the limit for large κ will reveal, the root of $\frac{\pi}{2}$ contains exactly two elements for κ large enough. In that case, denote them by $x_* < x^*$ and note that $x^* < \frac{\pi}{2}$ by Lemma 3.2.
If $|\theta^{-1}\{(\frac{\pi}{2})\}| \leq 1$, set $x_* = x^* = x_m$ where x_m is the global minimum of θ in (p, π). In this case it is possible that $x^* > \frac{\pi}{2}$.

We now turn to analyze the extrema of θ in (p, π). Again, the argument will rely on Lemma 2.6.

Lemma 3.6. *Let θ be a minimizing profile. Then $\theta - \mathrm{id}$ is monotonically decreasing on (p, π).*

For $x \in (0,\pi)$, it holds that $(\bar\theta - \mathrm{id})'(x) = \theta'(\pi - x) - 1 = (\theta - \mathrm{id})'(\pi - x)$. Therefore, the last Lemma implies:

Corollary 3.2. *Let θ be a minimizing profile. Then $(\theta - \mathrm{id})$ is monotonically decreasing on $(0,\pi)$.*

Proof of Lemma 3.6. Working backwards from π, we show that $\theta - \mathrm{id}$ does not assume a maximum. Since $\pi = (\theta - \mathrm{id})(0) > (\theta - \mathrm{id})(p) > 0 = (\theta - \mathrm{id})(\pi)$ for all $x \in (0,\pi)$, this proves the decay. The intervals that structure the proof are empty if $x_m \geq \frac{\pi}{2}$. However, since the arguments stay valid on the non-empty intervals, this will not be mentioned explicitly. For this proof, most arguments rely on the Euler-Lagrange equation for axisymmetric minimizers. Recall that this implies that the corresponding minimizing profile solves

$$\theta'' \sin x + \theta' \cos x = \frac{\sin(2\theta)}{\sin x} + \kappa \sin(2\theta - 2x)\sin x. \tag{3.1}$$

First interval: $\frac{3\pi}{4} < x < \theta < \pi$
On this interval, $\sin(2\theta) > \sin(2x)$ and $\cos x \leq 0$. Also, $2(\theta - x) \leq 2\frac{\pi}{4}$ so $\sin(2\theta - 2x) > 0$. Thus, if $\theta'(x) - 1 \geq 0$, it would follow that

$$\begin{aligned}
\theta''(x) &= \frac{1}{\sin x}\left(\frac{\sin(2\theta)}{2\sin x} - \cos x\theta' + \frac{\kappa}{2}\sin x \sin(2\theta - 2x)\right) \\
&\geq \frac{1}{\sin x}\left(\frac{2\sin x\cos x}{2\sin x} - \cos x + \frac{\kappa}{2}\sin x\sin(2\theta - 2x)\right) \\
&= \frac{1}{\sin x}\left(\frac{\kappa}{2}\sin x \sin(2\theta - 2x)\right) > 0
\end{aligned}$$

and therefore there can be no maximum of $\theta - \mathrm{id}$ on $(\frac{3\pi}{4}, \pi)$.

Second interval: $\frac{\pi}{2} < x < \frac{3\pi}{4}$
From the optimization of e_{II} it is known that the right-hand-side of (3.1) is positive for $\xi(x) < \theta(x)$ and $x \in (\frac{\pi}{2}, \pi)$ and negative while increasing in θ for θ between $\mathrm{id} = \xi + d_\kappa$ and ξ.

For $\theta(x) > \xi(x)$, this implies non-negativity of $\theta''(x)$. Since there are no minima of θ on $(\frac{\pi}{2}, \frac{3\pi}{4})$, the first derivative $\theta'(x)$ is non-negative and from the differential equation it follows with $\cos x < 0$ that

$$\theta''(x) = \frac{1}{\sin x}\left(-\cos x\,\theta'(x) + \frac{\sin(2\theta)}{2\sin x} + \frac{\kappa}{2}\sin(2\theta - 2x)\sin x\right) > 0.$$

Thus, there can be no maximum of $\theta - \mathrm{id}$ on this interval where $\theta > \xi$. The only remaining case is that of $x < \theta(x) < \xi(x)$. In this case, since the derivative of e_{II} with respect to θ is strictly monotonically increasing in θ for $\mathrm{id} < \theta < \xi$ as seen in the proof of Lemma 2.6, one finds

$$\frac{\sin(2\theta)}{2\sin x} + \frac{\kappa}{2}\sin(2\theta - 2x)\sin x > \frac{\sin(2x)}{2\sin x} + \frac{\kappa}{2}\sin(2x - 2x)\sin x = \cos x.$$

Therefore, if $\theta'(x) \geq 1$, the differential equation again implies positivity of the second derivative:

$$\begin{aligned}\theta''(x) &= \frac{1}{\sin x}\Big(- \cos x\theta' + \frac{\sin(2\theta)}{2\sin x} + \frac{\kappa}{2}\sin(2\theta - 2x)\sin x\Big) \\ &> \frac{1}{\sin x}\left(-\cos x + \cos x\right) = 0.\end{aligned}$$

Finally, the second derivative is positive for $x = \frac{\pi}{2}$ because $\theta(\frac{\pi}{2}) \in (\frac{\pi}{2}, \pi)$ and $\kappa > 1$.

$$\theta''\left(\frac{\pi}{2}\right) + 0 \cdot \theta'(x) = \frac{\sin\left(2\theta(\frac{\pi}{2})\right)}{2 \cdot 1} + \frac{\kappa}{2}\sin(2\theta - \pi) = (1 - \kappa)\sin(2\theta) > 0.$$

Since any maximum of $\theta - \mathrm{id}$ at y would imply $\theta'(y) = 1 \geq 1$ and $\theta''(y) \leq 0$, there are no such maxima within $(\frac{\pi}{2}, \frac{3\pi}{4})$.

Third interval: $\mathbf{x_m < x < \frac{\pi}{2}}$
On $(x_m, \frac{\pi}{2})$, the second derivative of a solution can become negative for $\theta' = 1$. However, we will show that this is only possible for θ close to $x + \frac{\pi}{2}$ and infer an energy estimate exceeding the lower bound of 8π from [37]. Thus, such maxima do not appear for minimizing profiles.

Assume there exists $\bar{x} \in (x_m, \frac{\pi}{2})$ such that $\theta'(\bar{x}) = 1$. Then the second derivative of θ at $\bar{x}$ is given by

$$\theta''(\bar{x})\sin\bar{x} = -\cos\bar{x} + \frac{\sin(2\theta(\bar{x}))}{2\sin\bar{x}} + \frac{\kappa}{2}\sin(2\theta(\bar{x}) - 2\bar{x})\sin\bar{x}.$$

Since the right hand side of this equation is zero if and only if

$$\theta(\bar{x}) = \bar{x} + \frac{n\pi}{2} - \frac{1}{2}\operatorname{arccot}\Big(\frac{\kappa}{2}\tan\bar{x} + \cot(2\bar{x}) + 1\Big) = \circledast_n, \quad n \in \mathbb{Z}$$

and positive for example for $\theta(\bar{x}) = \bar{x} + d_\kappa(\bar{x}) < \circledast_1$, $\theta''(\bar{x})$ can only be negative if $\circledast_1 < \theta(\bar{x}) < \circledast_2$. If $p \geq \frac{\pi}{2}$ the case of $x_m < x < \frac{\pi}{2}$ is irrelevant. Otherwise,

$$\theta\left(\frac{\pi}{2}\right) < \pi = \lim_{x\to\frac{\pi}{2}} x + \frac{\pi}{2} - \frac{1}{2}\operatorname{arccot}\Big(\frac{\kappa}{2}\tan x + \cot(2x) + 1\Big)$$

while $\theta(\bar{x}) > \bar{x} + \frac{\pi}{2} - \frac{1}{2}\operatorname{arccot}\left(\frac{\kappa}{2}\tan\bar{x} + \cot(2\bar{x}) + 1\right)$ so the intermediate value theorem implies existence of some $a \in (\bar{x}, \frac{\pi}{2})$ such that

$$\theta(a) = a + \frac{\pi}{2} - \frac{1}{2}\operatorname{arccot}(\frac{\kappa}{2}\tan a + \cot(2a) + 1) =: a + \frac{\pi}{2} - \frac{1}{2}\delta_\kappa(a).$$

Since $\delta_\kappa > d_\kappa$ we have $\theta(a) < \eta(a)$ and without loss of generality one may assume that $\theta(x) < \eta(x)$ on $(a, \frac{\pi}{2})$ by choosing $a = \max\{a \in (\bar{x}, \frac{\pi}{2}) | \theta(a) = a + \frac{\pi}{2} - \frac{1}{2}\delta_\kappa(a)\}$.

Using monotonicity of θ on $(a, \pi) \subset (x_m, \pi)$ we will now estimate the anisotropy part of the energy on $(a, \theta(a))$. Note that $\sin(\theta - x)$ is increasing in θ as long as it holds that

$x \le \theta(a) \le \theta(x) \le x + \frac{\pi}{2}$. By choice of a, this is true for $x \in (a, \theta(a))$. Therefore,

$$\begin{aligned}\int_a^\pi \sin^2(\theta - x)\sin x \, \mathrm{d}x &\ge \int_a^{\theta(a)} \sin^2(\theta(a) - x)\sin x \, \mathrm{d}x \\ &= \frac{1}{12}\left[\cos(2\theta(a) - 3x) - 3\left(\cos(2\theta(a) - x) + 2\cos x\right)\right]_a^{\theta(a)} \\ &= \frac{1}{12}\left(-8\cos(\theta(a)) - \cos(2\theta(a) - 3a) + 3\left(\cos(2\theta(a) - a) + 2\cos(a)\right)\right).\end{aligned}$$

Viewing this as a function of a, the expression has exactly one local maximum in $(0, \frac{\pi}{2})$, no minima, and the following values at the boundary:

$$\frac{1}{12}\left(-8\cos\left((\frac{\pi}{2}\right) + \cos(\pi - 0) - 3\left(\cos(\pi - 0) + 2\cos(0)\right)\right) = \frac{1}{3}$$

for $a = 0$, $\theta(a) = 0 + \frac{\pi}{2} - 0$ and

$$\frac{1}{12}\left(-8\cos(\pi) + \cos\left(2\pi - \frac{3\pi}{2}\right) - 3\left(\cos\left(2\pi - \frac{\pi}{2}\right) + 2\cos\left(\frac{\pi}{2}\right)\right)\right) = \frac{2}{3}$$

for $a = \frac{\pi}{2}$, $\theta(a) = \pi$. By a very rough estimate, neglecting $0 < x_m < a$, the energy of θ would therefore be larger than $\frac{\kappa}{3}$ if $\theta - \mathrm{id}$ had a local maximum at $\bar{x}$.

On the other hand, it is shown in [37] that the full energy $\frac{1}{2}\mathcal{E} = \pi E$ is bounded from above by 8π. The statement is repeated in Lemma 3.9. For $\kappa > 24$, the resulting bound for E is exceeded by $\frac{\kappa}{3}$ and $\theta - \mathrm{id}$ can therefore not have a local maximum in $(x_{\min}, \frac{\pi}{2})$.

Since θ' is negative and therefore smaller than 1 on $(p, x_{\min})$ this concludes the proof of Lemma 3.6. □

3.2.3. Behavior near π

The profile θ is well-defined and differentiable on $(0, \pi)$ as the coordinate function of the differentiable field $\mathbf{m}_\theta$. Also, with $\mathbf{m}_\theta(\pm\hat{\mathbf{e}}_3) = -\hat{\mathbf{e}}_3$, the limits $\lim_{x \searrow 0} \theta(x) = \pi$ and $\lim_{x \nearrow \pi} \theta(x) = \pi$ exist. However, differentiability up to the boundary is not clear. Indeed, the profile is expected to drop quickly near p and behave moderately near π where we have $\mathrm{id} < \theta < \pi$. We will further investigate this behavior .

Lemma 3.7. *Let θ be a minimizing profile and assume $x < \theta(x) < \pi$ on $(\pi - \varepsilon, \pi)$. Then the following statements are true:*

(1) $\theta'(\pi) := \lim_{x \to \pi} \theta'(x) \in (0, 1)$ *exists.*

(2) $\lim_{x \nearrow \pi} \sin x \ \theta''(x) = 0.$

Proof. 1. First of all, recall that in spherical coordinates Ψ on $\mathbb{S}^2 \setminus \{x \geq 0,\ y = 0\}$, $\mathbf{m}$ is differentiable if and only if $(0,\pi) \times (0, 2\pi) \ni (x, \varphi) \mapsto \mathbf{m} \circ \Psi \in \mathbb{R}^3$ is differentiable. In particular, $|\theta'(x)| = |\frac{d}{dx}\mathbf{m}(\psi(x,\varphi)|$ is smooth on $(0,\pi)$ since $\mathbf{m}$ is smooth. Moreover, since $\mathbf{m}_\theta$ is differentiable at $-\hat{\mathbf{e}}_3$, the limit

$$L := |\nabla_{\mathbb{S}^2}\mathbf{m}(-\hat{\mathbf{e}}_3)|^2 = \lim_{\mathbf{x}\to-\hat{\mathbf{e}}_3} |\nabla_{\mathbb{S}^2}\mathbf{m}(\mathbf{x})|^2 = \lim_{x\nearrow\pi} \left((\theta')^2(x) + \frac{\sin^2\theta}{\sin^2 x}\right)$$

exists. Being interested in only one of the terms, we need to show that both terms converge individually. If this is not the case then, due to the bounds $0 < \theta' < 1$ and $0 < \frac{\sin\theta}{\sin x} < \frac{\sin x}{\sin x} = 1$ on $(\pi - \varepsilon, \pi)$, they both oscillate, the oscillations annihilating each other. Let

$$I := \liminf_{x\nearrow\pi}(\theta'(x))^2 = \lim_{k\to\infty}(\theta'(y_k))^2 \text{ and } S := \limsup_{x\nearrow\pi}(\theta'(x))^2 = \lim_{k\to\infty}\left(\theta'(\bar{y}_k)\right)^2$$

for two sequences $(y_k)_{k\in\mathbb{N}}$ and $(\bar{y}_k)_{k\in\mathbb{N}}$. Then

$$L - S = \lim_{k\to\infty}\frac{\sin^2\theta(\bar{y}_k)}{\sin^2\bar{y}_k} = \liminf_{x\nearrow\pi}\frac{\sin^2\theta}{\sin^2 x} \text{ and } L - I = \lim_{k\to\infty}\frac{\sin^2\theta(y_k)}{\sin^2 y_k} = \limsup_{k\nearrow\pi}\frac{\sin^2\theta}{\sin^2 x}.$$

Indeed, it holds that $L = I + S$ which is shown by employing the mean value theorem for each y_k:

$$\frac{\sin^2\theta(y_k)}{\sin^2(y_k)} = \left(\frac{\sin\theta(y_k) - \sin\theta(\pi)}{\sin y_k - \sin\pi}\right)^2 = \left(\frac{\theta'(\xi_k)\cos\theta(\xi_k)}{\cos\xi_k}\right)^2 \text{ where } \xi_k \in (y_k, \pi).$$

As $k \to \infty$, equality implies that both sides converge and we conclude

$$S = \limsup_{x\to\pi}(\theta'(x))^2 \geq \lim_{k\to\infty}(\theta'(\xi_k))^2\frac{\cos^2\theta(\xi_k)}{\cos^2(\xi_k)} = \lim_{k\to\infty}\frac{\sin^2\theta(y_k)}{\sin^2(y_k)} = L - I.$$

Repeating the argument for the sequence $\bar{y}_k$ results in the inequality $I \leq L - S$. Combining both inequalities, we find $I + S \leq L \leq I + S$ and therefore $L = I + S$.

Having established the equality, consider the relation between θ' and $\frac{\sin(2\theta)}{\sin(2x)}$. In the first case, we have $\theta'(x) \leq \frac{\sin\theta}{\sin x}\frac{\cos\theta}{\cos x}$ for all $x \in (\pi - \varepsilon, \pi)$. Then

$$S = \lim_{k\to\infty}(\theta')^2(\bar{y}_k) \leq \lim_{k\to\infty}\left(\frac{\sin^2\theta(\bar{y}_k)}{\sin^2(\bar{y}_k)}\frac{\cos^2\theta(\bar{y}_k)}{\cos^2(\bar{y}_k)}\right) = \lim_{k\to\infty}\frac{\sin^2\theta(\bar{y}_k)}{\sin^2\bar{y}_k} = I$$

such that $\limsup_{x\nearrow\pi}(\theta'(x))^2 = S \leq I = \liminf_{x\nearrow\pi}\theta'(x)^2$ implies the existence of the limit.

In the second case there exists $x_0 \in (\pi - \varepsilon, \pi)$ such that $\theta'(x_0) > \frac{\sin(2\theta(x_0))}{\sin(2x_0)}$. Plugging this into the equation results in

$$\begin{aligned}\theta''(x_0)\sin x_0 &= -\cos(x_0)\theta'(x_0) + \frac{\sin(2\theta(x_0))}{2\sin(x_0)}\frac{\kappa}{2}\sin(2\theta(x_0) - 2x_0)\sin x_0 \\ &> -\cos x_0\frac{\sin(2\theta(x_0))}{2\sin x_0\cos x_0} + \frac{\sin(2\theta(x_0))}{2\sin x_0} + \frac{\kappa}{2}\sin(2\theta - 2x_0)\sin x_0 > 0.\end{aligned}$$

On the other hand,

$$\left(\frac{\sin(2\theta)}{\sin(2x)}\right)'\Big|_{x_0} = \frac{2\theta'\cos(2\theta)\sin(2x) - 2\sin(2\theta)\cos(2x)}{(\sin(2x)^2}\Big|_{x_0}$$
$$< 2\frac{\cos(2\theta(x_0))\sin(2\theta(x_0) - \sin(2\theta(x_0)))\cos(2\theta(x_0))}{\sin(2x_0)^2} = 0$$

due to $\cos(2x) < \cos(2\theta)$ for $x < \theta$ and $x, \theta \in (\frac{\pi}{2}, \pi)$ and also due to the inequality $\sin(2x_0)\theta'(x_0) < \sin(2\theta(x_0))$. On $(\frac{\pi}{2}, \pi)$, this inequality is reversed due to $\cos(x) < 0$. Thus, the original inequality is self-reinforcing and holds on (x_0, π), implying $\theta'' > 0$ on (x_0, π) such that θ' is monotonically increasing. Hence, $\theta'(\pi)$ exists.

For the upper bound, note that $\lim_{x\nearrow\pi}\theta'(x) \leq 1$ follows from Lemma 3.6. If this held with equality, the functions $f = \mathrm{id}$ and $g = \theta$ would fulfill the assumptions of Lemma 3.8 below with $\mathrm{id} \leq \theta$, $\mathrm{id}(\pi) = \theta(\pi)$ and $\mathrm{id}'(\pi) = 1 = \theta'(\pi)$. But then $\mathrm{id} \equiv \theta$ on $(0, \pi)$ which contradicts $Q(\mathbf{m}) = 0$.

2. After establishing the existence of θ', the second statement is easy to prove. In fact, the differential equation implies that

$$\lim_{x\to\pi}\theta''\sin x = \lim_{x\to\pi}\left(-\theta'\cos x + \frac{\sin(2\theta)}{2\sin x} + \frac{\kappa}{2}\sin(2\theta - 2x)\sin x\right)$$
$$= \lim_{x\to\pi}\left(-\theta'\cos x + \frac{\theta'(x)\cos(2\theta)}{\cos x}\right) = 0.$$

□

The fact that $\theta'(\pi) < 1$ will be important later on in Chapter 4. It relies on a partial uniqueness result for the ordinary differential equation at π which we prove via a maximum principle, following the strategy in [22].

Lemma 3.8. *Let f, $g \in C((\pi - \varepsilon, \pi])$ be solutions of (3.1) and assume that there exists $\varepsilon > 0$ such that $g \leq f$ on $(\pi - \varepsilon, \pi)$. If the boundary values satisfy $f(\pi) = g(\pi)$ and $\lim_{x\to\pi} f'(x) = \lim_{x\to\pi} g'(x) = a \in \mathbb{R}$, then $f \equiv g$ on $(\pi - \varepsilon, \pi]$.*

Proof. Assume that θ is a solution of (3.1). Then, expressing the stereographic projection $\pi_- : \mathbb{S}^2 \setminus \{-\hat{\mathbf{e}}_3\} \to \mathbb{R}^2$ in polar coordinates (r, φ) with $r(x) = \frac{\sin x}{1+\cos x}$, the profile $\theta_p(r(x)) = \theta(x) - r(x)$ is a solution of the equation

$$\theta_p' + r\theta_p'' = \sin(\theta_p)\left(\cos\theta_p\frac{r^4 - 6r^2 + 1}{r(r^2+1)^2} + \kappa\cos\theta_p\frac{4r}{(r^2+1)^2} - 4\sin\theta_p\frac{1-r^2}{(1+r^2)^2}\right)$$

with boundary values $\theta_p(0) = \pi$, $\lim_{r\to\infty}\theta_p(r) = \lim_{x\to\pi}\theta(x) - x = 0$.

Performing this transformation and then setting $\bar{f} = f_p(\frac{1}{t})$ and $\bar{g} = g_p(\frac{1}{t})$, $\bar{f}$ and $\bar{g}$ are each solutions of the ODE

$$\frac{1}{t}\varphi' + \varphi'' = \frac{\sin(2\varphi)}{2}\left(\frac{t^4 - 6t^2 + 1}{t^2(1+t^2)^2} + \kappa\frac{4}{(t^2+1)^2}\right) + 4\sin^2\varphi\left(\frac{t(1-t^2)}{(t^2+1)^2}\right).$$

Moreover, they satisfy $\bar{f}(0) = \bar{g}(0)$ and $\lim_{t\to 0}\frac{\bar{f}}{t} = a = \lim_{t\to 0}\frac{\bar{g}}{t}$. Setting $v = \bar{f} - \bar{g} \geq 0$ thus gives a solution to the equation

$$\begin{aligned} v'' + \frac{1}{t}v' - \frac{1}{t^2}\frac{1}{(1+t^2)^2}\frac{\sin(2\bar{f}) - \sin(2\bar{g})}{2} \\ = \frac{\sin(2\bar{f}) - \sin(2\bar{g})}{2}\left(\frac{t^2 - 6 + 4\kappa}{(1+t^2)^2}\right) + 2\left(\sin^2(\bar{f}) - \sin^2(\bar{g})\right)\frac{t(1-t^2)}{(1+t^2)^2} \end{aligned}$$

where the right-hand side can be estimated by $c(t)(f-g)$ for a non-negative and continuous function $c\colon [0,\tilde{\varepsilon}) \to \mathbb{R}$:

$$\sin(2\bar{f}) - \sin(2\bar{g}) = 2(\bar{f} - \bar{g})\cos(\xi_1) \leq 2(\bar{f} - \bar{g})$$

and

$$\sin^2(\bar{f}) - \sin^2(\bar{g}) = (\bar{f} - \bar{g})\sin(2\xi_2) \leq (\bar{f} - \bar{g}).$$

Employing the first inequality on the left hand side together with $\frac{1}{(1+t^2)^2} \leq 1$, we can conclude that v solves the following differential inequality:

$$v'' + \frac{1}{t}v' - \frac{1}{t^2}v - c(t)v \leq 0.$$

We are now in a situation to employ the method in [22] with $p = q = 1$: Set $w = \frac{v}{t}$. Then $w(0) = \lim_{t\to 0}\frac{\bar{f}-\bar{g}}{t} = 0$ and $w \geq 0$ on $[0,\varepsilon)$ for some $\varepsilon > 0$. Furthermore, w satisfies

$$w'' + \frac{3}{t}w' - c(t)w = \frac{1}{t}\left(v'' + \frac{1}{t}v' - \frac{1}{t^2}v - c(t)v\right) \leq 0 \text{ in } (0,\tilde{\varepsilon}).$$

By the extended maximum principle in the appendix of [22], w is identically 0 and therefore $f \equiv g$. □

3.2.4. The Limit for Large κ

In this section, we will investigate how some features of the profile depend on κ in the limit $\kappa \to \infty$. To do so, we will assume that the first statement of Proposition 3.1 has been fully proven. In particular, we will assume $p = 0$ [$p = \pi$] for lower [upper] minimizzing profiles – a statement that will be proven in the following chapter. However, since the behaviour for $\kappa \to \infty$ is not relevant for the analysis of $\mathcal{H}$ when κ is fixed, we are in no danger of circular reasoning.

The qualitative analysis of θ is based on an upper bound from [37]:

Lemma 3.9 ((Energy Bound, [37]). *For every $\kappa > 0$ there exists a co-rotational field $\mathbf{m} \in H^1(\mathbb{S}^2;\mathbb{S}^2)$ with $Q(\mathbf{m}) = 0$ and $E(\mathbf{m}) < 8\pi$.*

Note that by co-rotational the authors refer to fields of the type

$$\mathbf{m} = (\sin\theta\sin\varphi,\ \sin\theta\cos\varphi,\ \cos\theta)^T$$

which are called axisymmetric in this text. Furthermore, note that the energy in [37] is given by

$$E(\mathbf{m}) = \frac{1}{2}\int_{\mathbb{S}^2} |\nabla\mathbf{m}|^2 + \kappa(1-(\mathbf{m}\cdot\nu)^2)\,\mathrm{d}\sigma = \frac{1}{2}\mathcal{E}(\mathbf{m}) = \pi E(\theta).$$

As a direct consequence of the upper bound combined with the assumption $p = 0$ [$p = \pi$] we have

Corollary 3.3. *Given $\kappa > 0$, let $\theta^{(\kappa)}$ be a family of lower [upper] minimizing profiles. Then $\theta^\kappa(x) \to x$ as $\kappa \to \infty$ for all $x \in (0,\pi]$ [$\theta(x) \to \pi + x$ as $\kappa \to \infty$ for all $x \in [0,\pi)$].*

Proof. Pointwise convergence almost everywhere follows from the $L^1((0,\pi))$-convergence of $\sin^2(\theta^{(\kappa)} - x)\sin x$ and the fact that $\sin(y) \to 0$ and $0 < y < \pi$ implies $y \to 0$ or $y \to \pi$. However, $\theta - x$ has to converge to the same value everywhere on $(0,\pi)$ because $\theta \neq \pi$ for all x [$\theta \geq \pi$]. Smoothness of critical points and the corresponding continuity of θ^κ for each κ imply that the pointwise convergence holds on $(0,\pi)$ and convergence at π [at 0] is implied by the fixed boundary value. In contrast, there is no convergence at the north [south] pole where the Skyrmion forms. □

By establishing lower bounds for the anisotropy term we can improve the statement and give explicit bounds for $|\theta^{(\kappa)} - \mathrm{id}\,|$ on intervals bounded away from 0 [π].

Lemma 3.10. *Let θ^κ be a family of lower minimizing profiles.*

(1) For $\kappa > 94$, it holds that

$$\sup_{[x_m^\kappa,\pi]} |\theta^{(\kappa)}(x) - x| = \theta_m^{(\kappa)} - x_m^{(\kappa)} \leq \sqrt[4]{\frac{216}{\kappa}}.$$

(2) As $\kappa \to \infty$, $x_m^\kappa \to 0$.

For families of upper minimizing profiles, the same statement holds with $\theta^{\mathbf{m}} - (x^m + \pi)$ and $\pi - x^m$.

On the level of axisymmetric minimizers $\mathbf{m_0}^\kappa : \mathbb{S}^2 \to \mathbb{S}^2$, the following is an easy conclusion:

Corollary 3.4. *If $\mathbf{m}^{(\kappa)}$ is a family of axisymmetric minimizers of the same type then $\mathbf{m}^{\kappa} \to \nu$ uniformly for $\kappa \to \infty$ on every compact subset of $\mathbb{S}^2 \setminus \{\hat{\mathbf{e}}_3\}$.*

Proof. For the proof of the refined convergence analysis, we introduce the following notation: Let θ be an upper [lower] minimizing profile and $a \in [0, \pi]$ $[a \in [\pi, 2\pi]]$. Then set

$$y_a := (\theta - \mathrm{id})^{-1}(a) \left[y_a := (\mathrm{id} + \pi - \theta)^{-1}(a) \right].$$

Note that y_a is well defined due to the strict monotonicity of $\theta - \mathrm{id}$ that follows from Proposition 3.1(3).

(1) As a preliminary step, assume that $a \in [x_m, \pi)$ satisfies $\theta(a) - a \leq \frac{\pi}{2}$. Then θ is motonically increasing on $(a, \theta(a))$ by Lemma 3.4. On the other hand, $\theta - x$ is decreasing on $(0, \pi)$ such that, in total,

$$\theta(a) - x < \theta(x) - x < \theta(a) - a < \frac{\pi}{2}$$

for all $x \in (a, \theta(a))$. Therefore, $\sin(\theta - x) > \sin(\theta(a) - a)$ on $(a, \theta(a))$ and we may estimate

$$\begin{aligned} \int_a^{\theta(a)} \sin^2(\theta - x) \sin x \, \mathrm{d}x &> \int_a^{\theta(a)} \sin^2(\theta(a) - x) \sin x \, \mathrm{d}x \\ &= \frac{1}{12} \left[\cos(2\theta(a) - 3x) - 3\cos(2\theta(a) - x) - 6\cos(x) \right]_a^{\theta(a)} \\ &= \circledast \end{aligned}$$

Collecting terms of the form $\cos(\theta - x)$, this expression can be further simplified.

$$\begin{aligned} 12\circledast &= -8\cos(\theta(a)) - \cos(2\theta(a) - a) + 3\cos(2\theta(a) - a) + 6\cos a \\ &= \cos a \left(-8\cos(\theta(a) - a) + 2\cos(2\theta(a) - 2a) + 6 \right) \\ &\quad + \sin a \left(8\sin(\theta(a) - a) - 4\sin(2\theta(a) - 2a) \right) \\ &= 4\Big((1 - \cos(\theta(a) - a))^2 \cos a + 2\sin a \sin(\theta(a) - a) \left(1 - \cos(\theta(a) - a)\right) \Big). \end{aligned}$$

If $\theta_m - x_m > \frac{\pi}{2}$, then $a := y_{\frac{\pi}{2}} > x_m$ and $\theta(y_{\frac{\pi}{2}}) - y_{\frac{\pi}{2}} = \frac{\pi}{2}$. Therefore, by above computations and taking into account the energy bound of Lemma 3.9 as well as the special choice of a,

$$\begin{aligned} \frac{8}{\kappa} &\geq \int_{y_{\frac{\pi}{2}}}^{\theta(y_{\frac{\pi}{2}})} \sin^2(\theta - x) \sin x \, \mathrm{d}x \\ &\geq \frac{4}{12} \Big((1 - 0)^2 \cos y_{\frac{\pi}{2}} + 2\sin y_{\frac{\pi}{2}} \cdot 1 \, (1 - 0) \Big) \\ &> \frac{1}{3} \end{aligned}$$

for $y_{\frac{\pi}{2}} \in (0,\pi)$. For $\kappa \geq 24$, this is impossible and it follows that $\theta_m - x_m < \frac{\pi}{2}$ for $\kappa \geq 24$. Thus, x_m satisfies the requirements for a above and

$$\frac{8}{\kappa} \geq \frac{1}{3}\Big((1-\cos(\theta_m - x_m))^2 \cos x_m + 2\sin x_m \sin(\theta_m - x_m)\,(1-\cos(\theta_m - x_m))\Big).$$

If $\theta_m - x_m < \frac{\pi}{4}$, then $1-\cos(\theta_m - x_m) < \sin(\theta_m - x_m)$ and therefore

$$\frac{8}{\kappa} \geq \frac{1}{3}\,(1-\cos(\theta_m - x_m))^2\,(\cos x_m + 2\sin x_m) > \frac{1}{3}\,(1-\cos(\theta_m - x_m))^2\,.$$

Employing the inequality $1-\cos(x) \geq \frac{x^2}{3}$ on $(0,\frac{\pi}{2})$, this yields the upper bound of the Lemma.

On the other hand, if $\theta_m - x_m \geq \frac{\pi}{4}$, then taking $a := y_{\frac{\pi}{4}}$ with $\theta(y_{\frac{\pi}{4}}) - y_{\frac{\pi}{4}} = \frac{\pi}{4}$, we can conclude

$$\begin{aligned}
\frac{8}{\kappa} &\geq \int_{y_{\frac{\pi}{4}}}^{\theta(y_{\frac{\pi}{4}})} \sin^2(\theta - x)\sin x \,\mathrm{d}x \\
&\geq \frac{1}{3}\left(1-\frac{\sqrt{2}}{2}\right)\cos y_{\frac{\pi}{4}} + 2\sin y_{\frac{\pi}{4}}\frac{\sqrt{2}}{2}\left(1-\frac{\sqrt{2}}{2}\right) \\
&\geq \frac{3}{2} - \sqrt{2}
\end{aligned}$$

for $y_{\frac{\pi}{4}} \in (0,\frac{3\pi}{4})$. For $\kappa > 94$, this is false. Note that $\pi > \theta(y_{\frac{\pi}{4}}) = y_{\frac{\pi}{4}} + \frac{\pi}{4}$ implies that this is true independently of κ. Indeed, as $\kappa \to \infty$, there might be a more accurate bound for $y_{\frac{\pi}{4}}$, which would improve the constant and therefore reduce κ_0. However, as we are interested in the behavior for large κ, this does not affect the result very much.

(2) The convergence of x_m directly follows from the pointwise convergence of $\theta^{(\kappa)}$ on $(0,\pi]$. Given $\varepsilon > 0$, we show that $x_m < \varepsilon$ for κ sufficiently large. Indeed, choose $x < \frac{\varepsilon}{2}$. Then there exists $\kappa_0 > 0$ such that $\theta^{(\kappa)}(x) - x < \frac{\varepsilon}{2}$ for all $\kappa > \kappa_0$. On the other hand,

$$\theta^{(\kappa)}(\varepsilon) > \varepsilon > x + \frac{\varepsilon}{2} > \theta^{(\kappa)}(x).$$

Hence $\theta^{(\kappa)}(x) < \theta(\varepsilon)$ and $x < \varepsilon$. Due to the fact that $\theta^{(\kappa)}$ has exactly one minimum, there exists $a \in [x,\varepsilon)$ such that θ is increasing on $[a,\varepsilon]$ and therefore $x_m^{(\kappa)} < \varepsilon$.

□

4. Local Minimality of Symmetric Minimizers

In this chapter, we will show that minimizers in the class of axisymmetric fields are actually locally minimizing within the full topological sector. In particular, we prove the following Theorem:

Theorem 3. *Given $\kappa > 24$, let $\mathbf{m}_0 = \mathbf{m}_\theta$ be minimizing among all axisymmetric fields of degree 0. Then the Hessian of $\mathcal{E}$ at $\mathbf{m}_0$ is positive semidefinite. Furthermore, if the reduced energy is strictly convex at θ in the sense that $\frac{d^2}{dt^2}E(\theta+t\beta) > 0$ for all variations $\beta \in C_0^\infty((0,\pi)) \setminus \{0\}|_{t=0}$, then $\mathbf{m}_0$ is a local minimizer among all fields of degree 0 and there exist ε_0, $c > 0$ such that*

$$\mathcal{E}(\mathbf{m}) - \mathcal{E}(\mathbf{m}_0) \geq c \inf_{R \in SO(3)/SO(3)_{\hat{\mathbf{e}}_3}} \|\mathbf{m}_R - \mathbf{m}_0\|_{H^1}$$

for all $\|\mathbf{m} - \mathbf{m}_0\|_{H^1} < \varepsilon_0$.

Note: Here, $\|\mathbf{m}-\mathbf{m}_0\|_{H^1} = \|\nabla\mathbf{m}-\nabla\mathbf{m}_0\|_{L^2(\mathbb{S}^2;\mathbb{R}^3)}+\|\mathbf{m}-\mathbf{m}_0\|_{L^2(\mathbb{S}^2;\mathbb{R}^3)}$.

4.1. Non-Negativity of the Hessian

As discussed in the introduction, variations on the sphere imply that the Hessian is given by

$$\mathcal{H}(\mathbf{m})\langle\phi,\psi\rangle = \int_{\mathbb{S}^2} \nabla\phi\cdot\nabla\psi - \kappa(\psi\cdot\nu)(\phi\cdot\nu)\,\mathrm{d}\sigma - \int_{\mathbb{S}^2} (\phi\cdot\psi)\left(|\nabla\mathbf{m}|^2 - \kappa(\mathbf{m}\cdot\nu)^2\right)\,\mathrm{d}\sigma.$$

where $\phi, \psi \in H^1(\mathbb{S}^2, T_{\mathbf{m}}\mathbb{S}^2)$.

Since $\mathcal{H}$ is a quadratic form, it suffices to prove nonnegativity for $\mathcal{H}(\phi) := \delta^2\mathcal{E}(\mathbf{m})\langle\phi,\phi\rangle$. For axisymmetric minimizers, the kernel consists of tangent fields associated to joint rotations or changes in the profile θ:

Proposition 4.1. *Let $\kappa > 24$ and assume that $\mathbf{m}_0 \colon \mathbb{S}^2 \to \mathbb{S}^2$ minimises $\mathcal{E}$ among all axisymmetric fields of degree 0. Then $\mathcal{H}(\phi) \geq 0$ for all smooth tangent fields $\phi \in T_{\mathbf{m}_0}\mathbb{S}^2$. Furthermore, the kernel of $\mathcal{H}$ consists exactly of those fields*

$$\phi = \frac{d}{dt}\mathbf{m}_{0R_v(t)}\Big|_{t=0}$$

associated to a joint rotation around $v \in \mathbb{R}^3 \setminus \{0\}$ as well as those of the form

$$\phi(x,\varphi) = \alpha(x)\begin{pmatrix}\cos\theta(x)\cos\varphi\\ \cos\theta(x)\sin\varphi\\ -\sin\theta(x)\end{pmatrix} \quad \text{with} \quad \alpha \in C_0^\infty((0,\pi)), \quad \frac{d^2}{dt^2}E(\theta+t\alpha)\Big|_{t=0} = 0,$$

which result from non-convexity of the reduced functional.

When proving Proposition 4.1, we may assume without loss of generality that $\mathbf{m}$ is a type I Skyrmion, see figure 3.1, resulting in a lower minimizing profile θ that satisfies $\theta(0) = \theta(\pi) = \pi$. Otherwise, there is a transformation of $\mathbf{m}_0$ to a type one axisymmetric Skyrmion that carries over to the tangent fields. For example, consider the transformation $\bar{\mathbf{m}} \colon \mathbf{x} \mapsto -\mathbf{m}(-\mathbf{x})$ and note that $\phi(x) \in T_{\bar{\mathbf{m}}(x)}\mathbb{S}^2$ iff $\bar{\phi}(x) = -\phi(-x) \in T_{\mathbf{m}(x)}\mathbb{S}^2$. Furthermore,

$$\delta^2\mathcal{E}(\mathbf{m})\langle\phi,\phi\rangle = \delta^2\mathcal{E}(\bar{\mathbf{m}})\langle\bar{\phi},\bar{\phi}\rangle$$

and ϕ is associated to a joint rotation of $\mathbf{m}$ around v iff $\bar{\phi}$ is associated to a joint rotation of $\bar{\mathbf{m}}$ around v. Finally,

$$\phi(x,\varphi) = \alpha(x)\begin{pmatrix}\cos\theta\cos\varphi\\ \cos\theta\sin\varphi\\ -\sin\theta\end{pmatrix} \text{ iff } \bar{\phi}(x,\varphi) = \alpha(\pi - x)\begin{pmatrix}\cos\bar{\theta}\cos\varphi\\ \cos\bar{\theta}\sin\varphi\\ -\sin\bar{\theta}\end{pmatrix}$$

and $E(\theta + t\alpha) = E(\bar{\theta} - t\alpha(\pi - \cdot))$.

Due to this assumption, it is sufficient to consider tangent fields that are compactly supported on $\mathbb{S}^2 \setminus \{\hat{\mathbf{e}}_3\}$ because $\mathbf{m} \sim \nu$ near the south pole. This implies that a decomposition of $\mathbf{m}$ on $\mathbb{S}^2 \setminus \{\pm\hat{\mathbf{e}}_3\}$ via a moving frame can be naturally extended at $-\hat{\mathbf{e}}_3$ due to convergence results discussed in the appendix A.1.

The proof will be structured according to the strategy in [32], starting out with smooth tangent fields that have compact support on $\mathbb{S}^2 \setminus \{\hat{\mathbf{e}}_3\}$.

4.1.1. Non-Negativity for $C_c^\infty(\mathbb{S}^2 \setminus \{\hat{\mathbf{e}}_3\}; T_{\mathbf{m}_0}\mathbb{S}^2)$

Away from the poles, tangent fields can be expressed in a moving frame consisting of

$$\mathbf{m}_0 = \begin{pmatrix}\sin\theta\cos\varphi\\ \sin\theta\sin\varphi\\ \cos\theta\end{pmatrix} \quad X = \begin{pmatrix}\sin\varphi\\ -\cos\varphi\\ 0\end{pmatrix} \quad Y = \begin{pmatrix}\cos\theta\cos\varphi\\ \cos\theta\sin\varphi\\ -\sin\theta\end{pmatrix}.$$

It holds that $|\mathbf{m}_0| = |X| = |Y| = 1$ and $\mathbf{m}_0 \cdot X = \mathbf{m}_0 \cdot Y = X \cdot Y = 0$. Hence, X and Y span $T_{\mathbf{m}_0}\mathbb{S}^2$ on $\mathbb{S}^2 \setminus \{\pm\hat{\mathbf{e}}_3\}$ and $\phi \in C_c^\infty(\mathbb{S}^2 \setminus \{\pm\hat{\mathbf{e}}_3\}; T_{\mathbf{m}_0}\mathbb{S}^2)$ can be expressed as $\phi = u_1 X + u_2 Y$ where $u_i \in C_c^\infty(\mathbb{S}^2 \setminus \{\hat{\mathbf{e}}_3\})$. Note that $u_1 = X \cdot \phi$ and $u_2 = Y \cdot \phi$ imply a special structure of u_i that guarantees continuity at $\pm\hat{\mathbf{e}}_3$ where X, Y are degenerated. See A.1 for more details. In particular, this justifies the choice of ϕ with $\phi(-\hat{\mathbf{e}}_3) \neq 0$.

Since $\{\pm\hat{\mathbf{e}}_3\}$ is a zero set and irrelevant for integration, we can rewrite the Hessian in spherical coordinates, employing $\phi = u_1 X + u_2 Y$.

$$
\begin{aligned}
\mathcal{H}(\phi) &= \int\limits_{\mathbb{S}^2} \left(|\nabla \phi|^2 - \kappa (\phi \cdot \nu)^2 - |\phi|^2 \left(|\nabla m_0|^2 - \kappa (m_0 \cdot \nu)^2 \right) \mathrm{d}\sigma \right. \\
&= \int\limits_0^{2\pi} \int\limits_0^{\pi} \left(\left| \frac{\partial u}{\partial x} \right|^2 + \frac{1}{\sin^2 x} \left| \frac{\partial u}{\partial \varphi} \right|^2 + \frac{2 \cos\theta}{\sin^2 x} \left(u \times \frac{\partial u}{\partial \varphi} \right) \right. \\
&\qquad \left. + u_1^2 f_1(\theta, x) + u_2^2 f_2(\theta, x) \right) \sin x \, \mathrm{d}x \, \mathrm{d}\varphi
\end{aligned}
$$

where

$$
\begin{aligned}
f_1(\theta, x) &= \left(-(\theta')^2 + \frac{\cos^2\theta}{\sin^2 x} + \kappa \cos^2(\theta - x) \right) \\
f_2(\theta, x) &= \left(\frac{\cos(2\theta)}{\sin^2 x} + \kappa \cos(2\theta - 2x) \right).
\end{aligned}
$$

Next, dependence of u_i on the variables x, φ is separated by expanding u_i as a Fourier series in φ with coefficients a_k^i, $b_k^i \in C_c^\infty((0, \pi])$:

$$
u_i(x, \varphi) = \frac{1}{2} a_0^i(x) + \sum_{k=1}^{\infty} \left(a_k^i(x) \cos(k\varphi) + b_k^i(x) \sin(k\varphi) \right), \quad i = 1, 2,
$$

Some computations give

$$
\mathcal{H}(\mathbf{m}_0)\langle \phi, \phi \rangle = 2\pi \mathcal{H}_0(a_0^1, a_0^2) + \pi \sum_{k=1}^{\infty} \left(\mathcal{H}_k(a_k^1, b_k^2) + \mathcal{H}_k(b_k^1, -a_k^2) \right),
$$

where

$$
\begin{aligned}
\mathcal{H}_k(\alpha, \beta) = \int\limits_0^{\pi} &\left(\left(\alpha'\right)^2 + \left(\beta'\right)^2 + \frac{k^2}{\sin^2 x} \left(\alpha^2 + \beta^2 \right) \right. \\
&\left. + \frac{4k \cos\theta}{\sin^2 x} \alpha\beta + \alpha^2 f_1(\theta, x) + \beta^2 f_2(\theta, x) \right) \sin x \, \mathrm{d}x
\end{aligned}
$$

for $\alpha, \beta \in C_c^\infty(0, \pi]$ and $k \in \mathbb{N}_0$.

Conveniently, the Fourier modes $\mathcal{H}_k$ display the following monotonicity:

Lemma 4.1. *For fixed α, β and $k \geq 1$, $\mathcal{H}_k(\alpha,\beta)$ is strictly increasing in k unless $\alpha = \beta = 0$ in which case $\mathcal{H}_k(\alpha,\beta) = 0$ independently of k.*

Proof. Assume $\alpha, \beta \neq 0$.

$$\begin{aligned}
&\mathcal{H}_{k+1}(\alpha,\beta) - \mathcal{H}_k(\alpha,\beta)\\
&= \int_0^\pi \left(\frac{(k+1)^2}{\sin x} - \frac{k^2}{\sin x}\right)\left(\alpha^2+\beta^2\right) + \left(\frac{4(k+1)\cos\theta}{\sin x} - \frac{4k\cos\theta}{\sin x}\right)\alpha\beta \,\mathrm{d}x\\
&= \int_0^\pi \frac{2k+1}{\sin x}\left(\alpha^2+\beta^2\right) + \frac{4\cos\theta}{\sin x}\alpha\beta\,\mathrm{d}x\\
&> \int_0^\pi \left(2\alpha^2+2\beta^2-4\alpha\beta\right)\frac{1}{\sin x}\,\mathrm{d}x = \int_0^\pi \frac{2(\alpha-\beta)^2}{\sin x}\,\mathrm{d}x \geq 0.
\end{aligned}$$

If $\alpha = 0$ or $\beta = 0$ then the mixed term is identically 0 and the strict inequality becomes an equality. In particular, if $\alpha \cdot \beta = 0$ but either $\alpha \neq 0$ or $\beta \neq 0$ the last inequality is strict and the difference is again strictly positive. If $\alpha = \beta = 0$ however, this inequality becomes an equality as well and the difference is 0. Plugging $\alpha = \beta = 0$ into the expression for $\mathcal{H}_k$ immediately gives $\mathcal{H}_k(0,0) = 0$ for all k.

Note: The special structure of u_i implies $\alpha(\pi) = \beta(\pi)$ for Fourier coefficients so that the last integral has finite value in these cases. □

Non-negativity thus reduces to the problem of determining a signum for $\mathcal{H}_0$ and $\mathcal{H}_1$. As the Lemmata below show, $\mathcal{H}_1 > 0$ unless $\alpha, \beta = 0$ and positivity of $\mathcal{H}_0$ is equivalent to convexity of E. In conclusion, the following statement holds:

Proposition 4.2. *Let $\kappa > 4$ and $\phi \in C_c^\infty(\mathbb{S}^2 \setminus \{\hat{\mathbf{e}}_3\}; T_{m_0}\mathbb{S}^2)$. Then, $\mathcal{H}(\phi) \geq 0$ and $\mathcal{H}(\phi) = 0$ if and only if*

$$\phi(x,\varphi) = \alpha(x)\begin{pmatrix}\cos\theta\cos\varphi\\ \cos\theta\sin\varphi\\ \sin\theta\end{pmatrix} \quad \text{with } \alpha \in C_c^\infty(0,\pi) \text{ and } \frac{d^2}{dt^2}E(\theta+t\alpha)|_{t=0} = 0,$$

where θ is the polar profile associated to m_0.

Before getting into non-negativity of $\mathcal{H}_0$ and $\mathcal{H}_1$, we consider the following general Hardy-type decomposition, a variation of [23] :

Lemma 4.2. *Consider the integral*

$$\int_a^b \sin x|u'(x)|^2 + V(x)u(x)^2\ dx$$

where $u \in C^\infty(a,b) \cap C([a,b])$ *and choose* $\psi\colon (a,b) \to \mathbb{R}$ *smooth with* $\psi > 0$ *on* (a,b) *such that* $u^2 \frac{\psi'}{\psi} \sin x$ *can be continuously extended to* $\{a,b\}$*. Then it holds that*

$$\int_a^b V u^2 \, dx = \int_a^b \psi^2 V \left(\frac{u}{\psi}\right)^2 \, dx$$

and

$$\int_a^b \sin x (u')^2 \, dx = \int_a^b \psi^2 \sin x \left(\left(\frac{u}{\psi}\right)'\right)^2 \, dx + \int_a^b \left(\frac{u}{\psi}\right)^2 \left(-\sin x \psi'\right)' \psi \, dx$$
$$+ \left(\lim_{x \nearrow \pi} - \lim_{x \searrow 0}\right) \sin x \frac{\psi'}{\psi} u^2.$$

The proof is elementary by partial integration. As an application of Lemma 4.2, we first consider

$$\mathcal{H}_1(\alpha,\beta) = \int_0^\pi \left((\alpha')^2 + (\beta')^2\right) \sin x + \frac{4\cos\theta}{\sin x} \alpha\beta$$
$$+ \alpha^2 \left(\frac{1}{\sin x} - (\theta')^2 \sin x + \frac{\cos^2\theta}{\sin x} + \kappa \cos^2(\theta - x) \sin x\right)$$
$$+ \beta^2 \left(\frac{1}{\sin x} + \frac{\cos(2\theta)}{\sin x} + \kappa(2\theta - 2x) \sin x\right) dx.$$

Lemma 4.3. $\mathcal{H}_1(\alpha,\beta) > 0$ *for all* $\alpha, \beta \in C_c^\infty((0,\pi]) \setminus \{0\}$*.*

Proof. We perform separate decompositions for the α and β terms of the functional. First, apply Lemma 4.2 to the α part with $\psi(x) = \frac{\sin(\theta - x)}{\sin x}$ such that $\alpha = \frac{\xi \sin(\theta - x)}{\sin x}$. Since α is compactly supported away from 0, the partial integration at 0 holds trivially. Near π, we show that the limit of $\frac{\psi'}{\psi}$ exists.

$$\begin{aligned}
\lim_{x\to\pi} \frac{\psi'(x)}{\psi(x)} \sin x &= \lim_{x\to\pi} \frac{\sin^2 x}{\sin(\theta - x)} \left(\frac{(\theta' - 1)\cos(\theta - x)}{\sin x} - \frac{\sin(\theta - x)\cos x}{\sin^2 x}\right) \\
&= \lim_{x\to\pi} \left(-\cos x + \frac{\sin x(\theta' - 1)\cos(\theta - x)}{\sin(\theta - x)}\right) \\
&= \lim_{x\to\pi} \Bigg(-\cos x + \frac{\cos x(\theta' - 1)\cos(\theta - x) + \sin x \theta'' \cos(\theta - x)}{(\theta' - 1)\cos(\theta - x)} \\
&\qquad - \frac{\sin x(\theta' - 1)^2 \sin(\theta - x)}{(\theta' - 1)\cos(\theta - x)}\Bigg) \\
&= \lim_{x\to\pi} \left(\sin x \frac{\theta''}{\theta' - 1} - \sin x(\theta' - 1)\frac{\sin(\theta - x)}{\cos(\theta - x)}\right) \\
&= \lim_{x\to\pi} \sin x \frac{\theta''}{\theta' - 1}.
\end{aligned}$$

By Lemma 3.7, this final limit is 0. Hence, application of Lemma 4.2 yields, without any contribution from the boundary term at π, the following simplification:

$$\begin{aligned}
&\int_0^\pi (\alpha')^2 \sin x + \alpha^2 \left(\frac{1}{\sin x} - (\theta')^2 \sin x + \frac{\cos^2\theta}{\sin x} + \kappa \cos^2(\theta - x) \sin x \right) \, dx \\
&\quad = \int_0^\pi \frac{\sin^2(\theta - x)}{\sin x} (\xi')^2 \\
&\quad + \xi^2 \Bigg(\frac{\sin(2\theta - 2x)}{2\sin^2 x} \left(-\theta'' \sin x + (\theta' - 1)\cos x + \frac{\kappa}{2} \sin(2\theta - 2x) \sin x \right) \\
&\quad + \sin^2(\theta - x) \left(\frac{1}{\sin^3 x} + \frac{-(\theta')^2}{\sin x} + \frac{\cos^2\theta}{\sin^3 x} + \frac{(\theta' - x)^2}{\sin x} - \frac{1}{\sin x} - \frac{\cos^2 x}{\sin^3 x} \right) \Bigg) \, dx \\
&\quad \overset{\circledast}{=} \int_0^\pi \frac{\sin^2(\theta - x)}{\sin x} (\xi')^2 + \xi^2 \Bigg(\frac{\sin(2\theta - 2x)}{2\sin^2 x} \left((2\theta' - 1)\cos x - \frac{\sin(2\theta)}{2\sin x} \right) \\
&\quad + \sin^2(\theta - x) \left(\frac{-2\theta'}{\sin x} + \frac{1 + \cos^2\theta - \cos^2 x}{\sin^3 x} \right) \Bigg) \, dx \\
&\quad = \int_0^\pi \frac{\sin^2(\theta - x)}{\sin x} (\xi')^2 + \xi^2 \frac{2\sin(\theta - x)\cos\theta(\theta' - 1)}{\sin^2 x} \, dx.
\end{aligned}$$

Note that the differential equation (3.1), i.e.

$$\theta'' \sin x + \theta' \cos x = \frac{\sin(2\theta)}{2\sin x} + \frac{\kappa}{2} \sin(2\theta - 2x) \sin x,$$

was employed in $\circledast$ as well as several trigonometric identities in the last equality.

For the terms with β, use $\psi = \theta' - 1$ and set $\eta = \frac{\beta}{\psi}$ For the limit at π, we again have

$$\lim_{x \to \pi} \sin x \frac{\psi'}{\psi} = \lim_{x \to \pi} \sin x \frac{\theta''}{\theta' - 1} = 0$$

such that the contribution of the boundary terms in Lemma 4.2 is once more negligible. Applying the decomposition of Lemma 4.2 to the β-terms results in the expression below.

$$\begin{aligned}
&\int_0^\pi (\beta')^2 \sin x + \beta^2 \left(\frac{1}{\sin x} + \frac{\cos(2\theta)}{\sin x} + \kappa \cos(2\theta - 2x) \sin x \right) \, dx \\
&= \int_0^\pi (\eta')^2 \sin x (1-\theta')^2 + \eta^2 \Bigg((1-\theta')^2 \frac{1+\cos(2\theta)}{\sin x} \, dx \\
&\qquad + (1-\theta') \left(\cos x \theta'' + \sin x \theta''' + (1-\theta')\kappa \cos(2\theta - 2x) \sin x \right) \Bigg) \\
&\overset{\ominus}{=} \int_0^\pi (\eta')^2 \sin x (1-\theta')^2 + \eta^2 (1-\theta') \Bigg(\sin x \theta' + \frac{1-\theta'}{\sin x} - \frac{\sin(2\theta)\cos x}{2\sin^2 x} \\
&\qquad + \frac{\cos(2\theta)}{\sin x} + \left(\frac{\kappa}{2} \sin(2\theta - 2x) \sin x - \sin x \theta'' \right) \frac{\cos x}{\sin x} \Bigg) \, dx \\
&\overset{\circledast}{=} \int_0^\pi (\eta')^2 \sin x (1-\theta')^2 + \eta^2 (1-\theta') \Bigg(\sin x \theta' + \frac{1-\theta'}{\sin x} \\
&\qquad + \frac{\cos^2 x}{\sin x} \theta' - \frac{\sin(2\theta)\cos x}{\sin^2 x} + \frac{\cos(2\theta)}{\sin x} \Bigg) \, dx \\
&= \int_0^\pi (\eta')^2 \sin x (1-\theta\prime)^2 + \eta^2 (1-\theta') \left(-2 \frac{\sin(\theta - x)}{\sin^2 x} \cos\theta \right) \, dx.
\end{aligned}$$

The differential equation was again applied in $\circledast$, its derivative in $\ominus$.

Recalling $\alpha\beta = \xi\eta \frac{\sin(\theta - x)(1-\theta')}{\sin x}$ and collecting terms, the overall result of the decomposition is given by

$$\begin{aligned}
&\mathcal{H}_1(\alpha, \beta) \\
&\quad = \int_0^\pi \frac{\sin^2(\theta - x)}{\sin x} (\xi')^2 + (1-\theta')^2 \sin x (\eta')^2 - 2 \frac{\sin(\theta - x)(1-\theta')\cos\theta}{\sin^2 x} (\xi - \eta)^2 \, dx \\
&\quad =: \tilde{\mathcal{H}}_1(\xi, \eta).
\end{aligned}$$

With $\cos\theta$ taking both positive and negative values, further analysis becomes necessary for the last term. Using

$$\begin{aligned}
-2\sin(\theta - x)(1-\theta')\cos\theta &= -2\sin(\theta - x)(1-\theta') \left(\cos(\theta - x)\cos x - \sin(\theta - x)\sin x \right) \\
&= \left(\frac{d}{dx} \sin^2(\theta - x) \right) \cos x + 2\sin^2(\theta - x)(1-\theta')\sin x,
\end{aligned}$$

integration by parts leads to

$$\begin{aligned}\tilde{\mathcal{H}}_1(\xi,\eta) = \int_0^\pi & \frac{\sin^2(\theta-x)}{\sin x}(\xi')^2 + (1-\theta')^2 \sin x(\eta')^2 \\ & - \sin^2(\theta-x)(\xi-\eta)^2\left(\frac{-1-\cos^2 x}{\sin^3 x}\right) + 2\frac{\sin^2(\theta-x)(1-\theta')}{\sin x}(\xi-\eta)^2 \\ & - 2\frac{\sin^2(\theta-x)\cos x}{\sin^2 x}\xi'(\xi-\eta) + 2\frac{\sin^2(\theta-x)\cos x}{\sin^2 x}\eta'(\xi-\eta)\; dx \\ = \int_0^\pi & \left(\frac{\sin(\theta-x)}{\sqrt{\sin x}}\xi' - \frac{\sin(\theta-x)\cos x}{\sqrt{\sin x}^3}(\xi-\eta)\right)^2 \\ & + \left(\frac{\sin(\theta-x)\sqrt{1+2\sin x(1-\theta')}}{\sqrt{\sin x}^3}(\xi-\eta) + \frac{\sin(\theta-x)\cos x}{\sqrt{\sin x(1+2\sin x(1-\theta')}}\eta'\right)^2 \\ & + \left((1-\theta')^2\sin x - \frac{\sin^2(\theta-x)\cos^2 x}{\sin x(1+2\sin x(1-\theta')}\right)(\eta')^2\; dx.\end{aligned}$$

It remains to show that $(1-\theta')^2\sin x - \frac{\sin^2(\theta-x)\cos^2 x}{\sin x + 2\sin^2 x(1-\theta')}$ is non-negative. Since the denominator is strictly positive on $(0,\pi)$, it suffices to consider the numerator

$$(1-\theta')^2\sin^2 x - \sin^2(\theta-x)\cos^2 x + 2\sin^3 x(1-\theta')^3.$$

To do so, let $h(x) = (1-\theta')^2\sin^2 x - \sin^2(\theta-x)$. Then $h(0) = h(\pi) = 0$ and $h(x)$ is strictly smaller than $(1-\theta')^2\sin^2 x - \sin^2(\theta-x)\cos^2 x + 2\sin^3 x(1-\theta')^3$ since $|\cos x| < 1$ and $\theta - \mathrm{id}$ is monotonously decreasing. Furthermore,

$$\begin{aligned} h'(x) &= -2\theta''(1-\theta')\sin^2 x + (1-\theta')^2\sin(2x) + (1-\theta')\sin(2\theta-2x) \\ &= (1-\theta')\big((-2\theta''\sin x - 2\theta'\cos x)\sin x + \sin(2x) + \sin(2\theta-2x)\big) \\ &\overset{\circledast}{=} (1-\theta')\big(-\sin(2\theta) - \kappa\sin(2\theta-2x)\sin^2 x + \sin(2x) + \sin(2\theta-2x)\big) \\ &= 2(1-\theta')\sin(\theta-x)\sin x\Big(-\cos(\theta-x)(\kappa-2)\sin x + \cos x\sin(\theta-x)\Big),\end{aligned}$$

where equation (3.1) has once more been employed in $\circledast$, followed by some trigonometric identities.

Since $\sin(\theta-x) > 0$ on $(0,\pi)$, the last expression for the derivative implies that $h'(x) = 0$ if and only if $\cot(\theta-x)(\kappa-2) = \cot x$ where the right hand side is increasing for $x \in (0,\pi)$ and the left hand side is decreasing in x, due to the monotonicity of $\theta - \mathrm{id}$. Therefore, h' has only one zero, h has only one extremum in $(0,\pi)$ and $h(x) \neq 0 = h(0) = h(\pi)$ for all $x \in (0,\pi)$. Taking into account that $h'(x) > 0$ for very small x and $\kappa > 2$, it follows that $h(x) > 0$ for all $x \in (0,\pi)$ and therefore the factor of $(\eta')^2$ is non-negative, resulting in $\tilde{\mathcal{H}}_1(\xi,\eta) \geq 0$ for all $\xi,\eta \in C_c^\infty((0,\pi])$.

$\tilde{H}_1$, as a sum of three squares, can only be 0 if all three terms are identically 0. From the last term we can then deduce $\eta' \equiv 0$ and therefore, from the second term and since

$\sin(\theta - x) > 0$, we have $\xi - \eta \equiv 0$ which implies $\xi = \eta = const$. From the boundary data it follows that $\xi = \eta \equiv 0$. □

We next prove non-negativity of $\mathcal{H}_0$ which is given by

$$\mathcal{H}_0(\alpha,\beta) = \int_0^\pi (\alpha')^2 \sin x + \alpha^2 \left(-(\theta')^2 \sin x + \frac{\cos^2\theta}{\sin x} + \kappa \cos^2(\theta - x) \sin x \right) + (\beta')^2 \sin x + \beta^2 \left(\frac{\cos(2\theta)}{\sin x} + \kappa \cos(2\theta - 2x) \sin x \right) \, dx.$$

However, in contrast to $\mathcal{H}_1$, this requires stronger assumptions on α, β, which hold true for the relevant α, β.

Lemma 4.4. *Let $\alpha = a_0^1$ and $\beta = a_0^2$ be the zeroth-order Fourier coefficients of $u_1 = \phi \cdot X$ and $u_2 = \phi \cdot Y$, respectively. Then the limits of α and β at $0, \pi$ exist and are equal to* 0.

For the proof, we refer to the Lemma A.1 in the appendix, where the limits of Fourier coefficients at the poles are discussed in general. We now proceed to proving non-negativity of $\mathcal{H}_0$.

Lemma 4.5. *Let $\kappa > 24$. Then $\mathcal{H}_0(\alpha,\beta) \geq 0$ for all $\alpha, \beta \in C^\infty(0,\pi)$ such that $\alpha(0) = \beta(0) = 0$ and $\alpha(\pi) = \beta(\pi) = 0$. In particular,*

$$\int_0^\pi (\alpha')^2 \sin x + \alpha^2 \left(-(\theta')^2 \sin x + \frac{\cos^2\theta}{\sin x} + \kappa \cos^2(\theta - x) \sin x \right) > 0$$

for all such α and

$$\int_0^\pi (\beta')^2 \sin x + \beta^2 \left(\frac{\cos(2\theta)}{\sin x} + \kappa \cos(2\theta - 2x) \sin x \right) \, dx \geq 0$$

for all such β with equality if and only if $\frac{d^2}{dt^2} E(\theta + t\beta)|_{t=0} = 0$.

Proof. The first statement is once more proven by applying Lemma 4.2. First, assume that α is compactly supported in $(0,\pi)$ and consider ξ with $\alpha(x) = \xi(x)\sin(\theta - x)$. Due to the compact support, no boundary terms have to be considered.

$$\int_0^\pi \sin x |\alpha'|^2 + \alpha^2 \left(\sin x \left(-(\theta')^2 + \frac{\cos^2\theta}{\sin^2 x} + \kappa \cos^2(\theta - x) \right) \right)$$

$$= \int_0^\pi \sin^2(\theta - x) \sin x (\xi')^2 + \xi^2 \Bigg(\sin x (\theta' - 1)^2 \sin^2(\theta - x)$$

$$
\begin{aligned}
&- \cos x(\theta' - 1)\frac{\sin(2\theta - 2x)}{2} - \sin x\theta'' \frac{\sin(2\theta - 2x)}{2} \\
&+ \sin^2(\theta - x)\left(- \sin x(\theta')^2 + \frac{\cos^2(\theta)}{\sin x}\right) + \frac{\kappa}{4}\sin^2(2\theta - 2x)\sin x\Bigg)\, dx \\
\overset{\circledast}{=} \int_0^\pi & \sin^2(\theta - x)\sin x(\xi')^2 \\
&+ \xi^2\left(\frac{\sin(2\theta - 2x)}{2}\left(- \cos x(\theta' - 1) - \sin x\theta'' + \theta''\sin x + \theta'\cos x - \frac{\sin(2\theta}{2\sin x}\right)\right. \\
&\left.+ \sin^2(\theta - x)\left((\theta' - 1)^2\sin x - (\theta')^2\sin x + \frac{\cos^2(\theta)}{\sin x}\right)\right)\, dx \\
= \int_0^\pi & \sin^2(\theta - x)\sin x(\xi')^2 \ + \xi^2\sin(\theta - x)\Bigg(\cos(\theta - x)\left(\cos x - \frac{\sin(2\theta)}{2\sin x}\right) \\
&+ \sin^2(\theta - x)\left(\frac{\cos^2\theta}{\sin x} + \sin x - 2\theta'\sin x\right)\Bigg)\, dx \\
= \int_0^\pi & \sin^2(\theta - x)\sin x\left((\xi')^2 + 2\xi^2(1 - \theta')\right)\, dx
\end{aligned}
$$

Note that the differential equation was again applied in $\circledast$. For the last step, the identity

$$
\cos(\theta - x)\left(\cos x - \frac{\sin(2\theta)}{2\sin x}\right) = \sin(\theta - x)\left(\sin x - \frac{\cos^2\theta}{\sin x}\right)
$$

was used. It easily follows from trigonometric identities.

For $\kappa > 24$, Proposition 3.1 implies that $1 - \theta' > 0$ and therefore the integrand is non-negative for any ξ. Furthermore, it is zero if and only if $\xi \equiv 0$ and therefore $\alpha \equiv 0$, due to the boundary restriction.

The non-negativity of the integrand ensures that this property carries over to the limiting $\alpha \in C_0^\infty(0, \pi)$.

For the second statement, recall the reduced functional $E(\theta)$ and the minimality of θ. These imply that for $\mu, \nu \in C_0^\infty(0, \pi)$ and variations $\theta_s = \theta + s\mu + t\nu$, the second derivative

$$
\frac{\mathrm{d}^2}{\mathrm{d}s\,\mathrm{d}t}E(\theta_{st})\Big|_{s=t=0}
$$

is non-negative. Computing it, we find

$$\begin{aligned}\frac{d^2}{dsdt}E(\theta_{st})|_{s=t=0} &= \frac{d}{ds}\int_0^\pi 2\nu'\theta'_{st}\sin x + \frac{\nu\sin(2\theta_{st}}{\sin x} + \kappa\sin(2\theta_{st}-2x)\nu\sin x\;dx\Big|_{s=t=0}\\ &= 2\int_0^\pi \nu'\mu'\sin x + \frac{\cos(2\theta)}{\sin x}\nu\mu + \kappa\cos(2\theta-2x)\sin x\nu\mu\;dx\\ &= H(\mu,\nu)\end{aligned}$$

and therefore, choosing $\nu = \mu = \beta \in C_0^\infty((0,\pi))$,

$$\int_0^\pi (\beta')^2\sin x + \beta^2\left(\frac{\cos(2\theta)}{\sin x} + \kappa\cos(2\theta-2x)\sin x\right)\;dx = \frac{1}{2}H(\beta,\beta) =: \frac{1}{2}H(\beta) \geq 0$$

for all $\beta \in C_0^\infty((0,\pi))$. However, minimality alone can not imply a strict inequality for $\beta \neq 0$ and so the problem of degeneracy for $\mathcal{H}$ is reduced to whether or not the Hessian H of the reduced functional is degenerate. See Chapter 6 for a discussion on this topic. □

By performing the proof of Proposition 4.2 again under different assumptions, we can complete the proof of Proposition 3.1 by giving the missing statement about the range of minimizing profiles.

Corollary 4.1. *Let $\kappa > 24$ and assume that $\mathbf{m}_0\colon \mathbb{S}^2 \to \mathbb{S}^2$ minimizes $\mathcal{E}$ among all axisymmetric fields. Then the corresponding minimizing profile fulfils $\theta < \pi$ for all x if it is a lower minimizing profile and $\theta > \pi$ if it is an upper minimizing profile.*

Proof. Without loss of generality, consider a lower minimizing profile θ with $\theta(\frac{\pi}{2}) < \pi$. If not, $\bar{\theta}$ fulfills this claim.

In Lemma 3.7, the asymptotics of θ for $x \to \pi$ were analysed based only on the fact that $0 < \theta' < 1$ on an interval $(\pi - \varepsilon, \pi)$. If $\theta > \pi$ was true for any $x \in (0,\pi)$ then it would follow that $\theta > \pi$ on $(0,p) \subset (0,\frac{\pi}{2})$ and $\theta < \pi$ on (p,π). In particular, θ would have a unique maximum at $x_{\max} \in (0,p)$ and $0 < \theta' < 1$ on $(0,x_{\max})$, see Lemma 3.4. Thus the same techniques as in Lemma 3.7 could be applied to $\bar{\theta}$ to show that

$$\lim_{x\searrow 0}\sin x\frac{\theta''}{1-\theta'} = -\lim_{x\nearrow\pi}\sin x\frac{\bar{\theta}''}{1-\bar{\theta}'} = 0.$$

Therefore, if $p > 0$, the assumptions of Proposition 4.2 above could be weakened to allow any $\phi \in C^\infty(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2)$ because both boundary terms in the partial integration for $\mathcal{H}_1$ would be controlled.

This would mean that $\mathcal{H}(\phi) = 0$ only if $\phi = \alpha Y$ for some $\alpha \in C_0^\infty((0,\pi))$ with $H(\alpha) = 0$. In particular, given $v \in \mathbb{R}^3 \setminus \{0\}$, the tangent field associated to the joint rotation around v,

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathbf{m}_{0R_v(t)}|_{t=0},$$

which can be expressed in the moving frame as described below in Lemma 4.8, would not be an element of the kernel. But $t \mapsto \mathcal{E}(\mathbf{m}_{0R_v(t)})$ is constant so this is false. Thus the assumption of $\theta > \pi$ on $(0,p) \neq \emptyset$ must have been wrong.

Since $\theta \equiv \pi$ does not solve the equation, the profile has to fulfil $\theta < \pi$ on $(0,\pi)$. □

4.1.2. Non-Negativity for $C^\infty(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2)$

The nonnegativity result for $\mathcal{H}$ can be lifted from the compactly supported case to general tangent fields $\phi \in H^1(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2)$. In a first step, fields with $\phi(\hat{\mathbf{e}}_3) = 0$ are considered.

Lemma 4.6. *Let $\phi \in H^1(\mathbb{S}^2; T_{m_0}\mathbb{S}^2)$ with $\phi(\hat{\mathbf{e}}_3) = 0$ and assume $\kappa > 24$. Then $\mathcal{H}(\phi) \geq 0$. Moreover, $\mathcal{H}(\phi) = 0$ implies that ϕ is axisymmetric and*

$$\phi(x,\varphi) = \alpha(x) \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix} \quad \text{with } \alpha \in C_0^\infty((0,\pi)),\ H(\alpha) = 0.$$

Proof. Given $\varepsilon > 0$, consider a smoothing cut-off function $\rho_\varepsilon \colon [0,\pi] \to [0,1]$ with $\rho_\varepsilon \equiv 0$ on $[0,\varepsilon]$, $\rho_\varepsilon \equiv 1$ on $[2\varepsilon,\pi]$, and $|\rho_\varepsilon'| \leq \frac{2}{\varepsilon}$. Set $\phi_\varepsilon(x,\varphi) = \rho_\varepsilon(x)\phi(x,\varphi)$. Obviously $\phi_\varepsilon \to \phi$ pointwisely on $\mathbb{S}^2$. Furthermore, for $(x,\varphi) \in \mathbb{S}^2 \setminus \{\pm\hat{\mathbf{e}}_3\}$,

$$\begin{aligned}
\int_{\mathbb{S}^2} |\nabla\phi_\varepsilon|^2 \, dS &= \int_0^{2\pi}\int_0^{\pi} \left|\frac{\partial\phi_\varepsilon}{\partial x}\right|^2 + \frac{1}{\sin^2 x}\left|\frac{\partial\phi_\varepsilon}{\partial\varphi}\right|^2 \, dx \, d\varphi \\
&= \int_{\mathbb{S}^2} |\nabla\phi|^2 \, dx + \int_0^{2\pi}\int_0^{\pi} \left(2\phi\phi_x\rho_\varepsilon\rho\varepsilon' + \phi^2(\rho')^2\right)\sin x \, dx \, d\varphi.
\end{aligned}$$

Taking into account that ρ_ε is constant on $[0,\varepsilon] \cup [2\varepsilon,\pi]$, this implies

$$\begin{aligned}
\mathcal{H}(\phi_\varepsilon) = &\int_{\mathbb{S}^2} \rho_\varepsilon^2 \left(|\nabla\phi|^2 - \kappa(\phi\cdot\nu)^2 - |\phi|^2\left(|\nabla m_0|^2 - \kappa(m_0\cdot\nu)^2\right)\right) \, dS \\
&+ \int_0^{2\pi} \left(\int_\varepsilon^{2\varepsilon} \left(2\phi\partial_x\phi\rho_\varepsilon\rho_\varepsilon' + \phi^2(\rho_\varepsilon')^2\right)\sin x \, dx \right) \\
\longrightarrow &\mathcal{H}(\phi) \text{ as } \varepsilon \to 0
\end{aligned}$$

where the first term converges due to Lebesgue and the pointwise convergence $\rho_\varepsilon \to 1$ almost everywhere while the second term converges to 0 due to

$$\int_{\varepsilon}^{2\varepsilon} \left(2\phi\phi_x\rho_\varepsilon\rho_\varepsilon' + |\phi|^2(\rho_\varepsilon'|^2\right)\sin x \; dx \leq \int_{\varepsilon}^{2\varepsilon} 2\varepsilon \left(2 \sup_{[\varepsilon,2\varepsilon]}(\phi\phi_x)\frac{2}{\varepsilon} + (\sup_{[0,2\varepsilon]}(\phi))^2\frac{4}{\varepsilon^2}\right) \; dx$$
$$\leq \left(8\sup_{[0,1]}(\phi\phi_x) + (\sup_{[\varepsilon,2\varepsilon]}(\phi))^2\frac{8}{\varepsilon}\right)(2\varepsilon - \varepsilon)$$

and $\lim_{\epsilon\to 0}\sup_{[\varepsilon,2\varepsilon]}(\phi) = 0$.

For every $\varepsilon > 0$, ϕ_ε is compactly supported in $(0, \pi)$ and smooth. Therefore $\mathcal{H}(\phi_\varepsilon) \geq 0$ by Proposition 4.2, implying $\mathcal{H}(\phi) \geq 0$.

Now assume that $\mathcal{H}(\phi) = 0$. Then $\mathcal{H}(\phi_\varepsilon) \to 0$ as $\varepsilon \to 0$. Decomposing $\phi_\varepsilon = \rho_\varepsilon\phi$ via the moving frame and then into Fourier modes, the choice of ρ_ε implies that $(a_k^i)_\varepsilon = \rho_\varepsilon a_k^i$ and $(b_k^i)_\varepsilon = \rho_\varepsilon b_k^i$ for a_k^i, $b_k^i \in C^\infty((0,\pi)$ for all $k \in \mathbb{N}_0$ and $i = 1, 2$. Since $\mathcal{H}_k \geq 0$ for all k they all individually converge to 0. For $k = 0$ and again considering the α and β part separately, this implies

$$\int_0^\pi 2(\rho_\varepsilon\xi)^2\sin^2(\theta - x)\sin x \; dx = \int_0^\pi 2\rho_\varepsilon^2(a_0^1)^2\sin x(1-\theta') \; dx \to 0 \text{ as } \varepsilon \to 0.$$

On the other hand the above expression is increasing in ε and non-negative so the integral must be identically 0 and therefore $a_0^1 = 0$. For the β part where $\beta = a_0^2$, note that

$$|(\beta_\varepsilon')^2 - (\beta')^2\rho_\varepsilon^2| = |2\beta\beta'\rho_\varepsilon\rho_\varepsilon' + \beta^2(\rho_\varepsilon')^2| \leq \left(\sup_{[\varepsilon,2\varepsilon]}(\beta\beta')\frac{2}{\varepsilon} + \sup_{[\varepsilon,2\varepsilon]}(\beta)^2\frac{4}{\varepsilon^2}\right)\chi_{[\varepsilon,2\varepsilon]}.$$

As for $\mathcal{H}_0$, this implies , using dominated convergence and a similar reasoning as above, that

$$\begin{aligned}
0 &= \lim_{\varepsilon\to 0}\int_0^\pi (\beta_\varepsilon)^2\sin x + (\beta_\varepsilon)^2\left(\frac{\cos(2\theta)}{\sin x} + \kappa\cos(2\theta - 2x)\sin x\right) \; dx \\
&= \lim_{\varepsilon\to 0}\int_0^\pi \rho_\varepsilon^2\left((\beta')^2\sin x + \beta^2\left(\frac{\cos(2\theta)}{\sin x} + \kappa\cos(2\theta - 2x)\sin x\right)\right) \; dx \\
&\quad + \int_0^\pi \left(2\beta\beta'\rho_\varepsilon\rho_\varepsilon' + \beta^2(\rho_\varepsilon')^2\right)\sin x \; dx \\
&= \int_0^\pi (\beta')^2\sin x + \beta^2\left(\frac{\cos(2\theta)}{\sin x} + \kappa\cos(2\theta - 2x)\sin x\right) \; dx \\
&= H(\beta).
\end{aligned}$$

Thus, while a_0^2 does not have to be 0, we find that a_0^2 is in the kernel of the Hessian for the reduced energy.

For $\mathcal{H}_1$, recall that L^2 convergence implies convergence in measure and also that

$$\begin{aligned}\tilde{\mathcal{H}}_1(\xi,\eta) = &\left\| \frac{\sin(\theta - x)}{\sqrt{\sin x}}\xi' - \frac{\sin(\theta - x)\cos x}{\sqrt{\sin x}^3}(\xi - \eta) \right\|_{L^2}^2 \\ &+ \left\| \frac{\sin(\theta - x)\sqrt{1 + 2\sin x(1-\theta')}}{\sqrt{\sin x}^3}(\xi - \eta) + \frac{\sin(\theta - x)\cos x}{\sqrt{\sin x(1 + 2\sin x(1-\theta'))}}\eta' \right\|_{L^2}^2 \\ &+ \left\| \left((1-\theta')^2 \sin x - \frac{\sin^2(\theta - x)\cos^2 x}{\sin x(1 + 2\sin x(1-\theta')} \right)^{\frac{1}{2}} (\eta') \right\|_{L^2}^2 .\end{aligned}$$

From the third norm we can infer that η converges in measure to a constant while the second and first norm imply that the same holds true for ξ and that the constants are identical. Due to

$$\eta_\varepsilon = \rho_\varepsilon \frac{\beta}{1-\theta'} \text{ and } \lim_{x\to\pi} \frac{\beta(x)}{1-\theta'(x)} = \frac{\beta(\pi)}{1-\theta'(\pi)},$$

this constant must be identical to $0 = \beta(\pi)$, which follows from $\lim_{x\to\pi}\phi(x,\varphi) = 0$ for all φ and Lemma A.1.

For $\mathcal{H}_k$ where $k \geq 1$ let $\alpha_\varepsilon = \rho_\varepsilon\alpha$, $\beta_\varepsilon = \rho_\varepsilon\beta$ correspond to k-th Fourier coefficients of ϕ_ε. Then

$$0 = \lim_{\varepsilon\to 0} \mathcal{H}_k(\alpha_\varepsilon, \beta_\varepsilon) \geq \lim_{\varepsilon\to 0} \mathcal{H}_1(\alpha_\varepsilon, \beta_\varepsilon) = 0$$

so $\mathcal{H}_1(\alpha_\varepsilon, \beta_\varepsilon)$ converges to 0 and thus $\alpha_\varepsilon, \beta_\varepsilon$ are identically 0.

All together we have $\phi = u^1X + u^1Y = 0 + a_0^2 Y$ where the zero-order coefficient of u_2, $a_0^2 \in C_0^\infty(0,\pi)$, fulfills $H(a_0^2) = 0$. □

Before extending the result to general $\phi \in H^1(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2)$ we show that the tangent fields associated to joint rotations span the tangent space of $\mathbf{m}_0$ at $\hat{\mathbf{e}}_3$.

Definition 4. *For $v \in \mathbb{R}^3 \setminus \{0\}$, the field*

$$\phi_v = \frac{d}{dt} m_{R_v(t)}|_{t=0}$$

is called the tangent field associated to the joint rotation around v. For $v = \hat{\mathbf{e}}_1$ and $v = \hat{\mathbf{e}}_2$, set

$$\phi^{(1)} = \phi_{\hat{\mathbf{e}}_1} \textit{ and } \phi^{(2)} = \phi_{\hat{\mathbf{e}}_2}.$$

Note that $\phi_v = 0$ for vectors v parallel to the symmetry axis $\hat{\mathbf{e}}_3$. For other directions, the fields $\phi^{(1)}$ and $\phi^{(2)}$ have the following property:

Lemma 4.7. *The vectors* $\phi^{(1)}(\hat{\mathbf{e}}_3)$ *and* $\phi^{(2)}(\hat{\mathbf{e}}_3)$ *form a basis of* $T_{\mathbf{m}_0(\hat{\mathbf{e}}_3)}\mathbb{S}^2$*. Moreover, for any* $v \in \mathbb{R}^3 \setminus \{0\}$*, the tangent field associated to the joint rotation around* v *is given by*

$$\phi_v = v_1\phi^{(1)} + v_2\phi^{(2)}.$$

Proof. In general, for $\mathbf{x} \in \mathbb{S}^2$, direct computation gives

$$\phi_v(\mathbf{x}) = \mathbf{m}_0(\mathbf{x}) \times v + D\mathbf{m}_0(\mathbf{x})\langle v \times \mathbf{x}\rangle.$$

Since both terms are linear in v, this directly implies the representation of v in terms of $\phi^{(1)}$ and $\phi^{(2)}$. In the special case of $v = \hat{\mathbf{e}}_1$, $\mathbf{x} = \hat{\mathbf{e}}_3$ with $\mathbf{m}_0(\hat{\mathbf{e}}_3) = -\hat{\mathbf{e}}_3$, the expression yields

$$\phi^{(1)}(\hat{\mathbf{e}}_3) = -\hat{\mathbf{e}}_2 - \frac{\partial}{\partial\hat{\mathbf{e}}_2}\mathbf{m}_0 = -\hat{\mathbf{e}}_2\left(1 + \frac{\partial}{\partial\hat{\mathbf{e}}_2}m_2(\hat{\mathbf{e}}_3)\right)$$

since $\frac{\partial}{\partial\hat{\mathbf{e}}_2}m_1 = \frac{\partial}{\partial\hat{\mathbf{e}}_1}m_2 = 0$ and $\frac{\partial}{\partial\hat{\mathbf{e}}_1}m_3 = \frac{\partial}{\partial\hat{\mathbf{e}}_2}m_3 = 0$ for symmetry reasons. To compute the directional derivative, consider the curve $\gamma(t) = (0, t, \sqrt{1-t^2})^T$ for $t \in (-\varepsilon, \varepsilon)$ with $\gamma(0) = \hat{\mathbf{e}}_3$ and $\dot{\gamma}(0) = \hat{\mathbf{e}}_2$. Since $\mathbf{m}_0$ is differentiable at $\hat{\mathbf{e}}_3$, this implies

$$\frac{\partial}{\partial\hat{\mathbf{e}}_2}m_2 = \frac{d}{dt}m_2(\gamma(t))|_{t=0}.$$

Furthermore, $\gamma(t) = \Psi(\arcsin(t), \frac{3\pi}{2})$ for $t < 0$ and $\gamma(t) = \Psi(\arcsin(t), \frac{\pi}{2})$ for $t > 0$. While the spherical coordinates can not be used to compute the partial derivative at $\hat{\mathbf{e}}_3$, this together with the special form of $\mathbf{m}_0$ and $\theta < \pi$ implies $m_2(\gamma(t)) < 0$ for $t < 0$ and $m_2(\gamma(t)) > 0$ for $t > 0$. Thus,

$$\frac{\partial}{\partial\hat{\mathbf{e}}_2}m_2(\hat{\mathbf{e}}_3) \geq 0$$

and $\phi^{(1)}(\hat{\mathbf{e}}_3)$ is a non-zero multiple of $\hat{\mathbf{e}}_2$ Similar calculations yield

$$\phi^{(2)}(\hat{\mathbf{e}}) = \hat{\mathbf{e}}_1\left(1 + \frac{\partial}{\partial\hat{\mathbf{e}}_1}m_1\right)$$

and $\frac{\partial}{\partial\hat{\mathbf{e}}_1}m_1 \geq 0$, using the curve $\gamma(t) = (t, 0, \sqrt{1-t^2})^T$. □

Away from the poles tangential fields can be expressed by means of the moving frame X, Y defined above. While this is not relevant for the proof, it gives a motivation for the meaning of constant ξ, η in the expansion of $\tilde{\mathcal{H}}_1$.

Lemma 4.8. *Let* $v \in \mathbb{R}^3 \setminus \{0\}$ *be an arbitrary axis of rotation and* ϕ_v *the tangent field associated to* v *Then for* $(x, \varphi) \in \mathbb{S}^2 \setminus \{\pm\hat{\mathbf{e}}_3\}$*,*

$$\phi_v(x,\varphi) = -\frac{\sin(\theta - x)}{\sin x}\left(v_1\cos\varphi - v_2\sin\varphi\right)X + (\theta' - 1)\left(-v_1\sin\varphi + v_2\cos\varphi\right)Y$$

and for type I Skyrmions, this expression is continuous at the south pole $-\hat{\mathbf{e}}_3$ *.*

Note: Above expression for ϕ_v yields

$$\mathcal{H}(\phi_v) = \tilde{\mathcal{H}}_1(-v_1, -v_1) + \tilde{\mathcal{H}}_1(-v_2, -v_2),$$

i.e. constant ξ and η.

Proof. The representation of ϕ_v in spherical coordinates follows with some computations from the general form in Lemma 4.7. For the continuity, set

$$u_1 = -\frac{\sin(\theta - x)}{\sin x}(v_1 \cos\varphi - v_2 \sin\varphi) \quad \text{and} \quad u_2 = (\theta' - 1)(-v_1 \sin\varphi + v_2 \cos\varphi).$$

Then compute

$$\begin{aligned} u_1 X_1 + u_2 Y_1 &= v_1 \sin\varphi\cos\varphi \left(\frac{\sin(\theta - x)}{\sin x} - (\theta' - 1)\cos\theta \right) \\ &\quad + v_2 \left(\sin^2\varphi \frac{\sin(\theta - x)}{\sin x} + \cos^2\varphi(\theta' - 1)\cos\theta \right) \\ u_1 X_2 + u_2 Y_2 &= v_2 \sin\varphi\cos\varphi \left(\frac{\sin(\theta - x)}{\sin x} + (\theta' - 1)\cos\theta \right) \\ &\quad + v_1 \left(-\cos^2\varphi \frac{\sin(\theta - x)}{\sin x} - \sin^2\varphi(\theta' - 1)\cos\theta \right) \\ u_1 X_3 + u_2 Y_3 &= -\sin\theta\cos\theta(\theta' - 1)(-v_1 \sin\varphi + v_2 \cos\varphi). \end{aligned}$$

At $\hat{\mathbf{e}}_3$, a Skyrmion forms and nothing is known about the behaviour of θ' as $x \to 0$. In contrast, existence of the limit would lead to a similar contradiction as the one in Corollary 4.1. As $x \to \pi$, on the other hand, existence of the limit $\theta'(\pi)$ has been proven. By linearity of the limit, the third component converges to 0. For the first two components, due to

$$\begin{aligned} \lim_{x\to\pi} \frac{\sin(\theta - x)}{\sin x} &= \lim_{x\to\pi} (\theta' - 1)\frac{\cos(\theta - x)}{\cos x} = \lim_{x\to\pi} (\theta' - 1)\left(\cos\theta + \sin\theta\sin x\right) \\ &= \lim_{x\to\pi} (\theta' - 1)\cos\theta, \end{aligned}$$

the first line in each component converges to 0 while the second lines converge to $-v_2(\theta'(\pi) - 1) = -v_2\frac{a}{2}$ and $v_1(\theta'(\pi) - 1) = v_1\frac{a}{2}$, respectively, as $x \to \pi$. Comparing these to the expression of ϕ in Lemma 4.7 confirms the continuity. $\square$

From the expression of ϕ_v in spherical coordinates, we can easily deduce that tangent fields associated to joint rotations are not invariant under joint rotations around $\hat{\mathbf{e}}_3$. Instead, this results in a shift of the rotation axis.

Corollary 4.2. *For $v \in \mathbb{R}^3 \setminus \{0\}$ and $R \in SO(3)_{\hat{\mathbf{e}}_3}$, it holds that*

$$(\phi_v)_R = \phi_{R^{-1}v}.$$

Proof. In spherical coordinates, let $R \in SO(3)_{\hat{\mathbf{e}}_3}$ denote a rotation around $\hat{\mathbf{e}}_3$ with angle α. Then $R\Psi(x,\varphi) = \Psi(x,\varphi+\alpha)$ and $X_R = X$, $Y_R = Y$ on $\mathbb{S}^2 \setminus \{\pm\hat{\mathbf{e}}_3\}$. For ϕ_v, we have

$$\begin{aligned}(\phi_v)_R(\Psi(x,\varphi)) &= -\frac{\sin(\theta-x)}{\sin x}\left(v_1\cos(\varphi+\alpha) - v_2\sin(\varphi-\alpha)\right)X_R \\ &\quad + (\theta'-1)\left(-v_1\sin(\varphi-\alpha)+v_2\cos(\varphi+\alpha)\right)Y_R \\ &= -\frac{\sin(\theta-x)}{\sin x}\left(\tilde{v}_1\cos\varphi - \tilde{v}_2\sin\varphi\right)X + (\theta'-1)\left(-\tilde{v}_1\sin\varphi+\tilde{v}_2\cos\varphi\right)Y\end{aligned}$$

where $\tilde{v}_1 = (v_1\cos\alpha + v_2\sin\alpha)$, $\tilde{v}_2 = -v_1\sin\alpha + v_2\cos\alpha$ and therefore $\tilde{v} = R^{-1}v$. □

Combining the previous results for $\mathcal{H}$ with the knowledge about $T_{\mathbf{m}_0(\hat{\mathbf{e}}_3)}\mathbb{S}^2$, we can now prove nonnegativity of the Hessian for arbitrary tangent fields.

Proof of Proposition 4.1. Given $\phi \in T_{\mathbf{m}_0}\mathbb{S}^2$, note that $0 = \phi(\hat{\mathbf{e}}_3)\cdot\mathbf{m}_0(\hat{\mathbf{e}}_3) = -\phi_3(\hat{\mathbf{e}}_3)$. Set

$$\tilde{\phi} = \phi - \phi_1(\hat{\mathbf{e}}_3)\left(1+\frac{\partial}{\partial\hat{\mathbf{e}}_1}m_1(\hat{\mathbf{e}}_3)\right)^{-1}\phi^{(2)} + \phi_2(\hat{\mathbf{e}}_3)\left(1+\frac{\partial}{\partial\hat{\mathbf{e}}_2}m_2(\hat{\mathbf{e}}_3)\right)^{-1}\phi^{(1)}.$$

Then $\tilde{\phi}(\hat{\mathbf{e}}_3) = 0$ and Lemma 4.6 implies $\mathcal{H}(\tilde{\phi}) \geq 0$, hence

$$\mathcal{H}(\phi) = \mathcal{H}(\tilde{\phi}) - 0 \geq 0$$

and if $\mathcal{H}(\phi) = 0$ then $\mathcal{H}(\tilde{\phi}) = 0$ and again by Lemma 4.6, $\tilde{\phi}(x,\varphi) = \alpha(x)Y$ for some $\alpha \in H_0^1([0,\pi])$ so that $\phi = \phi_v + \alpha(x)Y$ for

$$v = \begin{pmatrix} -\phi_2(\hat{\mathbf{e}}_3)\left(1+\frac{\partial}{\partial\hat{\mathbf{e}}_2}m_2(\hat{\mathbf{e}}_3)\right)^{-1} \\ \phi_1(\hat{\mathbf{e}}_3)\left(1+\frac{\partial}{\partial\hat{\mathbf{e}}_1}m_1(\hat{\mathbf{e}}_3)\right)^{-1} \\ 0 \end{pmatrix}.$$

□

4.2. Minimality

We will now assume that E is strictly convex and infer minimality of $\mathbf{m}_0$. This is based on an identity for the energy difference where $\mathcal{H}$ is extended to a functional $\tilde{\mathcal{H}}$ on $H^1(\mathbb{S}^2;\mathbb{R}^3)$.

Lemma 4.9. *For a critical point* $\mathbf{m}_0$ *of* $\mathcal{E}$ *and* $\mathbf{m} \in H^1(\mathbb{S}^2;\mathbb{S}^2)$,

$$\mathcal{E}(\mathbf{m}) - \mathcal{E}(\mathbf{m}_0) = \frac{1}{2}\tilde{\mathcal{H}}(\mathbf{m}-\mathbf{m}_0).$$

Proof. Simple computations give

$$
\begin{aligned}
2\left(\mathcal{E}(\mathbf{m})-\mathcal{E}(\mathbf{m}_0)\right) &= \int_{\mathbb{S}^2} |\nabla \mathbf{m}|^2 - |\nabla \mathbf{m}_0|^2 + \kappa \left((\mathbf{m}_0\cdot\nu)^2 - (\mathbf{m}\cdot\nu)^2\right)\ d\sigma \\
&= \int_{\mathbb{S}^2} |\nabla(\mathbf{m}-\mathbf{m}_0)|^2 - \kappa\left((\mathbf{m}-\mathbf{m}_0)\cdot\nu\right)^2\ d\sigma \\
&\quad + 2\int_{\mathbb{S}^2} \nabla \mathbf{m}_0 \cdot (\nabla(\mathbf{m}-\mathbf{m}_0)) - \kappa(\mathbf{m}_0\cdot\nu)\left((\mathbf{m}-\mathbf{m}_0)\cdot\nu\right)\ d\sigma.
\end{aligned}
$$

The first integral is equal to the first term of $\mathcal{H}$ and needs no further computations. For the second integral, criticality of $\mathbf{m}_0$, pointwisely paired with the vector identity $a\cdot b = (a\cdot b)|\mathbf{m}_0|^2 = (a\times\mathbf{m}_0)\cdot(b\times\mathbf{m}_0) + (a\cdot\mathbf{m}_0)(b\cdot\mathbf{m}_0)$ gives

$$
\begin{aligned}
&\int_{\mathbb{S}^2} \nabla \mathbf{m}_0 \cdot (\nabla(\mathbf{m}-\mathbf{m}_0)) - \kappa(\mathbf{m}_0\cdot\nu)(\mathbf{m}-\mathbf{m}_0)\cdot\nu\ d\sigma \\
&\quad = \int_{\mathbb{S}^2} \left(-\Delta\mathbf{m}_0 - \kappa(\mathbf{m}_0\cdot\nu)\nu\right)(\mathbf{m}-\mathbf{m}_0)\ d\sigma \\
&\quad = \int_{\mathbb{S}^2} \left(-\Delta\mathbf{m}_0 - \kappa(\mathbf{m}_0\cdot\nu)\nu\right)\cdot\mathbf{m}_0(\mathbf{m}-\mathbf{m}_0)\cdot\mathbf{m}_0\ d\sigma \\
&\quad = \int_{\mathbb{S}^2} \left(|\nabla\mathbf{m}_0|^2 - \kappa(\mathbf{m}_0\cdot\nu)^2\right)\left(|\mathbf{m}-\mathbf{m}_0|^2 + (\mathbf{m}-\mathbf{m}_0)\cdot(\mathbf{m}+\mathbf{m}_0)\right)\ d\sigma.
\end{aligned}
$$

Since the last term vanishes identically due to $|\mathbf{m}|^2 - |\mathbf{m}_0|^2 = 1-1 = 0$, insertion into the first calculation proves the statement. □

Expressing the energy difference in terms of the Hessian is convenient because the Hessian is coercive on the level of tangent fields orthogonal to $\ker(\mathcal{H})$.

Lemma 4.10. *There exists $\lambda > 0$ such that*

$$\mathcal{H}(\phi) \geq \lambda \|\phi\|^2_{H^1}$$

for all $\phi \in H^1(\mathbb{S}^2, T_{\mathbf{m}_0}\mathbb{S}^2)$ with $\phi \perp \ker(\mathcal{H})$.

Here, orthogonality is understood with respect to the $L^2(\mathbb{S}^2)$-scalar product.

Proof. Consider

$$I = \inf\left\{\mathcal{H}(\phi)\ :\ \phi\in H^1(\mathbb{S}^2, T_{\mathbf{m}_0}\mathbb{S}^2),\ \phi\perp\ker(\mathcal{H}),\ \|\phi\|_{L^2} = 1\right\}.$$

By proposition 4.1, I is non-negative. Assume $I = 0$. Then, there exists a sequence $(\phi_k)_{k\in\mathbb{N}}$ with

$$\|\phi_k\|_{L^2} = 1, \quad \mathcal{H}(\phi_k) \leq \frac{1}{k} \quad \text{and } \phi_k \perp \ker(\mathcal{H}).$$

Due to $\mathcal{H} \lesssim \|\phi\|_{H^1}^2$, the sequence is uniformly bounded in H^1 and therefore admits a weakly convergent subsequence, strongly convergent in L^2 such that $\phi = \lim_{k\to\infty} \phi_k$ satisfies $\|\phi\|_{L^2} = 1$. On the other hand,

$$0 \leq \mathcal{H}(\phi) \leq \liminf_{k\to\infty} \mathcal{H}(\phi_k) = 0$$

and therefore $\phi \in \ker(\mathcal{H})$. But

$$\langle \phi, \phi^{(i)} \rangle_{L^2} = \lim_{k\to\infty} \langle \phi_k, \phi^{(i)} \rangle_{L^2} = 0$$

due to the strong L^2-convergence such that $\phi \in \ker(\mathcal{H}) \cup \ker(\mathcal{H})^{\perp} = \{0\}$. This contradicts $\|\phi\|_{L^2} = 1$.

Combining the lower bound with the estimate

$$\begin{aligned} \mathcal{H}(\phi) &= \|\phi\|_{H^1}^2 - \int_{\mathbb{S}^2} \left(1 + |\nabla \mathbf{m}_0|^2 - \kappa(\mathbf{m}_0 \cdot \nu)^2\right) + \kappa(\phi \cdot \nu)^2 \, \mathrm{d}\sigma \\ &\geq \|\phi\|_{H^1}^2 - \mu(\mathbf{m}_0, \kappa)\|\phi\|_{L^2}^2, \end{aligned}$$

we arrive at

$$\mathcal{H}(\phi) \geq \lambda \|\phi\|_{H^1}^2 + (-\lambda\mu + (1-\lambda)I) = \lambda \|\phi\|_{H^1}^2$$

for $\lambda = \frac{I}{\mu + I}$. □

The kernel of $\mathcal{H}$ contains of fields associated to joint rotations. Such variations due not affect the energy, implying that $\mathcal{E}(\mathbf{m}_R) - \mathcal{E}(\mathbf{m}_0) = \mathcal{E}(\mathbf{m}) - \mathcal{E}(\mathbf{m}_0)$ for all $R \in SO(3)$. Before applying the result for $\mathcal{H}$ to a decomposition of $\mathbf{m} - \mathbf{m}_0$ into tangential and normal parts, we therefore choose an optimal rotation of $\mathbf{m}$ such that $\mathbf{m}_R$ is orthogonal to the kernel of $\mathcal{H}$.

Lemma 4.11. *There exist $\rho > 0$ such that for all $\mathbf{m} \in H^1(\mathbb{S}^2; \mathbb{S}^2)$ with $\|\mathbf{m} - \mathbf{m}_0\|_{H^1} < \rho$, there exists $R \in SO(3)$ such that $\langle \mathbf{m}_R - \mathbf{m}_0, \phi^{(i)} \rangle = 0$ for $i = 1, 2$.*

Proof. Given $\mathbf{m} \in H^1(\mathbb{S}^2; \mathbb{S}^2)$, consider the mapping

$$SO(3) \ni R \mapsto \langle \mathbf{m}_R - \mathbf{m}_0, \phi^{(i)} \rangle_{L^2(\mathbb{S}^2)} \quad \text{for } i = 1, 2.$$

We want to use the implicit function Theorem to find a suitable R for any $\mathbf{m}$ such that the $\mathbf{m}_R - \mathbf{m}_0$ is L^2-orthogonal on the kernel of $\mathcal{H}$. Since $SO(3)$ is too large for the application of the Theorem and given the symmetry of $\mathbf{m}_0$, we will consider special representative of the equivalence classes in $SO(3)/SO(3)_{\hat{\mathbf{e}}_3}$. Note that for any $A \in SO(3)_{\hat{\mathbf{e}}_3}$, the $\hat{\mathbf{e}}_3$-equivariance of $\mathbf{m}_0$ implies

$$\langle \mathbf{m}_{RA} - \mathbf{m}_0, \phi^{(i)} \rangle = \langle \mathbf{m}_{RA} - \mathbf{m}_{0A}, \phi^{(i)} \rangle = \langle \mathbf{m}_R - \mathbf{m}_0, \phi^{(i)}_{A^{-1}} \rangle = \langle \mathbf{m}_R - \mathbf{m}_0, \phi_v \rangle$$

for $v = A\hat{\mathbf{e}}_i$. Thus, $\mathbf{m}_{RA} - \mathbf{m}_0$ is orthogonal on the kernel of $\mathcal{H}$ iff $\mathbf{m}_R - \mathbf{m}_0$ is orthogonal on it. Instead of working with the full equivalence classes, note that

$$\mathbb{S}^2 \cong SO(3)/SO(2) \cong SO(3)/SO(3)_{\hat{\mathbf{e}}_3}$$

where for $\nu \in \mathbb{S}^2$, a representative of $[R(\nu)]$ is given by

$$\begin{pmatrix} | & | & | \\ \tau_1 & \tau_2 & \nu \\ | & | & | \end{pmatrix} \in SO(3)$$

with arbitrary $\tau_1, \tau_2 \in T_\nu \mathbb{S}^2$ and $\tau_1 \perp \tau_2$. To see this, choose $A \in SO(2)$ arbitrary. Then $(\tau_1\, \tau_2\, \nu) \begin{pmatrix} A & \\ & 1 \end{pmatrix} = (\tilde{\tau}_1\, \tilde{\tau}_2\, \nu)$ so that the structure is unchanged. Furthermore, any orthogonal pair of τ_1, τ_2 complementing ν to a basis can be found via rotation.

Expressing ν in stereographic coordinates, τ_1 and τ_2 may be chosen in such a way that differentiability near 0 with $\pi^{-1}(0) = \hat{\mathbf{e}}_3$ is ensured. For example, one might take

$$\nu = \begin{pmatrix} \frac{2y_1}{1+|y|^2} \\ \frac{2y_2}{1+|y|^2} \\ \frac{1-|y|^2}{1+|y|^2} \end{pmatrix}, \quad \tau_1 = \frac{1}{\sqrt{1 + 4y_1^2 - 2|y|^2 + 4|y|^4}} \begin{pmatrix} 1 - |y|^2 \\ 0 \\ -y_1 \end{pmatrix}$$

$$\text{and} \quad \tau_2 = \nu \times \tau_1 = \frac{1}{\sqrt{1 + 4y_1^2 - 2|y|^2 + 4|y|^4}(1 + |y|^2)} \begin{pmatrix} -4y_1 y_2 \\ (1 - |y|^2)^2 + 4y_1^2 \\ -2y_2(1 - |y|^2) \end{pmatrix}.$$

Having found a way to express the desired R as a point in $\mathbb{R}^2$, we are now in a place to consider the mapping

$$F\colon \mathbb{R}^2 \times H^1(\mathbb{S}^2; \mathbb{R}^3) \to \mathbb{R}^2,\ (y, \xi) \mapsto \sum_{i=1,2} \langle \xi_{R(y)} + \mathbf{m}_{0R(y)} - \mathbf{m}_0, \phi^{(i)} \rangle \hat{\mathbf{e}}_i$$

and apply the infinite dimensional implicit function Theorem [59, Theorem 4.E] in the vicinity of $(0, 0)$. Recall that $H^1(\mathbb{S}^2; \mathbb{R}^3)$ is a Banach space. Here, ξ represents $\mathbf{m} - \mathbf{m}_0 \in H^1(\mathbb{S}^2; \mathbb{R}^3)$ with

$$\langle \xi_{R(y)} + \mathbf{m}_{0R(y)} - \mathbf{m}_0, \phi^{(i)} \rangle = \langle \mathbf{m}_{R(y)} - \mathbf{m}_0, \phi^{(i)} \rangle$$

such that above calculations still apply.

In order to differentiate F at $(0, 0)$, note that $R(y)$ is differentiable for $|y| < 1$ with

$$\frac{\partial}{\partial y_1} R(y)|_{(0,0)} = \frac{d}{dt} R_2(t)|_{t=0} \quad \text{and} \quad \frac{\partial}{\partial y_2} R(y)|_{(0,0)} = \frac{d}{dt} R_1(t)|_{t=0}.$$

Therefore, $\frac{d}{dy_i}F_j|_{(y,\xi)=(0,0)} = \langle \phi^{(i)}, \phi^{(j)} \rangle_{L^2(\mathbb{S}^2;\mathbb{R}^3)}$. By Cauchy-Schwarz, the Jacobian

$$\langle \phi^{(1)}, \phi^{(1)} \rangle \langle \phi^{(2)}, \phi^{(2)} \rangle - \langle \phi^{(1)}, \phi^{(2)} \rangle^2 \geq 0$$

is 0 if and only if $\phi^{(1)}$ and $\phi^{(2)}$ are linearly dependent. But since $\phi^{(1)}(\hat{\mathbf{e}}_3)$ and $\phi^{(2)}(\hat{\mathbf{e}}_3)$ are orthogonal for a axisymmetric minimizer $\mathbf{m}_0$, this can not be the case and it follows that $\det \frac{\partial F}{\partial y}|_{(y,\xi)=(0,0)} > 0$.

The implicit function theorem now ensures the existence of $\rho > 0$ and a C^1-function $g \colon B_\rho(0) \subset H^1(\mathbb{S}^2;\mathbb{R}^3) \to B_R(0) \subset R^2$, $\xi \mapsto y(\xi)$ such that $F(y(\xi),\xi) = F(0,0) = 0$ for all $\xi \in B_\rho(0)$ and where $y(\xi)$ is the only solution of this equation.

Uniqueness and existence of y correspond to existence of a unique coset $[R] = [R(y(\xi)] \in SO(3)/SO(2)$ for every $\xi \in B_\rho(0)$ and thus for every $\mathbf{m} \in B_\rho(\mathbf{m}_0)$ such that

$$\langle \mathbf{m}_P - \mathbf{m}_0, \phi^{(i)} \rangle = 0 \text{ for all } P \in [R(y(\xi)] \text{ and } i = 1, 2.$$

□

Aiming to employ the energy identity for $\mathcal{E}(\mathbf{m}) - \mathcal{E}(\mathbf{m}_0) = \mathcal{E}(\mathbf{m}_R) - \mathcal{E}(\mathbf{m}_0)$, we show that the distance of $\mathbf{m}$ from $\mathbf{m}_0$ is not heavily affected by the joint rotation.

Lemma 4.12. *There exists a constant c such that*

$$\|\mathbf{m}_R - \mathbf{m}_0\|_{H^1(\mathbb{S}^2)} < c\|\mathbf{m} - \mathbf{m}_0\|_{H^1(\mathbb{S}^2)} < c\rho$$

for the jointly rotated field from Lemma 4.11.

Proof. In a first step, the implicit function theorem which was employed above to find $R = R(y) = R(y(\xi))$ provides an estimate for $|y(\xi)|$, using

$$|y(\xi)| = |y(\xi) - y(0) - Dy(0)\langle\xi\rangle| + |Dy(0)\langle\xi\rangle| \leq c\|\xi\|_{H^1} + \left| \left(\frac{\partial F}{\partial y}|_{(0,0)} \right)^{-1} \langle \frac{\partial F}{\partial \xi} \langle \xi \rangle \rangle \right|.$$

With $F(0,\cdot) \colon \xi \mapsto \sum_{i=1,2} \langle \xi, \phi^{(i)} \rangle_{L^2} \hat{\mathbf{e}}_i$ and $\frac{\partial F}{\partial y}|_{(0,0)} = \sum \langle \phi^{(i)}, \phi^{(j)} \rangle_{L^2} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j$, the second term can be estimated by $c(\phi^{(i)})\|\xi\|_{L^2}$ and in total, one finds $|y(\xi)| \leq c\|\xi\|_{H^1}$.

In a second step, we estimate $\|\mathbf{m}_R - \mathbf{m}_0\|_{H^1}$ in terms of $|y|$ and thus in terms of $\|\mathbf{m} - \mathbf{m}_0\|_{H^1}$. Simple computations show $|R(y)v - v| \leq c|y||v|$ for a universal constant c and any $v \in \mathbb{R}^3$. This can be employed to conclude

$$\|\mathbf{m}_R - \mathbf{m}_0\|_{H^1} \leq \|\mathbf{m}_R - \mathbf{m}_{0R}\|_{H^1} + \|\mathbf{m}_{0R} - \mathbf{m}_0\|_{H^1} \leq \|\mathbf{m} - \mathbf{m}_0\|_{H^1} + c|y| \leq c\|\mathbf{m} - \mathbf{m}_0\|_{H^1}$$

where the details of the last two estimates are given in the following. Due to the orthogonality of $R(y)$, the chain rule implies $\|\xi_R\|_{H^1} = \|\xi\|_{H^1}$ and the estimate for the first term follows. For the second term, write

$$\begin{aligned}\|\mathbf{m}_{0R} - \mathbf{m}_0\|_{L^2} &= \|R^{-1}\mathbf{m}_0(R\cdot) - \mathbf{m}_0(R\cdot)\|_{L^2} + \|\mathbf{m}_0(R\cdot) - \mathbf{m}_0\|_{L^2} \\ &\leq c|y|\|\mathbf{m}_0(R\cdot)\|_{L^2} + \left(\int_{\mathbb{S}^2}\left(\int_0^1 \frac{d}{dt}\mathbf{m}_0\left(\frac{x+t(Rx-x)}{|x+t(Rx-x)|}\right)\,dt\right)^2 dx\right)^{\frac{1}{2}} \\ &\leq c|y|\|\mathbf{m}_0\|_{L^2} + c|y|\|\mathbf{m}_0\|_{H^1} \leq c(\mathbf{m}_0)|y|.\end{aligned}$$

The derivative part is slightly more delicate since the inner rotation affects the derivative. In stereographic coordinates π_p centered at p with $R(y)p = p$ there exists $\tilde{R} \in SO(2)$ such that $R\pi_p(x) = \pi_p(\tilde{R}x)$ for all $x \in \mathbb{R}^2$. In these coordinates, we have

$$\begin{aligned}\|\partial_i(\mathbf{m}_{0R} \circ \pi_p) &- \partial_i(\mathbf{m}_0 \circ \pi_p)\|_{L^2(\mathbb{R}^2)} \\ &= \|\partial_i\left(R^{-1}(\mathbf{m}_0 \circ \pi_p)\tilde{R}\right) - \partial_i\left(\mathbf{m}_0 \circ \pi_p\right)\|_{L^2(\mathbb{R}^2)} \\ &\leq \|R^{-1}\partial_i\left((\mathbf{m}_0 \circ \pi_p)\tilde{R}\right) - R^{-1}\left(\partial_i(\mathbf{m}_0 \circ \pi_p)\right)\|_{L^2(\mathbb{R}^2)} \\ &\quad + \|R^{-1}(\partial_i(\mathbf{m}_0 \circ \pi_p)) - \partial_i(\mathbf{m}_0 \circ \pi_p)\|_{L^2(\mathbb{R}^2)}\end{aligned}$$

With $\partial_i(\mathbf{m}_0 \circ \pi_p) \in \mathbb{R}^3$, the second term can directly be estimated by $c(\mathbf{m}_0)|y|$. On the other hand, the first one can be simplified, using orthogonality of R and $\tilde{R}$. From here on writing $\tilde{\mathbf{m}}_0 = \mathbf{m}_0 \circ \pi_p$ and $\|\cdot\|_{L^2}$ for $\|\cdot\|_{L^2(\mathbb{R}^2)}$ we estimate, for $i = 1$,

$$\begin{aligned}\|\partial_1(\tilde{\mathbf{m}}_0\tilde{R}) - \partial_1(\tilde{\mathbf{m}}_0)\|_{L^2} &= \|\partial_1\tilde{\mathbf{m}}_0(\tilde{R}\cdot)\tilde{R}_{11} + \partial_2\tilde{\mathbf{m}}_0(\tilde{R}\cdot)\tilde{R}_{12} - \partial_1\tilde{\mathbf{m}}_0\|_{L^2} \\ \leq \|\partial_1\tilde{\mathbf{m}}_0(\tilde{R}\cdot)\left(\tilde{R}_{11} - 1\right)\|_{L^2} &+ \|\partial_2\tilde{\mathbf{m}}_0(\tilde{R}\cdot)\tilde{R}_{12}\|_{L^2} + \|\partial_1\tilde{\mathbf{m}}_0(\tilde{R}\cdot) - \partial_1\tilde{\mathbf{m}}_0\|_{L^2}.\end{aligned}$$

Testing $\tilde{R}$ with unit vectors, the individual components of $\tilde{R}$ can be estimated in terms of $|y|$, for example

$$\begin{aligned}|\tilde{R}_{11} - 1| &= \left|\left(\tilde{R}\begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix}\right)\cdot\hat{\mathbf{e}}_1\right| \leq \left|\tilde{R}\begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix}\right| \\ &= \left|\pi_m^{-1}\left(R\left(\pi_p\left(\begin{pmatrix}1\\0\end{pmatrix}\right)\right)\right) - \pi_p^{-1}\left(\pi_p\left(\begin{pmatrix}1\\0\end{pmatrix}\right)\right)\right| \leq Lc|y|\end{aligned}$$

where L is the Lipschitz constant of π_p^{-1} on $\pi_p\left(B_2(0)\right) \subset \mathbb{S}^2$. Similar computations show $|\tilde{R}_{12}| < c|y|$, $|\tilde{R}_{21}| < c|y|$ and $|\tilde{R}_{22} - 1| < c|y|$. Using these estimates, it is obvious that the first two terms in the L^2-estimate of $\partial_1\tilde{m}$ can be bounded by $c|y|\|\mathbf{m}_0\|_{H^1(\mathbb{S}^2)}$ while the third term again involves an extra derivative. Recalling that $\mathbf{m}_0$ is smooth on the compact manifold $\mathbb{S}^2$, we can estimate

$$\|\partial_1\tilde{\mathbf{m}}_0(\tilde{R}\cdot) - \partial_1\tilde{\mathbf{m}}_0\|_{L^2}^2 \leq \left(\int_{\mathbb{R}^2}\int_0^1 \left|\frac{\mathrm{d}}{\mathrm{d}t}\partial_1\mathbf{m}_0(x+t(\tilde{R}x-x))\right|^2 \mathrm{d}t\,\mathrm{d}x\right)^{\frac{1}{2}}$$
$$\leq \left(\int_{\mathbb{R}^2} |D^2\tilde{\mathbf{m}}_0|^2|x|^2|y|^2\;dx\right)^{\frac{1}{2}} \leq |y|\|\mathbf{m}_0\|_{H^2(\mathbb{S}^2)}.$$

Note that $\|\mathbf{m}_0\|_{H^2(\mathbb{S}^2)}$ only adds to the constant c while $|y|$ is relevant for the estimate of $\|\mathbf{m}_R - \mathbf{m}_0\|_{H^1}$ in terms of $\|\mathbf{m} - \mathbf{m}_0\|_{H^1}$. Repeating the same estimates for $\partial_2\tilde{\mathbf{m}}_0$ completes the proof. □

Having collected all necessary estimates, we can now express the energy difference via the Hessian and proceed further to show the following:

Proposition 4.3. *Assume that* $\mathbf{m}_0$ *minimizes* $\mathcal{E}$ *in the set of axisymmetric fields of degree* 0*,* $\{\mathbf{m} \in H^1(\mathbb{S}^2;\mathbb{S}^2) : Q(\mathbf{m}) = 0,\ \mathbf{m}$ *axisymmetric*$\}$ *and that* E *is strictly convex at* θ*. Then* $\mathbf{m}_0$ *locally minimizes* $\mathcal{E}$ *in* $\{\mathbf{m} \in H^1(\mathbb{S}^2;\mathbb{S}^2)\ :\ Q(\mathbf{m}) = 0\}$*. In particular, there are constants* $\varepsilon_0 > 0$ *and* $c > 0$ *such that*

$$\mathcal{E}(\mathbf{m}) - \mathcal{E}(\mathbf{m}_0) \geq c \inf_{[R]\in SO(3)/SO(3)_{\hat{\mathbf{e}}_3}} \|\mathbf{m}_R - \mathbf{m}_0\|_{H^1}$$

for all $\|\mathbf{m} - \mathbf{m}_0\|_{H^1} < \varepsilon : 0$*.*

Proof. Due to the assumption of convexity, the kernel of $\mathcal{H}$ consists exactly of the tangent fields associated to joint rotations. Given $\mathbf{m}$ close to $\mathbf{m}_0$, Lemma 4.11 implies the existence of $R \in SO(3)$ such that $\mathbf{m}_R - \mathbf{m}_0$ is orthogonal to the kernel of $\mathcal{H}$. For this R we have

$$\mathcal{E}(\mathbf{m}) - \mathcal{E}(\mathbf{m}_0) = \mathcal{E}(\mathbf{m}_R) - \mathcal{E}(\mathbf{m}_0) = \tilde{\mathcal{H}}(\mathbf{m}_R - \mathbf{m}_0).$$

Writing $\xi = \mathbf{m}_R - \mathbf{m}$, we decompose $\xi = \xi^T + \xi^\perp$ into a tangential part ξ^T and a normal component $\xi^\perp$. Due to the bilinearity of $\mathcal{H}$, we can write

$$\tilde{\mathcal{H}}(\xi) = \mathcal{H}(\xi^T) + 2\tilde{\mathcal{H}}(\xi^T, \xi^\perp) + \tilde{\mathcal{H}}(\xi^\perp)$$

and estimate the three terms separately. For the first one, orthogonality of ξ with respect to $\phi^{(i)}$ implies orthogonality of ξ^T and the lower bound of Lemma 4.10 implies

$$\mathcal{H}(\xi^T) \geq g\|\xi^T\|_{H^1}^2.$$

Before analysing terms that involve the normal component $\xi^\perp$, note that

$$\xi^\perp = (\mathbf{m}_0 \cdot \xi)\mathbf{m}_0 = -\frac{1}{2}|\xi|^2\mathbf{m}_0$$

due to $|\xi|^2 = |\mathbf{m}|^2 + |\mathbf{m}_0|^2 - 2\mathbf{m}\cdot\mathbf{m}_0 = 2|\mathbf{m}_0|^2 - 2\mathbf{m}\cdot\mathbf{m}_0 = -2\xi\cdot\mathbf{m}$. In particular, this implies that $4|\xi^\perp|^2 = |\xi|^4$. For $\tilde{\mathcal{H}}(\xi^\perp)$, we compute

$$\begin{aligned}\tilde{\mathcal{H}}(\xi^\perp) &= \int_{\mathbb{S}^2} |\nabla\xi^\perp|^2 - \kappa(\nu\cdot\xi^\perp)^2 - |\xi^\perp|^2\left(|\nabla\mathbf{m}_0|^2 - \kappa(\mathbf{m}_0\cdot\nu)^2\right)\,\mathrm{d}\sigma\\ &= \int_{\mathbb{S}^2} |\nabla\xi^\perp|^2 - \frac{\kappa}{4}|\xi|^4 - \frac{1}{4}|\xi|^4\left(|\nabla\mathbf{m}_0|^2 - \kappa(\mathbf{m}_0\cdot\nu)^2\right)\,\mathrm{d}\sigma\\ &= \int_{\mathbb{S}^2} |\nabla\xi^\perp|^2 - \frac{1}{4}|\xi|^4|\nabla\mathbf{m}_0|^2\,\mathrm{d}\sigma\\ &= \|\xi^\perp\|_{H^1}^2 - c_1\|\xi\|_{L^4}^4.\end{aligned}$$

Finally, for the mixed term, we estimate

$$\begin{aligned}|\nabla\xi^T : \nabla\xi^\perp| &= \left|\nabla\left(\frac{1}{2}|\xi|^2\mathbf{m}_0\right) : \nabla\left(\mathbf{m}_0\times(\mathbf{m}_0\times\xi)\right)\right|\\ &= \left|\left((\xi\cdot\nabla\xi)\mathbf{m}_0 + \frac{1}{2}|\xi|^2\nabla\mathbf{m}_0\right) : \left(\nabla\xi - \nabla(\mathbf{m}_0\cdot\xi)\mathbf{m}_0 - (\mathbf{m}_0\cdot\xi)\nabla\mathbf{m}_0\right)\right|\\ &= \left|\sum_{i,j}(\xi\cdot\partial_i\xi)(\mathbf{m}_0\cdot\partial_j\xi) - (\xi\cdot\partial_i\xi)(\partial_j\mathbf{m}_0\cdot\xi) - (\xi\cdot\partial_i\xi)(\mathbf{m}_0\cdot\partial_j\xi)\right.\\ &\qquad\left. + \frac{1}{2}|\xi|^2(\partial_i\mathbf{m}_0\cdot\partial_j\xi) - \frac{1}{2}|\xi|^2(\mathbf{m}_0\cdot\xi)(\partial_i\mathbf{m}_0\cdot\partial_i\mathbf{m}_0\right|\\ &\lesssim \frac{3}{2}|\xi|^2|\nabla\xi||\nabla\mathbf{m}_0| + \frac{1}{2}|\xi|^3|\nabla\mathbf{m}_0|.\end{aligned}$$

Furthermore, we can estimate $|\xi^T| \le |\xi|$ and hence, noting that $\xi^T\cdot\xi^\perp = 0$, we find

$$\begin{aligned}\tilde{\mathcal{H}}(\xi^T,\xi^\perp) &= \int_{\mathbb{S}^2} \nabla\xi^T : \nabla\xi^\perp - \kappa(\nu\cdot\xi^T)(\nu\cdot\xi^\perp) - (\xi^T\cdot\xi^\perp)\left(|\nabla\mathbf{m}_0|^2 - \kappa(\mathbf{m}_0\cdot\nu)^2\right)\,\mathrm{d}\sigma\\ &\gtrsim -\int_{\mathbb{S}^2} |\xi|^2|\nabla\xi||\nabla\mathbf{m}_0| + |\xi|^2|\nabla\mathbf{m}_0| + \kappa|\xi|^3\,\mathrm{d}\sigma\\ &\gtrsim -\|\xi\|_{L^4}^2\|\nabla\xi\|_{L^2} - \|\nabla\mathbf{m}_0\|_{L^8}^{\frac{1}{2}}\|\xi\|_{L^4}^3 - \|\xi\|_{L^4}^2\|\xi\|_{L^2}^2\\ &\ge c_2\|\xi\|_{H^1}^3.\end{aligned}$$

Choosing ε small enough with $\|\xi\|_{H^1} \le c\|\mathbf{m}-\mathbf{m}_0\|_{H^1} < c\varepsilon$, the energy difference is bounded from below by

$$\begin{aligned}\mathcal{E}(\mathbf{m}) - \mathcal{E}(\mathbf{m}_0) &\ge \frac{1}{2}\left(\lambda\|\xi^T\|_{H^1}^2 + \|\xi^\perp\|_{H^1}^2 - c_1\|\xi\|_{H^1}^4 - c_2\|\xi\|_{H^1}^3\right)\\ &\ge \frac{1}{2}\left(\min\{\lambda,1\} - c_1\varepsilon^2 - c_2\varepsilon\right)\|\xi\|_{H^1}^2.\end{aligned}$$

□

5. Non-Symmetric Solutions of the Landau Lifshitz Equation

We now turn to the dynamics of Skyrmions on the sphere, which are governed by the Landau-Lifshitz equation

$$\partial_t \mathbf{m}(t) = \mathbf{m} \times (\Delta \mathbf{m} + \kappa(\mathbf{m} \cdot \nu)\nu)\,.$$

The axisymmetric critical points that were the subject of chapters two and three yield constant solutions of the equation. By jointly rotating them around $\hat{\mathbf{e}}_3$, they are formally time-dependent but actually constant due to their symmetry. On the other hand, invariance of the energy under joint rotations implies that any time-dependent solution constructed by rotating a specific profile automatically obeys the conservation of energy that is inherent in the equation. In order to yield non static solutions, such profiles should be non-equivariant and rotate at a non-zero speed. They can not be critical points of the energy because the right hand side of the equation also is invariant under joint rotations and thus

$$\mathbf{m} \times (\Delta \mathbf{m} + \kappa(\mathbf{m} \cdot \nu)\nu) = 0$$

would be preserved over time.

5.1. Rotating Solutions

In [37], the authors proved existence of constrained minimizers where the constraint is a means to control equivariance of the fields. These minimizers are then used to construct periodic solutions of the Landau-Lifshitz equation that are rotating at a certain speed $\omega \in \mathbb{R}$. However, while attainment of the energy minimum under the constraint was proven, there was no rigorous treatment of ω as Lagrange multiplier. In this section, we shall use the infinite dimensional Lagrange multiplier theorem from [59] to rigorously establish that constrained minimizers solve the equation

$$-\mathbf{m} \times \nabla \mathcal{E} = \sum_{i=1}^{3} \omega_i (-\mathbf{m} \times \nabla J_i)$$

with $\omega = (0, 0, \omega)$ in a special case. We will then apply the Łojasiewicz inequality proved in section 5.2 to prove $\omega \neq 0$, thus ruling out the possibility of constant in time solutions.

The final step will be done under the assumption that all minimizers are equivariant. This is conjectured to be true for large enough κ as increasing anisotropy forces $\mathbf{m}$ to align with the normal at least in the azimuthal direction.

The rest of the section is organized as follows. To start, we will briefly collect the main results of [37] and extend them to accomodate an arbitrary axis of rotation. Then we will prove that constrained minimizers fulfill the equation mentioned above and use the equation to extend the regularity result of Theorem 2 to constrained critical points. By a similar method, a convergence result for critical points will be shown. Finally, $\omega \neq 0$ follows by contradiction, due to the following Proposition:

Proposition 5.1. *Assume that $\mathbf{m}_0$ is a critical point of $\mathcal{E}$. Then there exist $\rho > 0, c > 0$ and $\gamma \in \left(0, \frac{1}{2}\right)$ such that $\|\mathbf{m} - \mathbf{m}_0\|_{H^2} < \rho$ implies*

$$\|(\nabla\mathcal{E}(\mathbf{m}))^{tan}\|_{L^2} \geq c|\mathcal{E}(\mathbf{m}) - \mathcal{E}(\mathbf{m}_0)|^{1-\gamma}.$$

The Proposition will be proved in section 5.2.

5.1.1. Preliminary Results

The key observation of [37] is that if $R(t) \in SO(3)_{\hat{\mathbf{e}}_3}$ is a rotation of angle t around $\hat{\mathbf{e}}_3$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{m}_{R(\omega t)} = \omega\left(-\mathbf{m} \times \nabla \mathbf{J}_3(\mathbf{m})\right)_{R(\omega t)}$$

such that static solutions of the equation

$$-\mathbf{m} \times \nabla\mathcal{E} = \omega\left(-\mathbf{m} \times \nabla J_3\right) \tag{5.1}$$

yield solutions of the Landau Lifshitz equation by setting $\mathbf{m}(x,t) = \mathbf{m}(x)_{R(\omega t)}$.

Here, $\mathbf{J} = \mathbf{S} + \mathbf{L}$ is the total angular momentum, where

$$\mathbf{S}(\mathbf{m}) = \int_{\mathbb{S}^2} \mathbf{m}\,\mathrm{d}\sigma \in \mathbb{R}^3 \quad \text{and} \quad \mathbf{L}(\mathbf{m}) = \int_{\mathbb{S}^2} \nu(\mathbf{m}^*\omega_{\mathbb{S}^2}) \in \mathbb{R}^3$$

are the spin and orbital angular momentum, respectively. For $\hat{\mathbf{e}}$-equivariant fields, the total angular momentum is determined by the values at $\pm\hat{\mathbf{e}}$.

Lemma 5.1. *Given $\hat{\mathbf{e}} \in \mathbb{S}^2$, assume that $\mathbf{m}\colon \mathbb{S}^2 \to \mathbb{S}^2$ is $\hat{\mathbf{e}}$-equivariant and continuous near $\pm\hat{\mathbf{e}}$. Then*

$$\mathbf{J}(\mathbf{m}) = 2\pi\left(\mathbf{m}(\hat{\mathbf{e}}) + \mathbf{m}(-\hat{\mathbf{e}})\right).$$

In particular, $\mathbf{J}(\mathbf{m}) = \pm 4\pi\hat{\mathbf{e}}$ if $Q(\mathbf{m}) = 0$ and $\mathbf{J}(\mathbf{m}) = 0$ if $Q(\mathbf{m}) = \pm 1$.

Proof. For $\hat{\mathbf{e}} = \hat{\mathbf{e}}_3$, this is the statement of Lemma 5 in [37]. For a general direction, the identity follows from the fact that

$$\mathbf{J}(\mathbf{m}_R) = R\mathbf{J}(\mathbf{m}) \text{ for all } R \in SO(3). \tag{5.2}$$

which is Lemma 1 of [37]. □

In order to find solutions to the static equation mentioned above, we will consider the constrained minimization problem $\mathcal{E}(\mathbf{m}) \to \min$ for $\mathbf{J}(\mathbf{m}) = \mathbf{J}_0$. For $\mathbf{J}_0 = J_3\hat{\mathbf{e}}_3$, attainment of the constrained minimum has been proven in [37] as well as an upper bound for the minimal energy of 8π. For general $\mathbf{J_0}$, these statements follow from (5.2) and the invariance of $\mathcal{E}$ under joint rotations. To prove that these constrained minimizers indeed yield a solution of (5.1), we will apply the following Theorem about Lagrange multipliers from Zeidler [59]:

Theorem
– from [59], p. 270, Proposition 1
Let $f\colon U(u) \subset X \to \mathbb{R}$ and $G\colon U(u) \subset X \to Y$ be C^1 on an open neighborhood of u, where X and Y are real Banach spaces. Suppose that u is a solution of

$$\begin{aligned} f(u) &\to \min! \\ G(u) &= 0 \end{aligned}$$

where $G'(u)\colon X \to Y$ is surjective. Then there exists a functional $\lambda \in Y^*$ such that

$$f'(u) + \lambda G'(u) = 0$$

holds true.

Aiming to work on Banach spaces, recall from section 2.1 that $H^1(\mathbb{S}^2;\mathbb{S}^2)$ with the norm $\|\cdot\|_{H^1}+\|\cdot\|_{L^\infty}$ is a Banach manifold which we called $X^1(\mathbb{S}^2;\mathbb{S}^2)$. Near a given $\mathbf{m}$, it can be parametrized over the Banach space $X^1(\mathbb{S}^2;T_{\mathbf{m}}\mathbb{S}^2)$ via $\psi\colon X^1(\mathbb{S}^2;T_{\mathbf{m}}\mathbb{S}^2) \to X^1(\mathbb{S}^2;\mathbb{S}^2)$. Hence we set $f = \mathcal{E} \circ \psi$ and $G = \mathbf{J} \circ \psi - \mathbf{J_0}$. In order to confirm the conditions for G' we henceforth compute the variation of $\mathbf{J}$.

In [37], the variation of $\mathbf{J}$ was only computed for the case $\mathbf{J} = J_3\hat{\mathbf{e}}_3$ and in stereographic coordinates. We introduce the angular derivative around an axis $\hat{\mathbf{e}} \in \mathbb{S}^2$ and note several of its properties in order to compute the variation in the general case and in a coordinate-free expression. Apart from resulting in the surjectivity of $(\mathbf{J} \circ \psi)'$, this will give a characterization of equivariance.

Definition 5. *The angular derivative $\partial_\chi\phi$ of $\phi \in C^\infty(\mathbb{S}^2;\mathbb{R}^3)$ around $\hat{\mathbf{e}}$ is given by*

$$\partial_\chi\phi(\mathbf{x}) = \frac{d}{dt}\phi(R_t\mathbf{x}).$$

Lemma 5.2 (Properties of the angular derivative)**.** *Consider an arbitrary axis* $\hat{\mathbf{e}} \in \mathbb{S}^2$.

(1) Let $\phi \in C^\infty(\mathbb{S}^2;\mathbb{R}^3)$ *and let* R_t *be a rotation around* $\hat{\mathbf{e}}$ *by angle* t*. Then,*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\phi)_{R(t)} = \hat{\mathbf{e}} \times \phi - \partial_\chi \phi.$$

(2) For smooth ϕ*, the angular derivative commutes with joint rotations.*

(3) The angular derivative is $L^2(\mathbb{S}^2;\mathbb{R}^3)$ *skew-adjoint.*

(4) $\mathbf{m} \in H^1(\mathbb{S}^2;\mathbb{R}^3)$ *is* $\hat{\mathbf{e}}$*-equivariant iff* $\partial_\chi \mathbf{m} = \hat{\mathbf{e}} \times \mathbf{m}$ *weakly in* $L^2(\mathbb{S}^2;\mathbb{R}^3)$.

Proof. (1) This follows pointwisely from direct computations and the definition of the angular derivative.

(2) Let $R \in SO(3)$ be an arbitrary rotation. Then

$$R\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} R_t\mathbf{x} = R\left(\hat{\mathbf{e}} \times \mathbf{x}\right) = \hat{\mathbf{e}} \times (Rx) = \frac{\mathrm{d}}{\mathrm{d}t}R_t(Rx)$$

and therefore

$$\partial_\chi \phi_R(x) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\phi_R(R_t\mathbf{x}) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}R\phi(R^{-1}R_t\mathbf{x}) = R\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\phi(R_t(R^{-1}\mathbf{x})).$$

(3) Again, this follows by a direct computation and from orthogonality of R.

$$\begin{aligned}\langle \partial_\chi \mathbf{m}, v\rangle_{L^2} &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\langle \mathbf{m}(R_t\mathbf{x}), v\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\langle \mathbf{m}(R_{-t}R_t(\mathbf{x}), v(R_{-t}\mathbf{x})\rangle \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\langle \mathbf{m}, v(R_{-t}\mathbf{x})\rangle = -\langle \mathbf{m}, \partial_\chi v\rangle.\end{aligned}$$

(4) Recall that $\mathbf{m}$ is $\hat{\mathbf{e}}$-equivariant if $\mathbf{m}_R = \mathbf{m}$ for all $R \in SO(3)_{\hat{\mathbf{e}}}$. In particular, $\mathbf{m}$ is equivariant iff the map $\mathbf{m} \mapsto \mathbf{m}_{R_t}$ is constant for R_t representing a rotation around $\hat{\mathbf{e}}$ by angle t. Testing with $\phi \in C^\infty(\mathbb{S}^2;\mathbb{R}^3)$, $\mathbf{m}$ is equivariant, iff

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{m}_{R_t}, \phi\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{m}, \phi_{R_{-t}}\rangle$$

for all such ϕ. Note that the second equality follows from the orthogonality of R as in (3). Together with the previous properties, this implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{m}_{R_t}, \phi\rangle = \langle \mathbf{m}, -\hat{\mathbf{e}} \times \phi + \partial_\chi \phi\rangle = \langle \hat{\mathbf{e}} \times \mathbf{m} - \partial_\chi \mathbf{m}, \phi\rangle$$

and therefore $\mathbf{m}$ is equivariant iff the last term in the equation is 0 for all smooth ϕ i.e. if $\partial_\chi = \hat{\mathbf{e}} \times \mathbf{m}$ weakly.

□

Using these properties, we can analyze the variation of $\mathbf{J}$.

Lemma 5.3. *Given* $\hat{\mathbf{e}} \in \mathbb{S}^2$, $\mathbf{m} \in H^1(\mathbb{S}^2;\mathbb{S}^2)$ *and* $v \in X^1(\mathbb{S}^2; T_{\mathbf{m}}\mathbb{S}^2)$, *the following holds:*

(1) $\hat{\mathbf{e}} \cdot (\delta\mathbf{J}(\mathbf{m})\langle v\rangle) = \delta\left(\hat{\mathbf{e}} \cdot \mathbf{J}\right)(\mathbf{m})\langle v\rangle = \langle \hat{\mathbf{e}} - \mathbf{m} \times \partial_\chi \mathbf{m}, v\rangle_{L^2}$.

(2) $\mathbf{m}$ *is equivariant around* $\hat{\mathbf{e}}$ *iff* $\delta\left(\hat{\mathbf{e}} \cdot \mathbf{J}\right)\langle v\rangle = 0$ *for all* $v \in X^1(\mathbb{S}^2; T_{\mathbf{m}}\mathbb{S}^2)$.

(3) If $\mathbf{m}$ *is not* $\hat{\mathbf{e}}$*-equivariant for any axis* $\hat{\mathbf{e}}$ *then the angular momentum* $J\colon \mathbb{S}^2 \to \mathbb{R}^3$ *has a surjective differential*

$$\delta\mathbf{J}(\mathbf{m})\colon X^1(\mathbb{S}^2; T_{\mathbf{m}}\mathbb{S}^2) \to \mathbb{R}^3.$$

Proof. (1) For $\hat{\mathbf{e}} = \hat{\mathbf{e}}_3$, the variation of J_3 was computed in [37]. By (5.2), this extends to the general case, using $\hat{\mathbf{e}} \cdot \mathbf{J} = (R\hat{\mathbf{e}}) \cdot (R\mathbf{J})$.

(2) From Lemma 5.2 above it follows that $\mathbf{m}$ is $\hat{\mathbf{e}}$-equivariant iff $\hat{\mathbf{e}} \times \mathbf{m} - \partial_\chi \mathbf{m} = 0$ weakly in $L^2(\mathbb{S}^2;\mathbb{R}^3)$. For arbitrary $\phi \in C^\infty(\mathbb{S}^2;\mathbb{R}^3)$, evaluate $\delta(\hat{\mathbf{e}} \cdot \mathbf{J})$ at $\mathbf{m} \times \phi$ to find that

$$\delta(\hat{\mathbf{e}} \cdot \mathbf{J})(\mathbf{m})\langle \mathbf{m} \times \phi\rangle = \langle \hat{\mathbf{e}} - \mathbf{m} \times \partial_\chi \mathbf{m}, \mathbf{m} \times \phi\rangle_{L^2} = \langle \hat{\mathbf{e}} \times \mathbf{m} - \partial_\chi \mathbf{m}, \phi\rangle_{L^2}.$$

Since any v can be smoothly approximated by tangent fields $v_k = \mathbf{m} \times \phi_k$, this implies the equivalence.

(3) Assume that $\mathbf{m}$ is not $\hat{\mathbf{e}}$-equivariant for any axis $\hat{\mathbf{e}}$. Then for all $\hat{\mathbf{e}} \in \mathbb{S}^2$ there exists $v^* \in X^1(\mathbb{S}^2; T_{\mathbf{m}}\mathbb{S}^2)$ such that

$$\hat{\mathbf{e}} \cdot (\delta\mathbf{J}(\mathbf{m})\langle v^*\rangle) = \alpha \neq 0.$$

Given $\mathbf{c} \in \mathbb{R}^3$, let $c \in [0,\infty)$ and $\hat{\mathbf{e}} \in \mathbb{S}^2$ be such that $\mathbf{c} = c\hat{\mathbf{e}}$. Then

$$\delta\mathbf{J}(\mathbf{m})\langle v\rangle = \mathbf{c} \text{ iff } \delta(\hat{\mathbf{e}} \cdot \mathbf{J})(\mathbf{m})\langle v\rangle = \langle \hat{\mathbf{e}} - \mathbf{m} \times \partial_\chi \mathbf{m}, v\rangle_{L^2} = c.$$

Recalling v^* from above, set $v_c = \frac{c}{|\alpha|} v^*$. For v_c, it holds that

$$\hat{\mathbf{e}} \cdot (\delta\mathbf{J}(\mathbf{m})\langle v_c\rangle) = \frac{c}{\alpha}\alpha = c$$

and thus $\delta\mathbf{J}(\mathbf{m})\langle v_c\rangle = \mathbf{c}$.

□

5.1.2. Lagrange Multiplier

We have now gathered all ingredients to prove that the constrained minimizers found in [37] satisfy the stationary equation (5.1) for some $\omega \in \mathbb{R}$.

Theorem 4. *For $\varepsilon > 0$ and a local minimizer $\mathbf{m} \in H^1(\mathbb{S}^2; \mathbb{S}^2)$ of $E = E(\mathbf{m})$ subject to $\mathbf{J}(\mathbf{m}) = -(4\pi+\varepsilon)\hat{\mathbf{e}}_3$ it holds that $\mathbf{m} \in C^\infty(\mathbb{S}^2; \mathbb{S}^2)$, and there exists a Lagrange multiplier $\omega \in \mathbb{R}$ such that $\{\mathbf{m}, E\} = \omega\{\mathbf{m}, J_3\}$.*

Proof. We apply the Lagrange multiplier theorem from [59] to the pulled back functionals

$$F = \mathcal{E} \circ \psi \quad \text{and} \quad \mathbf{G} = \mathbf{J} \circ \psi$$

on the closed subspace $X^1(\mathbb{S}^2; T_{\mathbf{m}}\mathbb{S}^2)$ of the Banach space $X^1(\mathbb{S}^2; \mathbb{R}^3)$. A constrained minimizer is given by $u_0 = 0$ with $\psi(0) = \mathbf{m}$. Then

$$DF(0)\langle v\rangle = \delta E(\mathbf{m})\langle v\rangle \quad \text{and} \quad D\mathbf{G}(0)\langle v\rangle = \delta\mathbf{J}(\mathbf{m})\langle v\rangle \quad \text{for } v \in X^1(\mathbb{S}^2; T_{\mathbf{m}}\mathbb{S}^2)$$

such that the differential $DG(0)\colon X^1(\mathbb{S}^2; T_{\mathbf{m}}\mathbb{S}^2) \to \mathbb{R}^3$ is surjective. It follows that there exists a Lagrange multiplier $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ such that

$$\delta E(\mathbf{m})\langle v\rangle = \sum_{j=1}^{3} \omega_j \delta J_j(\mathbf{m})\langle v\rangle.$$

Smoothness of $\mathbf{m}$ follows by the same methods as for non-constrained critical points. More details are given in Lemma 5.5 below. It remains to show that $\omega_1 = \omega_2 = 0$. Due to the smoothness, we may now write the Euler-Lagrange equations in its strong form as

$$\mathbf{m} \times \nabla E(\mathbf{m}) = \sum_{j=1}^{3} \omega_j \, \mathbf{m} \times \nabla J_j(\mathbf{m}).$$

Hence for every $1 \le i \le 3$, testing with ∇J_i gives

$$\langle \nabla J_i(\mathbf{m}), \mathbf{m} \times \nabla E(\mathbf{m})\rangle_{L^2} = \sum_{j=1}^{3} \omega_j \, \langle \nabla J_i(\mathbf{m}), \mathbf{m} \times \nabla J_j(\mathbf{m})\rangle_{L^2}$$

which may be written as

$$\{J_i, E\}(\mathbf{m}) = \sum_{j=1}^{3} \omega_j \{J_i, J_j\}(\mathbf{m})$$

by utilizing the poisson bracket defined in the introduction. As $\{J_i, J_j\} = \epsilon_{ijk} J_k$ and $\{J_i, \mathcal{E}\} = 0$ by Lemma 5.4 below it follows that

$$0 = \sum_{j,k=1}^{3} \epsilon_{ijk} \omega_j J_k(\mathbf{m})$$

so $\omega \times \mathbf{J}(\mathbf{m}) = 0$. But as $\mathbf{J}(\mathbf{m}) = J\hat{\mathbf{e}}_3 \neq 0$ it follows that $\omega_1 = \omega_2 = 0$. ☐

The following Lemmata justify some of the computations of the last proof.

Lemma 5.4.

(1) $\{J_i, J_j\} = \epsilon_{ijk} J_k$ *on* $X^2(\mathbb{S}^2; \mathbb{S}^2)$.

(2) $\{\mathcal{E}, J_k\} = 0$ *on* $X^2(\mathbb{S}^2; \mathbb{S}^2)$ *for* $k = 1, 2, 3$.

Proof. For the identity of $\mathbf{J}$, we first decompose the Poisson bracket into individual terms.

$$\{J_i, J_j\} = \{S_i, S_j\} + \{S_i, L_j\} + \{L_i, S_j\} + \{L_i, L_j\}.$$

Both of the mixed terms vanish due to

$$\{S_i, L_j\} = -\langle \hat{\mathbf{e}}_i, \partial_\chi^j \mathbf{m}\rangle_{L^2} = \langle \partial_\chi^j \hat{\mathbf{e}}_i, \mathbf{m}\rangle_{L^2} = 0,$$

where ∂_χ^j is the angular derivative around $\hat{\mathbf{e}}_j$. Then the final identity follows from general properties of the poisson bracket, see [37].

For the second identity, write $\mathcal{E} = D + \kappa A$ and $J_k = S_k + L_k$. We investigate each term separately using he expression for $\delta J_k = \delta(\mathbf{J} \cdot \hat{\mathbf{e}})$ from Lemma 5.3. By comparison with [37], $\delta S_k(\mathbf{m})\langle v\rangle = \langle \hat{\mathbf{e}}_k, v\rangle_{L^2}$ and $\delta L_k(\mathbf{m})\langle v\rangle = \langle -\mathbf{m} \times \partial_\chi \mathbf{m}, v\rangle_{L^2}$ where the angular derivative is around $\hat{\mathbf{e}}_k$. Hence we compute:

$$\begin{aligned}\{D, S_k\}(\mathbf{m}) &= \omega(X_D, X_{S_k})(\mathbf{m}) = \langle \mathbf{m} \times \Delta\mathbf{m}, \hat{\mathbf{e}}_k\rangle \\ &= \langle \nabla \cdot (\mathbf{m} \times \nabla\mathbf{m}), \hat{\mathbf{e}}_k\rangle = -\langle \mathbf{m} \times \nabla\mathbf{m}, \nabla\hat{\mathbf{e}}_k\rangle = 0.\end{aligned}$$

In stereographic coordinates centered at $\hat{\mathbf{e}}_k$, the angular derivative around $\hat{\mathbf{e}}_k$ is given by $\partial_\chi \mathbf{m} = (x^\perp \cdot \nabla)\mathbf{m} = x_1\partial_2\mathbf{m} - x_2\partial_1\mathbf{m}$ where $x = (x_1, x_2) \in \mathbb{R}^2$. Computing

$$\begin{aligned}\Delta\left((x^\perp \cdot \nabla)\mathbf{m}\right) &= ((\Delta x^\perp) \cdot \nabla)\mathbf{m} + 2\sum_{k=\{1,2\}} \partial_k x^\perp \cdot \nabla\partial_k\mathbf{m} + x^\perp \cdot \nabla\Delta\mathbf{m} \\ &= 0 + 2\left(-\partial_1\partial_2\mathbf{m} + \partial_2\partial_1\mathbf{m}\right) + (x^\perp \cdot \nabla)\Delta\mathbf{m}\end{aligned}$$

we confirm that the angular derivative and the laplace operator commute. Thus, for smooth $\mathbf{m}$,

$$\begin{aligned}\{D, L_k\}(\mathbf{m}) &= \langle \Delta\mathbf{m}, -\mathbf{m} \times (\mathbf{m} \times \partial_\chi\mathbf{m})\rangle \\ &= \langle \Delta\mathbf{m}, \partial_\chi\mathbf{m}\rangle = \langle \mathbf{m}, \Delta\partial_\chi\mathbf{m}\rangle = \langle \mathbf{m}, \partial_\chi\Delta\mathbf{m}\rangle = -\langle \partial_\chi\mathbf{m}, \Delta\mathbf{m}\rangle.\end{aligned}$$

On the other hand, it follows from symmetry of the scalar product that

$$\{D, L_k\}(\mathbf{m}) = \langle \Delta\mathbf{m}, \partial_\chi\mathbf{m}\rangle = \langle \partial_\chi\mathbf{m}, \Delta\mathbf{m}\rangle.$$

Thus, $\{D, L_k\} = -\{D, L_k\} = 0$. For $\mathbf{m} \in X^2(\mathbb{S}^2; \mathbb{S}^2)$, the statement follows by approximation.

For the anisotropy we make use of the $\hat{\mathbf{e}}$-equivariance of ν for every axis ν, implying $\partial_\chi \nu = \hat{\mathbf{e}}_k \times \nu$ weakly in L^2. Thus, for smooth $\mathbf{m} \in X^2(\mathbb{S}^2; \mathbb{S}^2)$,

$$\begin{aligned}
\{A, J_k\}(\mathbf{m}) = \omega(X_A, X_{J_k}) &= \langle \mathbf{m} \times (\mathbf{m} \cdot \nu)\nu, \hat{\mathbf{e}}_k - \mathbf{m} \times \partial_\chi \mathbf{m} \rangle \\
&= \langle (\mathbf{m} \cdot \nu)\mathbf{m}, \nu \times \hat{\mathbf{e}}_k \rangle - \langle (\mathbf{m} \cdot \nu)\nu, \partial_\chi \mathbf{m} \rangle \\
&= \langle (\mathbf{m} \cdot \nu)\mathbf{m}, \nu \times \hat{\mathbf{e}}_k \rangle + \langle (\partial_\chi (\mathbf{m} \cdot \nu))\, \nu, \mathbf{m} \rangle + \langle (\mathbf{m} \cdot \nu)\partial_\chi \nu, \mathbf{m} \rangle \\
&= \langle (\mathbf{m} \cdot \nu)\mathbf{m}, -\hat{\mathbf{e}}_k \times \nu + \partial_\chi \nu \rangle + \langle \partial_\chi (\mathbf{m} \cdot \nu)\nu, \mathbf{m} \rangle \\
&= \langle (\partial_\chi (\mathbf{m} \cdot \nu))\, \nu, \mathbf{m} \rangle_{L^2(\mathbb{S}^2; \mathbb{R}^3)} \\
&= \langle \partial_\chi (\mathbf{m} \cdot \nu), \mathbf{m} \cdot \nu \rangle_{L^2(\mathbb{S}^2)}.
\end{aligned}$$

As for $\{D, L_k\}$, The last expression is identically 0 by the symmetry of the scalar product and the skew-symmetry of the angular derivative. $\square$

Note: Again writing $\mathcal{E} = D + \kappa A$, we have shown that $\{D, S_k\} = 0 = \{D, L_k\}$. Thus, the spin and orbital angular momentum would be individually conserved if $\kappa = 0$. This corresponds to the invariance of D under individual rotations in domain or target space. However, for the anisotropy term, the individual terms were not zero but annihilated each other. Thus, for $\kappa > 0$, spin and orbital angular momentum are not individually conserved.

Using the Hamiltonian framework, preservation of the angular momentum is an easy consequence of the result for the Poisson bracket.

Corollary 5.1. *If* $\mathbf{m}$ *is a solution of the Landau Lifshitz equation, then* $J(\mathbf{m})$ *is conserved over time.*

Proof. From the introduction, recall that for any smooth function $K \colon P \to \mathbb{R}$, the flow for the Hamiltonian system satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} K(F(t, x_0)) = \{K, H\}(F(t, x_0)).$$

Here, $K = J_k$ and $H = \mathcal{E}$ satisfy $\{J_k, \mathcal{E}\} = 0$ such that the right-hand side is identically 0. $\square$

The following Lemma fills the gap of regularity in the proof of Theorem 4. Furthermore, by extending the techniques already used in 2, we obtain a convergence result for constrained critical points.

Lemma 5.5. *Fix* $\kappa > 0$ *and* $\mathbf{J}_0 \in \mathbb{R}^3$.

(1) Constrained critical points are smooth.

(2) *Let $(\mathbf{m}^k)_{k\in\mathbb{N}}$ be a sequence of (constrained) critical points of $\mathcal{E}$ with Lagrange multipliers $\omega_k = 0$. Assume that $\|\mathbf{m}^k - \mathbf{m}_0\|_{H^1(\mathbb{S}^2;\mathbb{R}^3)} \to 0$ for some $\mathbf{m}_0 \in H^1(\mathbb{S}^2;\mathbb{S}^2)$. Then*

$$\|\mathbf{m}^k - \mathbf{m}_0\|_{H^2(\mathbb{S}^2;\mathbb{R}^3)} \to 0$$

and $\mathbf{m}_0$ is a smooth critical point of $\mathcal{E}$.

Proof. For both statements, we proceed as in Theorem 2

(1) In stereographic coordinates centered at $\hat{\mathbf{e}}_3$, the Euler-Lagrange equation is given by

$$-\Delta\mathbf{m} = \Omega\colon \nabla\mathbf{m} + f$$

with $\Omega(\mathbf{m}) = \mathbf{m}\otimes\nabla\mathbf{m} - \nabla\mathbf{m}\otimes\mathbf{m}$ and

$$\begin{aligned} f = {} & \lambda^2(x)\mathbf{m}\times\left(\kappa(\nu\cdot\mathbf{m})\nu\times\mathbf{m} + \sum_{i=1}^{3}\omega_i\left(\hat{\mathbf{e}}_i\times\mathbf{m} - \tilde{x}_i\left(x^\perp\cdot\nabla\mathbf{m}\right)\right)\right) \\ & + \lambda(x)\mathbf{m}\times(\omega_1\partial_2\mathbf{m} - \omega_2\partial_1\mathbf{m}) \end{aligned}$$

where $\tilde{x} = (x_1, x_2, 1)$. As in the unconstrained case, $f \in L^2(\mathbb{R}^2)$ with bounds that only depend on κ, ω and $\|\nabla\mathbf{m}\|_{L^2}$. In particular, note that $|\tilde{x}_i||x|\lambda^2(x) \le \lambda(x)$ for all $x\in\mathbb{R}^2$. From here on, the proof is identical to the proof of Theorem 2 and we omit the details.

(2) Proving $H^2(\mathbb{S}^2; T_{\mathbf{m}}\mathbb{S}^2)$-convergence for strongly $H^1(\mathbb{S}^2; T_{\mathbf{m}}\mathbb{S}^2)$-convergent sequences again follows the same strategy as the proof of regularity in Theorem 2. However, since we are now interested not only in bounds but in convergence, we will give more details.

Firstly, note that every $\mathbf{m}^k$ is smooth since they are all critical points and write $\mathbf{m}^k = \mathbf{m}^k\circ\pi^{-1}$ for the stereographic projection π. Then, $\nabla\mathbf{m}^k \to \nabla\mathbf{m}$ strongly in $L^2(\mathbb{R}^2)$ and $\mathbf{m}^k\lambda(x) \to \mathbf{m}\lambda(x)$ strongly in $L^2(\mathbb{R}^2)$. Furthermore, each $\mathbf{m}^k$ satisfies

$$-\Delta\mathbf{m}^k = \Omega^k : \nabla\mathbf{m}^k + f^k$$

with Ω^k and f^k as in the Lemma and, defining Ω and f in terms of the limit $\mathbf{m}$,

$$\begin{aligned} \|\Omega^k - \Omega\|_{L^2} &= \|(\mathbf{m}^k - \mathbf{m})\otimes\nabla\mathbf{m} + \mathbf{m}\otimes(\nabla\mathbf{m}^k - \nabla\mathbf{m})\|_{L^2} \\ &\le \|(\mathbf{m}^k - \mathbf{m})\otimes\nabla\mathbf{m}\|_{L^2} + \|\nabla\mathbf{m}^k - \nabla\mathbf{m}\|_{L^2} \to 0 \text{ as } k\to\infty \end{aligned}$$

where the first term converges due to dominated convergence with

$$|(\mathbf{m}^k - \mathbf{m})\otimes\nabla\mathbf{m}(x)|^2 \to 0 \text{ pointwisely a.e. on } \mathbb{R}^2$$

and

$$|\mathbf{m}^k - \mathbf{m}|^2|\nabla\mathbf{m}|^2 \le 2\|\nabla\mathbf{m}\|^2 \in L^1(\mathbb{R}^2).$$

For f, $|\mathbf{m}| = 1$ implies

$$|f^k - f| \leq \lambda(x)^2 \kappa \cdot 3|\mathbf{m}^k - \mathbf{m}| \to 0 \text{ in } L^2(\mathbb{R}^2).$$

Working locally, we decompose each $\mathbf{m}_k$ into $\mathbf{m}_k = u_k + v_k$ such that u_k satisfies $-\Delta u_k = f^k + g^k = f^k + \Omega_1^k : \nabla \mathbf{m}^k \in L^2$. By the Calderón-Zygmund inequality and via the estimates in Theorem 2 we find, for each $q \in (1, 2)$,

$$\|D^2 u^k\|_{L^q} \leq \|f^k + g^k\|_{L^q(B_R)}$$

where g^k is uniformly bounded in L^q since the estimates go back on $\|\nabla \mathbf{m}^k\|_{L^2(\mathbb{R}^2)}$. Thus, proceeding as in Theorem 2, u^k is uniformly bounded in $C^\alpha(B_R)$ for every $\alpha \in (0, 1)$, yielding locally uniform convergence of a subsequence due to Arzela Ascoli.

On the other hand, $\mathbf{m}^k - u^k = v^k$ satisfies

$$-\Delta v^k = \Omega_0^k : \nabla \mathbf{m}^k$$

where $\Omega_0^k \to \Omega_0$ in L^2 due to the convergence of Ω^k and continuity of the Helmholtz projection. By the same Hardy space arguments as in the proof of regularity, this yields local uniform convergence $v^k \to v$ and thus, up to a subsequence, locally uniform convergence $\mathbf{m}^k \to \mathbf{m}$.

Fixing $B = B_{1+\varepsilon}(0)$ for some $\varepsilon > 0$, standard estimates for semilinear equations with quadratic growth [53], combined with bootstrapping yield uniform estimates for $\|\mathbf{m}^k\|_{H^l(U)}$ for all $l > 0$ and $U \subset\subset B$. These estimates are uniform not only in k but also in Ω for $\Omega = B_\delta(x) \subset\subset B$. By compactness of $\overline{B_{1+\varepsilon}(0)}$, this yields uniform $H^l(\mathbb{R}^2)$ bounds on $B_{1+\varepsilon}(0)$ and $H^2(\mathbb{R}^2)$ convergence of $\mathbf{m}^k \circ \pi^{-1}$ on $B_{1+\varepsilon}(0)$ follows for a subsequence due to Rellich.

Since $1 + |x|^2$ is bounded by a constant on $B_{1+\varepsilon}(0)$, this yields H^2 convergence of the original sequence $\mathbf{m}^k \colon \mathbb{S}^2 \to \mathbb{S}^2$ on $\mathbb{S}^2{}_{\geq 0}$. Repeating the procedure on $B_{1+\varepsilon}(0) = \pi_-(\mathbb{S}^2{}_{\leq 0})$ proves the $H^2(\mathbb{S}^2; \mathbb{R}^3)$ convergence of $\mathbf{m}^k$ to $\mathbf{m}$.

From the local $H^2(\mathbb{S}^2; \mathbb{R}^3)$ convergence of $\mathbf{m}^k$ to $\mathbf{m}$ it follows that $\mathbf{m}$ also satisfies the equation

$$\Delta \mathbf{m} = \Omega(\mathbf{m}) : \nabla \mathbf{m} + f$$

and thus is smooth as a critical point of $\mathcal{E}$.

□

We will apply the convergence result to show that if the constrained minimizers are also critical points of $\mathcal{E}$, meaning that they solve (5.1) with $\omega = 0$, then they converge to an unconstrained critical point in H^2. In order to prove the H^1-convergence we need to compare the constrained energy minimum with the global minimal value within the topological sector $Q = 0$. We thus consider special functions constructed from an equivariant $\mathbf{m}$ such that the total angular momentum is controlled.

Definition 6. *Given* $\mathbf{m}\colon \mathbb{S}^2 \to \mathbb{S}^2$*,* $s > 0$ *and an axis* $\hat{\mathbf{e}}$*, define* $\mathbf{m}_s\colon \mathbb{S}^2 \to \mathbb{S}^2$ *via*

$$\mathbf{m}_s = \tilde{\mathbf{m}}_s \circ \pi_{\hat{\mathbf{e}}}, \qquad \tilde{\mathbf{m}}_s(x_1, x_2) = (\mathbf{m} \circ \pi_{\hat{\mathbf{e}}})((1+s)x_1, s_2).$$

$\mathbf{m}_s$ *is called an elliptic distortion of* $\mathbf{m}$ *around* $\hat{\mathbf{e}}$*.*

It was shown in [37] that the size of the angular momentum is strictly increased by elliptical distortion for $s > 0$. In fact, this increase is strictly monotonous, resulting in the following Lemma:

Lemma 5.6. *Given* $\hat{\mathbf{e}} \in \mathbb{S}^2$*, let* $\mathbf{m}\colon \mathbb{S}^2 \to \mathbb{S}^2$ *be* $\hat{\mathbf{e}}$*-equivariant. Then there exists* $\delta > 0$ *such that* $\tilde{J}\colon \mathbb{R} \to \mathbb{R}$*,* $s \mapsto J(\mathbf{m}_s)$ *admits a continuous, strictly monotonic inverse on* $[0, \delta_0]$*. In particular,* $\tilde{J} \to 4\pi$ *implies* $\mathcal{E}(\mathbf{m}_t) \to \mathcal{E}(\mathbf{m})$*.*

Proof. For $\hat{\mathbf{e}} = \hat{\mathbf{e}}_3$, Lemma 6 in [37] implies that $\tilde{J}' < 0$ on $(0, \delta)$ for some $\delta > 0$. Furthermore, $\mathbb{R} \ni s \mapsto \mathbf{m}_s \in H^1(\mathbb{S}^2; \mathbb{S}^2)$ and $H^1(\mathbb{S}^2; \mathbb{S}^2) \ni \mathbf{m} \mapsto \mathbf{J}(\mathbf{m}) \in \mathbb{R}$ are continuous. Thus, $\tilde{J}$ is continuous and strictly monotonous on $[0, \delta]$ and the existence of a continuous, strictly monotonous inverse $\tilde{J}^{-1}$ follows on $[0, \delta_0]$ for every $\delta_0 < \delta$.

□

For $\|\mathbf{m}^\varepsilon - \mathbf{m}_0\|_{H^2}$ sufficiently small, the Łojasiewicz inequality of Proposition 5.1 applies and relates the energy difference to the norm of the tangential gradient. Having prepared the convergence, we are now able to prove that the Lagrange multiplier ω in the constrained minimization problem is nontrivial, given the assumption that all minimizers of degree 0 are equivariant. This results in non-static rotating solutions of the Landau Lifshitz equation.

Theorem 5. *Assume that all minimizers of* $\mathcal{E}$ *with* $Q = 0$ *are equivariant. Then there exists* $\varepsilon_0 > 0$ *such that for all* $0 < \varepsilon < \varepsilon_0$ *there exist* $\omega \neq 0$ *and* $\mathbf{m} \in C^\infty(\mathbb{S}^2; \mathbb{S}^2)$ *such that* $\mathbf{J}(\mathbf{m}) = -(4\pi + \varepsilon)\hat{\mathbf{e}}_3$ *and*

$$\mathbf{m}(x,t) := \mathbf{m}_{R(\omega t)}(x)$$

is a nontrivial periodic solution of the Landau Lifshitz equation.

Proof. For ε small enough, let $\mathbf{m}_\varepsilon$ be the constrained minimizer from Theorem 4, solving

$$\mathbf{m}_\varepsilon \times \nabla\mathcal{E}(\mathbf{m}_\varepsilon) = \omega_\varepsilon\, (\mathbf{m}_\varepsilon \times \mathbf{J}(\mathbf{m}_\varepsilon))$$

and satisfying $|\mathbf{J}(\mathbf{m}_\varepsilon)| = 4\pi + \varepsilon$. We show that there exists $\varepsilon_0 > 0$ such that $\omega_\varepsilon \neq 0$ for all $\varepsilon < \varepsilon_0$. Otherwise there exists a sequence $\varepsilon_k \to 0$ such that $\omega_{\varepsilon_k} = 0$ and thus

$$\mathbf{m}_{\varepsilon_k} \times \nabla\mathcal{E}(\mathbf{m}_{\varepsilon_k}) = 0.$$

Let $\mathbf{m}_0$ be a global minimizer. By assumption, $\mathbf{m}_0$ is equivariant. Furthermore, for every $\varepsilon > 0$, let ${\mathbf{m}_0}^\varepsilon$ be its elliptic distortion. Then for all ε_k, the constrained minimizers $\mathbf{m}_{\varepsilon_k}$ satisfy

$$\mathcal{E}(\mathbf{m}_{\varepsilon_k}) \leq \mathcal{E}({\mathbf{m}_0}^\varepsilon) \leq \mathcal{E}(\mathbf{m}_0) + o(\varepsilon)$$

by Lemma 5.6. Since $\mathcal{E}(\mathbf{m}) \geq \|\mathbf{m}\|_{H^1} - 4\pi$, this implies that the sequence $(\mathbf{m}_{\varepsilon_k})_{k\in\mathbb{N}}$ is uniformly bounded in H^1 and admits a weakly convergent subsequence $\mathbf{m}_k \rightharpoonup \mathbf{m}^*$ which is strongly convergent in L^2. Combining local minimality of $\mathbf{m}$ with lower semicontinuity of $\mathcal{E}$ and the estimate above, we find

$$\mathcal{E}(\mathbf{m}_0) \leq \mathcal{E}(\mathbf{m}^*) \leq \liminf_{k\to\infty} \mathcal{E}(\mathbf{m}_{\varepsilon_k}) \leq \mathcal{E}(\mathbf{m}_0)$$

such that $\mathbf{m}^*$ is locally minimizing and therefore equivariant by assumption. More importantly, the resulting convergence of the energy implies strong H^1-convergence of $\mathbf{m}_{\varepsilon_k}$ and thus H^2 convergence by Lemma 5.5. For $\|\mathbf{m}_{\varepsilon_k} - \mathbf{m}^*\|_{H^2}$ small enough, the Łojasiewicz inequality can be applied:

$$0 = \|\mathbf{m}_\varepsilon \times \nabla\mathcal{E}(\mathbf{m}_{\varepsilon_k})\|_{L^2} \geq \left(\mathcal{E}(\mathbf{m}_{\varepsilon_k}) - \mathcal{E}(\mathbf{m}^*)\right)^{1-\gamma} \geq 0$$

Therefore, $\mathcal{E}(\mathbf{m}_{\varepsilon_k}) = \mathcal{E}(\mathbf{m}^*)$ and $\mathbf{m}_{\varepsilon_k}$ is an unconstrained minimizer of $\mathcal{E}$. By assumption it follows that $\mathbf{m}_{\varepsilon_k}$ is equivariant, which contradicts $|\mathbf{J}(\mathbf{m}_{\varepsilon_k})| = 4\pi + \varepsilon$. □

5.2. Łojasiewicz-Simon Inequality

In this section, we will follow the framework of Haraux and Jendoubi [19] to prove the Łojasiewicz inequality for critical points of $\mathcal{E}$ stated in Proposition 5.1.

5.2.1. Preliminaries: Analyticity and Fredholm Property

One of the key ingredients of the finite dimensional Łojasiewicz inequality is analyticity and although generalizations exist [18],[4], it is usually required in the infinite dimensional case as well. As in the application of the Lagrange Multiplier theorem, we thus consider the pull-back of $\mathcal{E}$ via the chart defined in the introduction, section 2.1.

$$F\colon H^2(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2) \to \mathbb{R},\ F = \mathcal{E} \circ \psi,$$

where $\psi(v) = \frac{\mathbf{m}_0 + v}{|\mathbf{m}_0 + v|}$. Here, $\mathbf{m}_0$ denotes a critical point of $\mathcal{E}$. Then $\mathcal{E}(\mathbf{m}_0) = F(0)$ and, as computed in section 2.1,

$$DF(0)\langle\phi\rangle = \delta\mathcal{E}(\mathbf{m}_0)\langle\phi\rangle = \langle\nabla\mathcal{E}(\mathbf{m}_0)^{\tan}, \phi\rangle_{L^2} = 0$$

for all $\phi \in H^2(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2)$.

Moreover, F is analytic in the sense of definition 7.

Lemma 5.7. *There exists* $\rho > 0$ *such that*

$$F\colon B_\rho(0) \subset H^2(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2) \to \mathbb{R}$$

is analytic.

Proof. Consider the continuation of $\mathcal{E}$ to a functional on $H^2(\mathbb{S}^2;\mathbb{R}^3)$ given by

$$\tilde{\mathcal{E}}(\mathbf{m}) = \int_{\mathbb{S}^2} |\nabla \mathbf{m}|^2 + \kappa(1 - (\mathbf{m}\cdot\nu)^2)\,\mathrm{d}\sigma$$

such that $\tilde{\mathcal{E}}(\mathbf{m}) = \mathcal{E}(\mathbf{m})$ for $\mathbf{m}\colon \mathbb{S}^2 \to \mathbb{S}^2$. Furthermore, write $\tilde{\psi}$ for the non-surjective function $\tilde{\psi}\colon H^2(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2) \to H^2(\mathbb{S}^2;\mathbb{R}^3)$ with $\tilde{\psi}(v) = \psi(v)$. Then $\tilde{E}$ and $\tilde{\psi}$ are analytic as functionals between Banach spaces, hence

$$F = \mathcal{E}\circ\psi = \tilde{E}\circ\tilde{\psi}$$

is analytic due to theorem A.2.

For the analyticity of $\tilde{\psi}$ in a neighborhood of 0, apply Lemma A.4 to

$$f_a\colon \mathbb{R}^3 \to \mathbb{R},\ x \mapsto \frac{a+x}{|a+x|}$$

where f_a is analytic on $B_r(0) \subset \mathbb{R}^3$ with r independent of the choice of $a \in \mathbb{S}^2$.

On the other hand, the continuation of E is the sum of a norm and an integral over a polynomial in $\mathbf{m}$, both of which are analytic due to Lemma A.2 and Corollary A.1. □

In contrast to finite dimensions, analyticity alone is not sufficient to prove a Łojasiewicz type inequality in infinite dimensions. One way to fill the gap is via a Fredholm property of the second variation. In the present case, the Hessian of the pulled back functional $F = \mathcal{E}\circ\psi$,

$$D^2F\colon H^2(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2) \times H^2(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2) \to \mathbb{R}$$

gives rise to a bounded linear symmetric operator A on L^2 with domain $H^2(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2)$ satisfying

$$D^2F(0)\langle\phi,\cdot\rangle = \langle A\phi,\cdot\rangle_{L^2}, \qquad \text{i.e.} \quad A = D\left(\nabla_{L^2}F\right)(0)$$

because

$$D^2F(0)\langle\phi,\cdot\rangle = D\left(DF(0)\langle\cdot\rangle\right)\langle\phi\rangle = D\langle\nabla_{L^2}F(0),\cdot\rangle\langle\phi\rangle = \langle D\nabla_{L^2}F(0)\langle\phi\rangle,\cdot\rangle.$$

Furthermore, the following holds:

Lemma 5.8. *If* 0 *is a critical point of* F *then the operator*

$$A\colon H^2(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2) \to L^2(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2)$$

is a Fredholm operator of index 0.

Proof. We first show invertibility of $A + \lambda\iota\colon H^2(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2) \to L^2(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2)$ for some $\lambda > 0$ to be determined. Here, ι denotes the compact embedding $H^2 \hookrightarrow L^2$. Smoothness of $\mathbf{m}_0$ on $\mathbb{S}^2$ implies the existence of some $\mu > 0$ such that

$$\begin{aligned}\mathcal{H}(\phi) &= \|\phi\|^2_{H^1(\mathbb{S}^2;\mathbb{R}^3)} - \int_{\mathbb{S}^2} \phi^2 \left(1 + |\nabla \mathbf{m}_0|^2 - \kappa(\mathbf{m}_0 \cdot \nu)^2\right) + \kappa(\phi\cdot\nu)^2 \,\mathrm{d}\sigma \\ &\geq \|\phi\|^2_{H^1} - \mu\|\phi\|^2_{L^2(\mathbb{S}^2)}.\end{aligned}$$

On the other hand, $\mathcal{H}(\phi) + \mu\|\phi\|^2_{L^2} \leq c(\mathbf{m}_0, \kappa)\|\phi\|^2_{H^1}$ such that $\mathcal{H}(\cdot) + \mu\|\cdot\|^2_{L^2}$ defines a norm on $H^1(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2)$. We consider the Hilbert space $\tilde{H}^1(\mathbb{S}^2, T_{\mathbf{m}_0}\mathbb{S}^2)$ consisting of functions in $H^1(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2)$ and equipped with the scalar product

$$(\phi, \psi) \mapsto \mathcal{H}(\phi, \psi) + \mu\langle \phi, \mu\rangle_{L^2}.$$

Given $f \in L^2(\mathbb{S}^2, T_{\mathbf{m}_0}\mathbb{S}^2)$, we apply the Riesz representation theorem on $\tilde{H}^1(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2)$ to the linear functional

$$\tilde{H}^1(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2) \to \mathbb{R},\ \psi \mapsto \langle f, \psi\rangle_{L^2(\mathbb{S}^2)}$$

to find a unique $v \in H^1(\mathbb{S}^2, T_{\mathbf{m}_0}\mathbb{S}^2)$ such that

$$\langle f, \psi\rangle_{L^2} = \mathcal{H}(v, \psi) + \mu\langle v, \psi\rangle_{L^2}.$$

By considering the pullback via the inverse stereographic projection, we can proceed as in [32] to show $v \in H^2(\mathbb{S}^2, T_{\mathbf{m}_0}\mathbb{S}^2)$:

Given $\eta \in C^\infty(\mathbb{S}^2; \mathbb{R}^3)$, let $P_{\mathbf{m}_0}\eta = -\mathbf{m} \times \mathbf{m} \times \eta$ denote the projection of η onto $T_{\mathbf{m}_0}\mathbb{S}^2$. Then $P_{\mathbf{m}_0}\eta \in C^\infty(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2)$ and

$$\begin{aligned}\langle f, \eta\rangle_{L^2(\mathbb{S}^2;\mathbb{R}^3)} &= \langle f, P_{\mathbf{m}_0}\eta\rangle_{L^2(\mathbb{S}^2;T_{\mathbf{m}_0}\mathbb{S}^2)} = \mathcal{H}(v, P_{\mathbf{m}_0}\eta) + \mu\langle v, P_{\mathbf{m}_0}\eta\rangle_{L^2(\mathbb{S}^2;T_{\mathbf{m}_0}\mathbb{S}^2)} \\ &= \int_{\mathbb{S}^2} \nabla v : \nabla(P_{\mathbf{m}_0}\eta) - \kappa(v\cdot\nu)(P_{\mathbf{m}_0}\eta\cdot\nu) \\ &\qquad - v\cdot P_{\mathbf{m}_0}\eta\left(|\nabla\mathbf{m}_0|^2 - \kappa(\mathbf{m}_0\cdot\nu)^2 - \mu\right)\,\mathrm{d}\sigma \\ &= \langle \nabla v, \nabla P_{\mathbf{m}_0}\eta\rangle_{L^2(\mathbb{S}^2;\mathbb{R}^3)} - \langle g, \eta\rangle_{L^2(\mathbb{S}^2;\mathbb{R}^3)}\end{aligned}$$

where $g = P_{\mathbf{m}_0}\left(\kappa(v\cdot\nu)\nu - \left(|\nabla\mathbf{m}_0|^2 - \kappa(\mathbf{m}_0\cdot\nu)^2 - \mu\right)v\right) \in L^2(\mathbb{S}^2;\mathbb{R}^3)$ due to the smoothness of $\mathbf{m}_0$.

If v was smooth then

$$\begin{aligned}P_{\mathbf{m}_0}\Delta_{\mathbb{S}^2}v &= \Delta_{\mathbb{S}^2}v - (\mathbf{m}_0\cdot\Delta_{\mathbb{S}^2}v)\mathbf{m}_0 \\ &= \Delta_{\mathbb{S}^2}v - \Delta_{\mathbb{S}^2}(\mathbf{m}_0\cdot\nu)\mathbf{m}_0 + \nabla\cdot(\nabla\mathbf{m}_0\cdot v)\,\mathbf{m}_0 + (\nabla\mathbf{m}_0 : \nabla v)\,\mathbf{m}_0\end{aligned}$$

where $\Delta_{\mathbb{S}^2}(\mathbf{m}_0\cdot\nu)\mathbf{m}_0 = 0$ since $v \in T_{\mathbf{m}_0}\mathbb{S}^2$. Therefore, a smooth v would satisfy

$$\begin{aligned}\langle\nabla v, \nabla P_{\mathbf{m}_0}\eta\rangle_{L^2(\mathbb{S}^2)} &= -\langle\Delta v, P_{\mathbf{m}_0}\eta\rangle_{L^2(\mathbb{S}^2)} = -\langle P_{\mathbf{m}_0}\Delta v, \eta\rangle_{L^2(\mathbb{S}^2)} \\ &= -\langle\Delta v, \eta\rangle + \langle\nabla\cdot(\nabla\mathbf{m}_0\cdot v)\mathbf{m}_0, \eta\rangle + \langle(\nabla\mathbf{m}_0 : \nabla v)\mathbf{m}_0, \eta\rangle \\ &= \langle\nabla v, \nabla\eta\rangle_{L^2(\mathbb{S}^2;\mathbb{R}^3)} + \langle g_1, \eta\rangle_{L^2(\mathbb{S}^2;\mathbb{R}^3)}\end{aligned}$$

with $g_1 \in L^2(\mathbb{S}^2)$ involving only first derivatives of v. Approximating $v \in H^1(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2)$ by smooth functions, the identity thus carries over and implies that $v \in H^1(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2)$ weakly solves

$$-\Delta_{\mathbb{S}^2} v = f + g - g_1 \quad \text{on } \mathbb{S}^2.$$

Once more arguing via the pullback $v \circ \pi_\pm^{-1} \in H^1(\mathbb{R}^2; T_{\mathbf{m}_0}\mathbb{S}^2)$ on the upper and lower half sphere respectively, $v \in H^2(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2)$ follows from the regularizing properties of the Poisson equation on bounded domains in $\mathbb{R}^2$.

Having shown that for every $f \in L^2(\mathbb{S}^2, T_{\mathbf{m}_0}\mathbb{S}^2)$ there exists a unique $v \in H^2(\mathbb{S}^2, T_{\mathbf{m}_0}\mathbb{S}^2)$ such that

$$\langle f, \psi \rangle_{L^2} = \mathcal{H}(v, \psi) + \mu \langle v, \psi \rangle_{L^2} = \langle Av + \mu v, \psi \rangle_{L^2} \text{ for all } \psi \in H^1(\mathbb{S}^2, T_{\mathbf{m}_0}\mathbb{S}^2),$$

we conclude by the open mapping theorem that

$$(A + \mu\iota)^{-1} \colon L^2(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2) \to H^2(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2)$$

is a bounded linear operator where the embedding $\iota \colon H^2(\mathbb{S}^2, T_{\mathbf{m}_0}\mathbb{S}^2) \to L^2(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2)$ is compact. Therefore, the operators

$$\mathbb{1}_{H^2} - (A + \mu\,\iota)^{-1} A = \mu (A + \mu\,\iota)^{-1} \iota$$
$$\text{and } \mathbb{1}_{L^2} - A(A + \mu\,\iota)^{-1} = \mu\,\iota (A + \mu\,\iota)^{-1}$$

are compact and A is Fredholm with index 0.

□

5.2.2. Proof of Proposition 5.1

We have now gathered all ingredients to prove the inequality for F. From the statement for F, Proposition 5.1 can be deduced as follows:

Firstly, given $\mathbf{m} \in H^2(\mathbb{S}^2; \mathbb{S}^2)$, we have $\mathbf{m} = \Pi(\mathbf{m}_0 + v)$ for

$$v = \frac{\mathbf{m}}{\mathbf{m} \cdot \mathbf{m}_0} - \mathbf{m}_0$$

where $\mathbf{m} \cdot \mathbf{m}_0 > 0$ if $|\mathbf{m} - \mathbf{m}_0|^2 < 2$ and it follows by construction that $v \in T_{\mathbf{m}_0}\mathbb{S}^2$. Therefore,

$$F(v) - F(0) = \mathcal{E}(\mathbf{m}) - \mathcal{E}(\mathbf{m}_0).$$

For the derivative, recall from section 2.1 that

$$\frac{d}{dt} \Pi(\mathbf{m}_0 + t\phi) = \phi$$

and therefore

$$DF(v)\langle \phi \rangle = \delta\mathcal{E}(\Pi(\mathbf{m}_0 + v))\langle \phi \rangle = \delta\mathcal{E}(\mathbf{m})\langle \phi \rangle$$

for any $\phi \in H^2(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2)$ and v as above. This implies

$$\begin{aligned}\|\nabla F(v)\|_{L^2(\mathbb{S}^2;T_{\mathbf{m}_0}\mathbb{S}^2)} &= \sup_{\|\phi\|_{L^2(\mathbb{S}^2;T_{\mathbf{m}_0}\mathbb{S}^2)}=1} \langle \nabla F(v), \phi\rangle_{L^2(\mathbb{S}^2;T_{\mathbf{m}_0}\mathbb{S}^2)} \\ &= \sup_{\|\eta\|_{L^2(\mathbb{S}^2;\mathbb{R}^3)}=1} \langle \mathbf{m}_0 \times \nabla\mathcal{E}(\mathbf{m}_0), \eta\rangle_{L^2(\mathbb{S}^2;\mathbb{R}^3)} = \|(\nabla\mathcal{E}(\mathbf{m}_0))^{\tan}\|_{L^2(\mathbb{S}^2;\mathbb{R}^3)}.\end{aligned}$$

Here, we have used that for any $\eta \in L^2(\mathbb{S}^2;\mathbb{R}^3)$, the field $\phi_\eta = -\mathbf{m}_0 \times \eta$ satisfies the pointwise identity $|\phi_\eta(x)| = |\eta(x)|$ and therefore

$$\|\eta\|_{L^2(\mathbb{S}^2;\mathbb{R}^3)} = \|\phi_\eta\|_{L^2(\mathbb{S}^2;T_{\mathbf{m}_0}\mathbb{S}^2)}.$$

Also, given $\phi \in L^2(\mathbb{S}^2;\mathbb{R}^3)$, setting $\eta = -\mathbf{m}_0 \times \phi \in L^2(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2) \subset L^2(\mathbb{S}^2, \mathbb{R}^3)$ implies that ϕ is of the form $\phi = -\mathbf{m}_0 \times \eta$ for a function $\eta \in L^2(\mathbb{S}^2;\mathbb{R}^3)$.

All together, given $\mathbf{m} \in H^2(\mathbb{S}^2;\mathbb{S}^2)$ and setting $v := \frac{P_{\mathbf{m}_0}(\mathbf{m}-\mathbf{m}_0)}{\mathbf{m}\cdot\mathbf{m}_0} \in T_{\mathbf{m}_0}\mathbb{S}^2$, a Łojasiewicz-type inequality for F would imply

$$\|\nabla\mathcal{E}(\mathbf{m})^{\tan}\|_{L^2(\mathbb{S}^2;\mathbb{R}^3)} = \|\nabla F(v)\|_{L^2(\mathbb{S}^2;T_{\mathbf{m}_0}\mathbb{S}^2)} \geq (F(v) - F(0))^{1-\gamma} = (\mathcal{E}(\mathbf{m}) - \mathcal{E}(\mathbf{m}_0))^{1-\gamma}.$$

Passing on to F, let $N \subset H^2(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2)$ be the kernel of A and P_N the orthogonal projection onto it. Set $k = \dim(N)$ and note $k < \infty$ by Lemma 5.8. Moreover, if the reduced energy in chapter 2 is strictly convex, than $k = 2$ and N is spanned by the tangent fields associated to joint rotations.

The proof of the inequality for F is structured into three parts. First, we will show that

$$\mathcal{M} = P_N + \nabla F \colon H^2(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2) \to L^2(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2)$$

admits an analytic inverse in a neighborhood of 0. In a second step, this enables a reduction to finite dimensions by defining an analytic function $\Gamma\colon \mathbb{R}^3 \to \mathbb{R}$ by composing F on $N \sim \mathbb{R}^k$ with the inverse of $\mathcal{M}$. Finally, estimates on $\|\nabla F\|$ and $|F - \Gamma|$ will allow to establish a Łojasiwiz-Simon inequality for F, based on the inequality for Γ.

Invertibility of $\mathcal{M}$

Consider the bounded linear operator $\mathcal{L} = P_N + A\colon H^2(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2) \to L^2(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2)$. Since $\dim(N) < \infty$, the orthogonal projection onto N is a compact operator and thus $\mathcal{L}$ is Fredholm as the sum of a Fredholm operator and a compact operator. Furthermore, we have $\ker \mathcal{L} = \{0\}$ since $Av = -P_N v$ for $v \in H^2(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2)$ implies

$$\|Av\|_{L^2}^2 = -\langle Av, P_N v\rangle = -\langle v, AP_N v\rangle = 0$$

and therefore $Av = P_N v = 0$ and $v \in N \cap N^\perp = \{0\}$. Since $\mathcal{L}$ is Fredholm, the image $\mathcal{R}(\mathcal{L})$ is closed. Moreover, $\mathcal{L}$ is onto because it is Fredholm with index 0 and hence

$\dim(\ker \mathcal{L}) = 0 = \operatorname{codim}\mathcal{R}(\mathcal{L})$. Thus, the operator $\mathcal{L}\colon H^2(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2) \to L^2(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2)$ is invertible and by the closed graph theorem, $\mathcal{L}^{-1}$ is bounded.

Now consider $\mathcal{M} = P_N + \nabla F\colon H^2(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2) \to L^2(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2)$ where $\nabla F = \nabla_{\mathbb{S}^2;\mathbb{R}^3}$ is given by

$$\nabla F(v) = P_{\mathbf{m}_0}\left(-\Delta\psi(v) - \kappa(\psi(v)\cdot\nu)\nu\right).$$

This functional is analytic in a neighbourhood of 0 since the projection onto $T_{\mathbf{m}_0}\mathbb{S}^2$ is given by a polynomial in $\mathbf{m}_0$ while $\nabla F = \nabla E \circ \psi$ is analytic as the composition of $\tilde{\psi}$ and $\nabla\tilde{E}$ which are both analytic by the same reasoning as in Proposition 5.7.

Since $A = D(\nabla F)_{L^2}(0)$,

$$D\mathcal{M}(0) = D(P_N + \nabla F)(0) = P_N + \nabla DF(0) = P_N + A = \mathcal{L}$$

admits a bounded linear inverse. Hence, by the analytic inverse function theorem [56], there exist neighborhoods W_1, W_2 of $0 \in H^2(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2)$ and $0 = \mathcal{M}(0) \in L^2(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2)$, respectively, such that $\mathcal{M}\colon W_1 \to W_2$ admits an analytic inverse $\mathcal{M}^{-1}$. In particular, $\mathcal{M}^{-1}$ is bounded and differentiable.

Reduction to finite dimensions

Let $\phi_i \in H^2(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2)$, $1 \le i \le k$ be an orthonormal basis of N and choose $r > 0$ small enough to ensure $\sum_{l=1}^k \xi_l\phi_l \in W_2 \subset L^2(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2)$ for all $\xi \in B_r(0) \subset \mathbb{R}^k$. This is possible since $0 \in N \subset L^2(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2)$. Define $\Gamma\colon B_r(0) \to \mathbb{R}$ by

$$\Gamma(\xi) = F\left(\mathcal{M}^{-1}\left(\sum_{l=1}^k \xi_l\phi_l\right)\right).$$

Γ is the composition of the analytic functionals F, $\mathcal{M}^{-1}$, and a polynomial and thus analytic, too. Furthermore, it holds that

$$\begin{aligned}\partial_i\Gamma(\xi) &= DF\left(\mathcal{M}^{-1}(\sum_{j=1}^k \xi_j\phi_j)\right)\langle D\mathcal{M}^{-1}(\sum_{j=1}^k \xi_j\phi_j)\langle\phi_i\rangle\rangle \\ &= \left\langle \nabla F\left(\mathcal{M}^{-1}(\sum_{j=1}^k \xi_j\phi_j)\right), D\mathcal{M}^{-1}(\sum_{j=1}^k \xi_j\phi_j)\langle\phi_i\rangle\right\rangle_{L^2(\mathbb{S}^2;\mathbb{R}^3)}.\end{aligned}$$

Recall that $\mathcal{M}(0) = P_N(0) + \nabla F(0) = 0 \in L^2$ and thus $\mathcal{M}^{-1}(0) = 0$. For Γ, this implies

$$\Gamma(0) = F(\mathcal{M}^{-1}(0)) = F(0)$$

as well as

$$\nabla\Gamma(0) = \sum_{i=1}^k \langle\nabla F(\mathcal{M}^{-1}(0)), D\mathcal{M}^{-1}(0)\langle\phi_k\rangle\rangle\hat{\mathbf{e}}_k = \sum_{i=1}^k \langle\nabla F(0), D\mathcal{M}^{-1}(0)\langle\phi_k\rangle\rangle\hat{\mathbf{e}}_k = 0.$$

Hence, the classical Łojasiewicz inequality can be applied and it follows that there are are $\rho \in (0, r]$, $c > 0$ and $\gamma \in (0, \frac{1}{2})$ such that

$$|\Gamma(\xi) - \Gamma(0)|^{1-\gamma} \leq c|\nabla\Gamma(\xi)|$$

for all $\xi \in B_\rho(0)$.

Estimates relating Γ and F

Returning to the basis on N, define the coefficient vector $\mathbf{K}\colon W_1 \to \mathbb{R}^n$ componentwise by $K_i(u) = \langle u, \phi_i \rangle_{L^2}$ such that

$$P_N u = \sum_{i=1}^{k} K_i(u)\phi_i \in H^2(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2).$$

There is an open neighbourhood $U \subset W_1$ of 0 such that $K(u) \in B_\rho(0)$ for all $u \in U$. For $u \in U$, the Łojasiewicz inequality for Γ applies to $\xi = K(u)$ and

$$\begin{aligned} |F(u) - F(0)|^{1-\gamma} &\leq |F(u) - \Gamma(K(u))|^{1-\gamma} + |\Gamma(K(u)) - \Gamma(0)|^{1-\gamma} \\ &\leq |F(u) - \Gamma(K(u))|^{1-\gamma} + c|\nabla\Gamma(K(u))|, \end{aligned}$$

indicating that estimates on $|\nabla\Gamma(K(u))|$ and $|F - \Gamma \circ K|$ will lead to a Łojasiewicz type inequality for F. To reduce notation, we will use the notation $L^2 = L^2(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2)$ and $H^2 = H^2(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2)$ for the rest of the proof.

Since ∇F, $\mathcal{M}$ and $\mathcal{M}^{-1}$ are all analytic, they also are all continuously differentiable. As a result, after possibly reducing W_1 and $W_2 = \mathcal{M}(W_1)$, there are constants C_F and $C_{\mathcal{M}^{-1}}$ for which the following estimates hold:

$$\begin{aligned} \|\nabla F(u)\|_{L^2} &\leq \|\nabla F(0)\|_{L^2} + 1 = 1 && \text{for all } u \in W_1, \\ \|\nabla F(u) - \nabla F(v)\|_{L^2} &\leq C_F \|u - v\|_{H^2} && \text{for all } u, v \in W_1, \\ \|D\mathcal{M}^{-1}(u)\|_{\mathcal{L}(L^2;H^2)} &\leq \|D\mathcal{M}^{-1}(0)\|_{\mathcal{L}(L^2;H^2)} + 1 \\ &= \|(D\mathcal{M}(0))^{-1}\|_{\mathcal{L}(L^2;H^2)} + 1 =: C_{\mathcal{M}^{-1}} && \text{for all } u \in W_2, \end{aligned}$$

and, finally

$$\begin{aligned} \|\mathcal{M}^{-1}(u) - \mathcal{M}^{-1}(v)\|_{H^2} &\leq \|D\mathcal{M}^{-1}(u)\|_{\mathcal{L}(L^2;H^2)} \|u - v\|_{L^2} \\ &\leq C_{\mathcal{M}^{-1}} \|u - v\|_{L^2} && \text{for all } u,\ v \in W_2. \end{aligned}$$

Furthermore, W_1 is reduced to a ball centered at 0 to ensure convexity and W_2 and U from above are adapted as necessary to ensure $U \subset W_1$. Since all estimates hold for the center $u = 0$, U is still an open neighborhood of 0 in $H^2(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2)$.

For arbitrary $u \in W_1$, the estimate on $\mathcal{M}^{-1}$ gives

$$\begin{aligned}\|\mathcal{M}^{-1}(P_N u) - u\|_{H^2} &= \|\mathcal{M}^{-1}(P_N u) - \mathcal{M}^{-1}(\mathcal{M}(u))\|_{H^2} \\ &\leq C_{\mathcal{M}^{-1}}\|P_N u - \mathcal{M}(u)\|_{L^2} = C_{\mathcal{M}^{-1}}\|\nabla F(u)\|_{L^2}.\end{aligned}$$

For the gradient of Γ, evaluated at $K(u)$ where $u \in U$, the bounds on $\|D\mathcal{M}^{-1}\|$ and $\|\nabla F(u)\|_{L^2}$ give

$$\begin{aligned}|\nabla\Gamma(K(u))| &= \left|\sum_{i=1}^{k}\langle \nabla F(\mathcal{M}^{-1}(P_N u), D\mathcal{M}^{-1}(P_N u)\phi_i\rangle \hat{\mathbf{e}}_i\right| \\ &\leq kC_{\mathcal{M}^{-1}}\|\nabla F(\mathcal{M}^{-1}(P_N u))\|_{L^2} \\ &\leq kC_{\mathcal{M}^{-1}}\left(\|\nabla F(u)\|_{L^2} + \|\nabla F\left(\mathcal{M}^{-1}(P_N u)\right) - \nabla F(u)\|_{L^2}\right) \\ &\leq kC_{\mathcal{M}^{-1}}\|\nabla F(u)\|_{L^2} + kC_{\mathcal{M}^{-1}}C_F\|\mathcal{M}^{-1}(P_N u) - u\|_{H^2} \\ &\leq kC_{\mathcal{M}^{-1}}(1 + C_F C_{\mathcal{M}^{-1}})\|\nabla F(u)\|_{L^2} =: c_1\|\nabla F(u)\|_{L^2}.\end{aligned}$$

Lastly, convexity of W_1 ensures $u_t = u + t(\mathcal{M}^{-1}(P_N u) - u) \in W_1$ for all $t \in [0,1]$ and the absolute difference can be estimated by

$$\begin{aligned}|F(u) - \Gamma(K(u))| &= |F(u) - F(\mathcal{M}^{-1}(P_N u))| \\ &\leq \int_0^1 \left|\left\langle \nabla F(u_t), \mathcal{M}^{-1}(P_N u) - u\right\rangle_{L^2(\mathbb{S}^2;\mathbb{R}^3)}\right|\, dt \\ &\leq \int_0^1 \|\nabla F(u_t)\|_{L^2}\|\mathcal{M}^{-1}(P_N u) - u\|_{H^2}\, dt \\ &\leq C_{\mathcal{M}^{-1}}\|\nabla F(u)\|_{L^2}\int_0^1 \|\nabla F(u)\|_{L^2} + \|\nabla F(u_t) - \nabla F(u)\|_{L^2}\, dt \\ &\leq C_{\mathcal{M}^{-1}}\|\nabla F(u)\|_{L^2}\int_0^1 \|\nabla F(u)\|_{L^2} + C_F\|t\mathcal{M}^{-1}(P_N u) + u)\|_{L^2}\, dt \\ &\leq C_{\mathcal{M}^{-1}}\|\nabla F(u)\|_{L^2}(1 + \frac{C_F}{2}C_{\mathcal{M}^{-1}})\|\nabla F(u)\|_{L^2} =: c_2\|\nabla F(u)\|_{L^2}^2\end{aligned}$$

Combining the estimates for $|(\nabla\Gamma)\circ K|$ and $|F - \Gamma\circ K|$ with the Łojasiewicz inequality for Γ, one finds that for $\tilde{c} = cc_1 + c_2 > 0$, $\gamma \in (0, \frac{1}{2})$, and all $u \in U$, which is an open neighbourhood of 0 in $H^2(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2)$,

$$\begin{aligned}|F(u) - F(0)|^{1-\gamma} &\leq |\Gamma(K(u)) - \Gamma(0)|^{1-\gamma} + |F(u) - \Gamma(K(u))|^{1-\gamma} \\ &\leq c|\nabla\Gamma(K(u))| + |F(u) - \Gamma(K(u))|^{1-\gamma} \\ &\leq cc_1\|\nabla F(u)\|_{L^2} + c_2\|\nabla F(u)\|_{L^2}^{2(1-\gamma)} \\ &\leq \tilde{c}\|\nabla F(u)\|_{L^2},\end{aligned}$$

where $2(1-\gamma) > 1$ and $\|\nabla F(u)\|_{L^2} \leq 1$ where used in the last inequality.

5.3. Observations

In this section, we gather some further consequences of the Łojasiewiz inequality.

Combining the Łojasiewicz inequality with the energy inequality near minimizers from Theorem 3, we observe the following property of critical points:

Corollary 5.2. *Let $\mathbf{m}_0 = \mathbf{m}_\theta$ be an axisymmetric minimizer such that E is strictly convex at θ. Then there exists $\rho > 0$ such that all critical points $\mathbf{m}$ with $\|\mathbf{m}-\mathbf{m}_0\|_{H^2} < \rho$ are joint rotations of $\mathbf{m}_0$. In particular, if $\|\mathbf{m}-\mathbf{m}_0\|_{H^2} < \rho$ and $\mathbf{m}$ is critical, then $\mathbf{m}$ is $\hat{\mathbf{e}}$-axisymmetric for some $\hat{\mathbf{e}} \in \mathbb{S}^2$.*

Proof. Let ρ be small enough that both inequalities hold. Then any critical point $\mathbf{m}$ satisfies

$$0 = \|(\nabla\mathcal{E})^{\tan}(\mathbf{m})\|_{L^2} \geq (\mathcal{E}(\mathbf{m}) - \mathcal{E}(\mathbf{m}_0))^{1-\gamma} \geq c \inf_{R\in SO(3)/SO(3)_{\hat{\mathbf{e}}_3}} \|\mathbf{m}_R - \mathbf{m}_0\|_{H^1}^{1-\gamma}$$

and therefore there exists $R \in SO(3)$ such that $\mathbf{m}_R = \mathbf{m}_0$. Thus, $\mathbf{m} = (\mathbf{m}_0)_{R^{-1}}$ and $\mathbf{m}$ is $R\hat{\mathbf{e}}_3$-axisymmetric by Lemma 2.3. □

The next observation is inspired by a work of Rupflin [45] on harmonic maps and concerns the critical values of the energy.

Proposition 5.2. *For any $\varepsilon > 0$, the set*

$$\{E \in [0, 8\pi - \varepsilon] \ : \ \exists \mathbf{m} \in H^1(\mathbb{S}^2;\mathbb{S}^2) \text{ with } Q(\mathbf{m}) = 0,\ \mathcal{E}(\mathbf{m}) = E, \text{ and } \nabla\mathcal{E}(\mathbf{m})^{tan} = 0\}$$

is discrete.

Proof. Assume that the statement is wrong. Then there exists a sequence of distinct values E_k and corresponding critical points $\mathbf{m}_k$ with $Q(\mathbf{m}_k) = 0$, $\mathcal{E}(\mathbf{m}_k) = E_k$ such that $E_k \to E \in [0, 8\pi - \varepsilon]$.

Convergence of the energy implies uniform H^1 boundedness of $(\mathbf{m}_k)_{k\in\mathbb{N}}$ and thus the existence of a weakly in H^1, strongly in L^2 and pointwise almost everywhere convergent subsequence $\mathbf{m}_k \rightharpoonup \mathbf{m}$. As in the proof of Lemma 1, it follows from theorem $E1$ in [3] and Lemma 4.3(ii) in [35] that there exists integers $q_1, \dots q_N$ such that

$$\lim_{k\to\infty} Q(\mathbf{m}_k) = Q(\mathbf{m}) + \sum_{i=1}^N q_i \quad \text{and} \quad \lim_{k\to\infty} \mathcal{E}(\mathbf{m}_k) = \mathcal{E}(\mathbf{m}) + 4\pi \sum_{i=1}^N |q_i|.$$

If $q_i = 0$ for all $1 \leq i \leq N$ then the convergence $\mathcal{E}(\mathbf{m}_k) \to \mathcal{E}(\mathbf{m})$ implies strong H^1 convergence which we will deal with below. Otherwise, if at least one index is non-zero, it follows that

$$\mathcal{E}(\mathbf{m}) + 4\pi \sum_{i=1}^{n} |q_i| \geq 4\pi + 4\pi = 8\pi,$$

where we have employed the lower bound of [10]. But

$$E = \lim_{k\to\infty} \mathcal{E}(\mathbf{m}_k) \leq 8\pi - \varepsilon < 8\pi,$$

a contradiction.

If the convergence of the subsequence is strong due to the norm convergence that's implied by the energy convergence, then strong convergence in H^2 follows by Lemma 5.5 and it holds that $\mathbf{m}$ is a critical point. But then the Łojasiewicz inequality at $\mathbf{m}$ implies

$$0 = \|\nabla\mathcal{E}^{\tan}(\mathbf{m})\|_{L^2} \geq \frac{1}{c}\Big(\mathcal{E}(\mathbf{m}) - \mathcal{E}(\mathbf{m}^k)\Big)^{1-\gamma}$$

and therefore $E_k \equiv E$ for all k with $\|\mathbf{m}_k - \mathbf{m}\| < \rho$, a contradiction. $\square$

6. Outlook

In the last chapters, we have discussed axisymmetric minimizers and shown that they are locally minimizing among all magnetizations $\mathbf{m} \in H^1(\mathbb{S}^2; \mathbb{S}^2)$. Going further, it would be interesting to investigate the Cauchy problem for the (LL)-equation, as minimality would imply the orbital stability of constant axisymmetric solutions. However, it would be important to first answer the question of strict convexity of the reduced energy at minimizing profiles.

In terms of energy methods, the assumption of convexity is supported by the fact that for θ a minimizing profile and $\beta \in C_c^\infty((0,\pi)) \setminus \{0\}$, the energy difference can be expressed as

$$E(\theta+\beta) - E(\theta) = \int_0^\pi (\beta')^2 \sin x + \sin^2\beta \left(\frac{\cos(2\theta)}{\sin x} + \kappa\cos(2\theta - 2x)\sin x\right) \mathrm{d}x,$$

implying that the right hand side is strictly positive for $\beta \neq 0$. On the other hand, this is closely related to

$$\frac{d^2}{dt^2}E(\theta + t\beta)|_{t=0} = \int_0^\pi (\beta')^2 \sin x + \beta^2 \left(\frac{\cos(2\theta)}{\sin x} + \kappa\cos(2\theta - 2x)\sin x\right) \mathrm{d}x,$$

the only difference being that β appears quadratically or as $\sin\beta$. However, due to $\sin^2\beta \leq \beta^2$, the possibly negative expression in parentheses is given a larger weight in the reduced Hessian than in the energy difference and no conclusion can be made.

On the other hand, due to the similarity between the present work and the stability analysis of axisymmetric Skyrmions on the plane [32] it seems strange that in the Fourier expansion of the Hessian, the zero-mode $\mathcal{H}_0$ behaves differently. In [32], the corresponding term can be controlled by employing lemma 4.2 and using properties of the profile θ. Indeed, expressing the β-part for $\mathcal{H}_0$ in terms of $\Theta = \theta - \mathrm{id}$, we have

$$\begin{aligned} I_\beta &= \int_0^\pi (\beta')^2 \sin x + \beta^2 \left(\frac{\cos(2\theta)}{\sin x} + \kappa\cos(2\theta - 2x)\sin x\right) \mathrm{d}x \\ &= \int_0^\pi (\beta')^2 + \beta^2 \left(\cos(2\Theta)\left(\frac{\cos(2x)}{\sin x} + \kappa \sin x\right) - 2\sin(2\Theta)\cos x\right) \mathrm{d}x \end{aligned}$$

which is reminiscent of the term in [32], given by

$$\int_0^\infty (\beta')^2 r + \beta^2 \left(\frac{\cos(2\theta)}{r} + h\cos(\theta) - 2\sin(2\theta) \right) \, \mathrm{d}r.$$

As observed in section 2.2, the Skyrmion is expected to behave similar to the planar case at its center with possible differences in the tail. However, note that

$$\begin{aligned} &\cos(2\Theta) \left(\frac{\cos(2x}{\sin x} + \kappa \sin x \right) - 2\sin(2\Theta)\cos x \\ &= \cos(2\Theta) \left(\frac{\cos^2 x}{\sin x} + (\kappa - 1)\sin x \right) - 2\sin(2\Theta)\cos x \\ &> \frac{\sqrt{2}}{2} \left(\frac{\cos^2 x}{\sin x} + (\kappa - 1)\sin x - 2\cos x \right) \\ &= \frac{\sqrt{2}}{2} \left(\frac{(\cos x - \sin x)^2}{\sin x} + (\kappa - 2)\sin x \right) > 0 \end{aligned}$$

for $\Theta < \frac{\pi}{8}$, $x < \frac{\pi}{2}$ and $\kappa > 2$, implying that the integrand is positive on $(y_{\frac{\pi}{8}}, \frac{\pi}{2})$. Furthermore, the expression of the first line is obviously positive for $\Theta < \frac{\pi}{4}$ and $x > \frac{\pi}{2}$. It follows that positivity of I_β only depends on the behavior of the Skyrmion for small x, i.e. at its center. This suggests that one might be able to follow a similar strategy as in [31].

Applying Lemma 4.2 with $\beta = (1 - \theta')\eta$ and then again with $\psi = \sqrt{\tan(\frac{x}{2})}$, one arrives at

$$\begin{aligned} I_\beta = &\int_0^\pi \sin x \tan(\frac{x}{2})(1 - \theta')^2 \left(\left(\frac{\eta}{\tan(\frac{x}{2})} \right)' \right)^2 \\ &+ \eta^2 \frac{(\theta' - 1)}{\sin x} \left(-\frac{1}{4}(\theta' - 1) + \sin(\theta - x) \left(\frac{\cos(\theta - x)\cos x}{\sin x} - (\kappa - 2)\cos(\theta - x)\sin x \right) \right) \\ &+ \eta^2 \frac{(1 - \theta')^2}{\sin x} \left((1 - \cos x) \left((1 - \theta')\cos x - \frac{\sin(2\theta - 2x)}{2} \frac{\cos x}{\sin x} + 2\sin^2(\theta - x) \right) \right) \mathrm{d}x. \end{aligned}$$

Here, the integral in the first two lines is again similar to

$$\int_0^\infty (\theta')^2 \left(\left(\frac{\eta}{\sqrt{r}} \right)' \right)^2 + \eta^2 \frac{\theta'}{r} \left(-\frac{\theta'}{4} + \sin\theta \left(\frac{\cos\theta}{r} - hr \right) \right) \mathrm{d}r$$

while the integrand of the second integral is positive on $[y_{\frac{3\pi}{4}}, y_{\frac{\pi}{4}}]$ for $\kappa > 24$ because

$$\begin{aligned} (1 - \theta')\cos x - &\frac{\sin(2\theta - 2x)}{2} \frac{\cos x}{\sin x} + 2\sin^2(\theta - x) \\ &> \frac{\sin(\theta - x)}{\sin x}\cos x - \frac{\sin(\theta - x)}{\sin x}\cos(\theta - x)\cos x + 2\sin^2(\theta - x) \\ &= \frac{\sin(\theta - x)}{\sin x}\cos x \, (1 - \cos(\theta - x)) + 2\sin^2(\theta - x) > 0 \end{aligned}$$

for $x < \frac{\pi}{2}$. However, the bound for x follows from $x < y_{\frac{\pi}{4}} < \frac{\pi}{2}$ by the estimate in the proof of Lemma 3.10.

If the techniques of [31] are sufficient to prove non-negativity of the integral in the first line, at least for small x, then positivity of I_β for arbitrary β would follow by a partition of unity.

Once positivity of I_β has been established, there are two possible directions to continue the research. Focussing on the static case, on might investigate the question whether $Q = 0$ axisymmetric fields are globally minimizing within the set of equivariant fields with $Q = 0$. To do so, a first step might be to consider the system of coupled ODE (2.1), (2.2) and find out whether they have a solution (θ, χ) such that χ is non-constant.

On the other hand, the easily obtained orbital stability of axisymmetric minimizers, which is due to the energy conservation of the Hamiltonian system makes it attractive to first consider the dynamical properties. To establish the local and then global existence of soultions in $H^2(\mathbb{S}^2; \mathbb{S}^2)$, an adaptation of the methods in [17] or [7] seems promising. However, both of these works only deal with an energy consisting of Dirichlet term and anisotropy. They do not involve any non-squared derivatives like the (DMI)-like terms appearing after application of the stereographic projection, see the end of section 2.2. Thus, a careful analysis of the mixed terms is necessary.

If solutions of (1.1) would exist for initial values close to $\mathbf{m}^*$ where $\mathbf{m}^*$ is a axisymmetric minimizer then one could obtain

Lemma. *Let $\kappa > 24$. There exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$, there exists $\delta > 0$ such that $\|\mathbf{m} - \mathbf{m}^*\|_{H^1(\mathbb{S}^2;\mathbb{R}^3)} < \delta$ implies*

$$\inf_{R \in SO(3)/SO(3)_{\hat{\mathbf{e}}_3}} \|\mathbf{m}(t) - \mathbf{m}^*\|_{H^1(\mathbb{S}^2;\mathbb{R}^3)} < \varepsilon$$

for all $t > 0$ where $\mathbf{m}(t) \in C^0([0,\infty); H^2(\mathbb{S}^2; \mathbb{S}^2))$ solves (1.1) with $\mathbf{m}(0) = \mathbf{m}$.

Proof. For the proof, take $\varepsilon_0, c > 0$ from proposition 3. For every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|\mathbf{m} - \mathbf{m}^*\|_{H^1} < \delta$ implies $|\mathcal{E}(\mathbf{m}) - \mathcal{E}(\mathbf{m}^*)| = \mathcal{E}(\mathbf{m}) - \mathcal{E}(\mathbf{m}^*) < c\varepsilon$.

$$\begin{aligned} E(\mathbf{m}) - E(\mathbf{m}^*) &= \int_{\mathbb{S}^2} |\nabla \mathbf{m}|^2 - |\nabla \mathbf{m}^*|^2 \, \mathrm{d}\sigma + \int_{\mathbb{S}^2} (\mathbf{m}^* \cdot \nu)^2 - (\mathbf{m} \cdot \nu)^2 \, \mathrm{d}\sigma \\ &\leq \|\nabla \mathbf{m} - \nabla \mathbf{m}^*\|_{L^2} \left(\|\nabla \mathbf{m}\|_{L^2} + \|\nabla \mathbf{m}^*\|_{L^2} \right) \\ &\quad + \int_{\mathbb{S}^2} \left((\mathbf{m}^* - \mathbf{m}) \cdot \nu \right) \left((\mathbf{m}^* + \mathbf{m}) \cdot \nu \right) \mathrm{d}\sigma \\ &\leq \|\mathbf{m} - \mathbf{m}^*\|_{H^1(\mathbb{S}^2)} \left(2\|\nabla \mathbf{m}^*\|_{L^2} + \delta + 8\pi\kappa \right). \end{aligned}$$

Choose $\varepsilon > 0$ and $\mathbf{m} \in H^2(\mathbb{S}^2; \mathbb{S}^2)$ with $\|\mathbf{m} - \mathbf{m}^*\|_{H^1} < \delta$ with $\delta > 0$ from above. Let $\mathbf{m}(t)$ be the solution of (1.1) with initial value $\mathbf{m}(0) = \mathbf{m}$. Since $\mathbf{m}(t) \in X^2(\mathbb{S}^2; \mathbb{S}^2)$ in the notation of 2.1, it holds that $\mathcal{E}(\mathbf{m}(t)) = \mathcal{E}(\mathbf{m})$ for all $t \in [0, \infty)$. Therefore, as long as $\|\mathbf{m}(t) - \mathbf{m}^*\|_{H^1} < \varepsilon$, it follows that

$$\inf_{R \in SO(3)/SO(3)_{\hat{\mathbf{e}}_3}} \|\mathbf{m}(t) - \mathbf{m}^*\|_{H^1} \leq \frac{1}{c} \left(\mathcal{E}(\mathbf{m}(t) - \mathcal{E}(\mathbf{m}^*)\right) = \frac{1}{c} \left(\mathcal{E}(\mathbf{m}) - \mathcal{E}(\mathbf{m}^*)\right) < \varepsilon < \varepsilon_0.$$

Thus, the inequality remains valid on an open interval $(t, t + \tau)$ and subsequently on $(0, \infty)$. Note that $\delta < \varepsilon < \varepsilon_0$ establishes the bound for small t. □

In Hamiltonian systems, the conservation of energy generally prevents asymptotic stability of solutions because convergence would infer the convergence of energy which is impossible. Furthermore, in the presence of symmetry, one can not expect to rule out the infimum over all equivalence classes $[R] \in SO(3)/SO(3)_{\hat{\mathbf{e}}_3}$. Instead, it would be interesting to use the framework of [15] for stability in the presence of symmetry to investigate the stability of a set of non-static solutions.

The Landau Lifshitz equation is of the form

$$\frac{d}{dt}\mathbf{m}(t) = \mathcal{J} E'(\mathbf{m}(t))$$

with $\mathcal{J} \colon T_{\mathbf{m}}\mathbb{S}^2 \to T_{\mathbf{m}}\mathbb{S}^2$ given by $\mathcal{J}\xi = \mathbf{m} \times \xi$. It is invariant under the operation $(\mathbf{m}, R) \mapsto \mathbf{m}_R$ of the one-parameter group $SO(3)_{\hat{\mathbf{e}}} \simeq SO(2)$. The paper [15] then provides the framework to study the stability of solitonic solutions

$$\mathbf{m}(t) = \mathbf{m}_{R(\omega t)} = R(\omega t)^{-1}\mathbf{m}(R(\omega t)\cdot)$$

where $R(t)$ is a rotation by angle t around $\hat{\mathbf{e}}_3$. Furthermore, the angular momentum $\mathbf{J}$ is a conserved quantity of the equation and a generator of joint rotations [37]. In particular, J_3 is a generator of joint rotations around $\hat{\mathbf{e}}_3$ and invariant under them. It can therefore function as the conserved functional Q in [15].

In [15], the authors state three assumptions under which they than prove the stability of „bound states“, time-dependent solutions of the equation which are obtained by applying the group operation to static profiles. While the first assumption, existence of solutions to the Cauchy problem, is still open for (1.1), the second assumption deals with the existence of bound states.

In [37] together with chapter 4 of this work it has been proven that solutions of

$$\mathbf{m} \times \nabla\mathcal{E}(\mathbf{m}) = \omega\mathbf{m} \times \nabla J_3(\mathbf{m})$$

exist for $|J_3(\mathbf{m})|$ close to 4π where the $|J_3(\mathbf{m})| = 4\pi$ solution is equivariant with $\omega = 0$. However, [15] requires existence for $\omega \in (\omega_1, \omega_2)$ with $0 \in (\omega_1, \omega_2)$. In order to capitalize on our existence result, one would need to determine the relationship between $J_3(\mathbf{m}) - 4\pi$

and ω, aiming for continuity. If this could be done, then the third step would be to investigate the eigenvalues of

$$H_\omega(\mathbf{m}) = \mathcal{H}(\mathbf{m}) - \omega\delta^2\mathbf{J}(\mathbf{m})$$

in order to verify the third assumption of [15]. For $\omega = 0$, we have already proved for axisymmetric solutions that $H_0 = \mathcal{H}(\mathbf{m})$ has no negative eigenvalues and that its kernel is spanned by tangent fields associated to the operation of $SO(3)$. Aiming to extend this to $\omega \neq 0$, i.e. to constrained minimizers, it could be interesting to consider the effect that elliptical distortion has on $\mathcal{H}(\mathbf{m})$ where $\mathbf{m}$ is an axisymmetric minimizer. Furthermore, an expression of these minimizers in spherical coordinates would be helpful in adapting the methods of the axisymmetric case.

A. Appendix

A.1. Spherical Coordinates and Moving Frame

In section 2.2, we have found representation in spherical coordinates that are specific to equivariant and axisymmetric fields. Furthermore, in chapter 3, we have used this expression to define a moving frame on $\mathbb{S}^2 \setminus \{\pm\hat{\mathbf{e}}_3\}$. We have extended both the coordinate representation and the decomposition via the moving frame to the poles. In the following, we will give further justification for this process and prove that the coefficient functions of the Fourier decomposition of tangent fields are well-defined at the poles.

First, consider the regular parametrization

$$\Psi\colon (0,\pi)\times(0,2\pi) \to \mathbb{S}^2 \setminus \{(y_1,y_2,y_3) \in \mathbb{S}^2 \ : \ y_1 \geq 0,\ y_2 = 0\},\ \Psi(x,\varphi) = \begin{pmatrix} \sin x \cos\varphi \\ \sin x \sin\varphi \\ \cos x \end{pmatrix}.$$

For computation purposes, we may extend it to $\Psi\colon (0,\pi)\times[0,2\pi) \to \mathbb{S}^2\setminus\{\pm\hat{\mathbf{e}}_3\}$, which is still injective. At the poles, the injectivity degenerates. However,

$$\lim_{x\searrow 0} \Psi(x,\varphi) = \hat{\mathbf{e}}_3 \quad \text{and} \quad \lim_{x\nearrow \pi} (x,\varphi) = -\hat{\mathbf{e}}_3$$

independently of φ. Thus, setting $\Psi(0,\varphi) = \hat{\mathbf{e}}_3$ and $\Psi(\pi,\varphi) = -\hat{\mathbf{e}}_3$ for all $\varphi \in [0,2\pi)$ results in a continuous and surjective map $\Psi\colon [0,\pi]\times[0,2\pi) \to \mathbb{S}^2$.

Given $\mathbf{m} \in C(\mathbb{S}^2;\mathbb{S}^2)$ and using the same coordinates on the target sphere, we find functions

$$\tilde{\theta}\colon (0,\pi)\times[0,2\pi) \to (0,\pi), \quad \tilde{\phi}\colon (0,\pi)\times[0,2\pi) \to [0,2\pi),$$

not necessarily continuous, such that

$$\mathbf{m}(\Psi(x,\varphi)) = \begin{pmatrix} \sin\tilde{\theta}(x,\varphi)\cos\tilde{\phi}(x,\varphi) \\ \sin\tilde{\theta}(x,\varphi)\sin\tilde{\phi}(x,\varphi) \\ \cos\tilde{\theta}(x,\varphi) \end{pmatrix} \quad \text{for} \quad \Psi(x,\varphi) \in \mathbb{S}^2 \setminus \{\mathbf{m} = \pm\hat{\mathbf{e}}_3\}.$$

To ensure continuity of the coordinate functions θ, ϕ, observe that $\tilde{\theta}$, $\tilde{\phi}$ are as regular as $\mathbf{m}$ on any relatively open subset G of $\mathbb{S}^2$ such that $\mathbf{m}(\mathbf{x}) \in \mathbb{S}^2 \setminus \{y_1 \geq 0,\ y_2 = 0\}$ and $\mathbf{x} \in \mathbb{S}^2 \setminus \{x_1 \geq 0,\ x_2 = 0\}$ for all $\mathbf{x} \in G$. Discontinuities occur when $\tilde{\phi}$ approaches 0 or 2π and when $\tilde{\theta}$ approaches 0 or π.

Instead of $\tilde{\phi}\colon (0,\pi) \times [0,2\pi) \to (0,2\pi)$, it is more convenient to consider the sphere without poles as a cylinder over the circle:

$$\mathbb{S}^2 \setminus \{\pm\hat{\mathbf{e}}_3\} = (0,\pi) \times \mathbb{S}^1.$$

In this representation, regularity of $\tilde{\phi}\colon (0,\pi) \times \mathbb{S}^1 \to \mathbb{S}^1$ immediately follows from regularity of $\mathbf{m}$ at $\mathbb{S}^2 \setminus \{\pm\hat{\mathbf{e}}_3\}$. Opening up the circle at $\varphi = 0$, we can furthermore construct a continuous coordinate function $\phi\colon (0,\pi) \times (0,2\pi) \to \mathbb{R}$ by adding 2π for every covering of $\mathbb{S}^1$ by $\tilde{\phi}$.

The more difficult problem is posed by the discontinuities for $\tilde{\theta}$ approaching 0 or π. In general, the coordinate functions could not be extended. In the equivariant case however, the set $\{\mathbf{m} = \pm\hat{\mathbf{e}}_3\}$, which is responsible for the discontinuities of $\tilde{\theta}$, consists of rings of the form $\Psi(\{x\}\times[0,2\pi))$. Indeed, if $\mathbf{m}(\Psi(x,\varphi)) = \pm\hat{\mathbf{e}}_3$, then

$$\mathbf{m}(\Psi(x,\varphi+\alpha)) = R_\alpha R_\alpha^{-1}\mathbf{m}\left(R_\alpha\Psi(x,\varphi)\right) = R_\alpha\mathbf{m}(x,\varphi) = R_\alpha \pm \hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_3,$$

where $R_\alpha \in SO(3)_{\hat{\mathbf{e}}_3}$ is a rotation by angle α. Thus, starting at the north pole, we can construct a continuous function $\theta\colon (0,\pi) \to \mathbb{R}$ by adding multiples of π on each connectivity component of $\mathbb{S}^2 \setminus \{\mathbf{m} = \pm\hat{\mathbf{e}}_3\}$. Recall that for minimizing axisymmetric fields, $\theta < 2\pi$ was a very easy consequence of minimality as shown in Lemma 3.1 and the following.

With continuous coordinate functions on $\mathbb{S}^2 \setminus \{x_1 \geq 0,\ x_2 = 0\}$, the computations of section 2.2 are justified and the special form of $\phi(x,\varphi) = \chi(x) + \varphi$ allows to extend the representation to $\mathbb{S}^2 \setminus \pm\{\hat{\mathbf{e}}_3\}$. Furthermore, the symmetry implies $\mathbf{m}(\pm\hat{\mathbf{e}}_3) \in \{\hat{\mathbf{e}}_3, -\hat{\mathbf{e}}_3\}$ and therefore the value of θ has to be a multiple of π at the poles. Since θ is independent of φ, this can be done such that $\theta\colon [0,\pi] \to \mathbb{R}$ is continuous. The values of χ at the poles can not be determined.

Differentiability of the coordinate functions θ and χ on $\mathbb{S}^2\setminus\{\pm\hat{\mathbf{e}}_3\}$ follows by the same arguments as the continuity and the same holds for higher order derivatives.

We now consider the moving frame

$$X = \begin{pmatrix} \sin\varphi \\ -\cos\varphi \\ 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix},$$

pointwisely spanning $T_{\mathbf{m}_0}\mathbb{S}^2$. On $\mathbb{S}^2 \setminus \{\pm\hat{\mathbf{e}}_3\}$, these are well defined. At $\pm\hat{\mathbf{e}}_3$ however, the parametrization Ψ is not injective such that X, Y can not be defined. However, if $\phi\colon \mathbb{S}^2 \to T_{\mathbf{m}_0}\mathbb{S}^2$ is continuous at $\hat{\mathbf{e}}_3$, then for fixed $\varphi \in [0,2\pi)$ it holds that

$$\phi(\hat{\mathbf{e}}_3) = \lim_{x\searrow 0}\phi(x,\varphi) = \lim_{x\searrow 0}(\phi(x,\varphi)\cdot X(x,\varphi))X(x,\varphi) + (\phi(x,\varphi)\cdot Y(x,\varphi))Y(x,\varphi).$$

This can also be verified in the coordinate representation. Write $u_1 = \phi{\cdot}X$ and $u_2 = \phi{\cdot}Y$. Then

$$\begin{aligned} u_1 &= \phi\cdot X = \sin\varphi\phi_1 - \cos\varphi\phi_2, \\ u_2 &= \phi\cdot Y = \cos\theta\cos\varphi\phi_1 + \cos\theta\sin\varphi\phi_2 - \sin\theta\phi_3. \end{aligned}$$

and, again for fixed φ,

$$
\begin{aligned}
&u_1 X + u_2 Y(x,\varphi) \\
&= \begin{pmatrix} \sin^2\varphi\phi_1 - \sin\varphi\cos\varphi\phi_2 + \cos^2\theta\cos^2\varphi\phi_1 + \cos^2\theta\cos\varphi\sin\varphi\phi_2 - \sin\theta\cos\theta\phi_3 \\ -\cos\varphi\sin\varphi\phi_1 + \cos^2\varphi\phi_2 + \cos^2\theta\cos\varphi\sin\varphi\phi_1 + \cos^2\theta\sin^2\varphi\phi_2 - \sin\theta\cos\theta\phi_3 \\ \sin^2\theta\cos\theta\phi_1 + \sin\theta\cos\theta\sin\varphi\phi_2 + \sin^2\theta\phi_3 \end{pmatrix} \\
&= \begin{pmatrix} \phi_1\left(\sin^2\varphi + \cos^2\theta\cos^2\varphi\right) + \phi_2\sin\varphi\cos\varphi(-1+\cos^2\theta) - \phi_3\sin\theta\cos\theta \\ \phi_2\left(\cos^2\varphi + \cos^2\theta\sin^2\varphi\right) + \phi_1\sin\varphi\cos\varphi\left(-1+\cos^2\theta\right) - \phi_3\sin\theta\cos\theta \\ \sin\theta\left(\sin\theta\cos\theta\phi_1 + \cos\theta\sin\varphi\phi_2 + \sin\theta\phi_3\right) \end{pmatrix} \\
&\longrightarrow \begin{pmatrix} \phi_1(\hat{\mathbf{e}}_3) \\ \phi_2(\hat{\mathbf{e}}_3) \\ 0 \end{pmatrix} = \phi(\hat{\mathbf{e}}_3),
\end{aligned}
$$

because $\phi_3(\hat{\mathbf{e}}_3) = \phi \cdot \hat{\mathbf{e}}_3 = -\phi \cdot m(\hat{\mathbf{e}}_3) = 0$ and because $\lim_{x\to 0}\theta(x) = 0$. The same holds for $x \to \pi$.

Unfortunately, the individual coefficients u_i do not converge as $x \to 0$ or $x \to \pi$ because they explicitly depend on φ. On the other hand, if the dependence of φ is evened out by considering a Fourier decomposition of ϕ with respect to φ, then convergence can be guaranteed.

Lemma A.1. *Given $\phi \in C^\infty(\mathbb{S}^2; T_{\mathbf{m}_0}\mathbb{S}^2)$ with $\phi = u_1 X + u_2 Y$, let $a_k^{(i)}$, $b_k^{(i)} \colon (0,\pi) \to \mathbb{R}$ be functions such that*

$$u_i(x,\varphi) = a_0^{(i)}(x) + \sum_{k=1}^{\infty}\left(a_k^{(i)}(x)\cos(k\varphi) + b_k^{(i)}\sin(k\varphi)\right), \quad i = 1,2$$

almost everywhere. Then

$$\lim_{x\to a} a_k^{(i)}(x) = 0 = \lim_{x\to a} b_k^{(i)}(x), \quad i = 1,2$$

for $a \in \{0,\pi\}$ and for $k \in \mathbb{N}_{>1}$. Furthermore, $\lim_{x\to a} a_0^{(i)}(x) = 0$ for $a \in \{0,\pi\}$ and $i = 1,2$ and

$$
\begin{aligned}
\lim_{x\to a} a_1^{(1)}(x) &= -\phi_2(\Psi(a)), & \lim_{x\to a} a_1^{(2)}(x) &= -\phi_1(\Psi(a)), \\
\lim_{x\to a} b_1^{(1)}(x) &= \phi_1(\Psi(a)), & \lim_{x\to a} b_1^{(2)}(x) &= -\phi_2(\Psi(a)),
\end{aligned}
$$

where again $a \in \{0,\pi\}$ and $\Psi(0) = \hat{\mathbf{e}}_3$, $\Psi(\pi) = -\hat{\mathbf{e}}_3$.

Proof. As Fourier coefficients, the $a_k^{(i)}$ and $b_k^{(i)}$ are pointwisely given by

$$a_k^{(i)}(x) = \frac{1}{\pi}\int_0^{2\pi} u_i(x,\varphi)\cos(k\varphi)\,\mathrm{d}\varphi \quad \text{and} \quad b_k^{(i)}(x) = \frac{1}{\pi}\int_0^{2\pi} u_i(x,\varphi)\sin(k\varphi)\,\mathrm{d}\varphi,$$

where again $u_1 = \phi \cdot X$ and $u_2 = \phi \cdot Y$. Being parameter integrals, they are as regular as ϕ for $x \in (0, \pi)$. Furthermore, by dominated convergence, the $a_k^{(i)}$ satisfy

$$\begin{aligned}\lim_{x\to\pi} a_k^{(1)} &= \lim_{x\to\pi} \frac{1}{\pi} \int_0^{2\pi} \phi(x,\varphi) \cdot X(x,\varphi) \cos(k\varphi)\,\mathrm{d}\varphi \\ &= \frac{1}{\pi} \int_0^{2\pi} \phi(-\hat{\mathbf{e}}_3) \cdot \begin{pmatrix} \sin\varphi\cos(k\varphi) \\ -\cos\varphi\cos(k\varphi) \\ 0 \end{pmatrix} \mathrm{d}\varphi\end{aligned}$$

and

$$\begin{aligned}\lim_{x\to\pi} a_k^{(2)} &= \lim_{x\to\pi} \frac{1}{\pi} \int_0^{2\pi} \phi(x,\varphi) \cdot Y(x,\varphi) \cos(k\varphi)\,\mathrm{d}\varphi \\ &= \frac{1}{\pi} \int_0^{2\pi} \phi(-\hat{\mathbf{e}}_3) \cdot \begin{pmatrix} -\cos\varphi\cos(k\varphi) \\ -\sin\varphi\cos(k\varphi) \\ 0 \end{pmatrix} \mathrm{d}\varphi.\end{aligned}$$

Here, we have used that ϕ and $|X|$, $|Y|$ are bounded near π and that for every fixed $\varphi \in [0, 2\pi)$, the limits $\lim_{x\to\pi} X(x,\varphi)$ and $\lim_{x\to\pi} Y(x,\varphi)$ exist. The problem in extending X, Y to the poles only lies in the fact that these limits are not independent of φ such that the limit for $\mathbf{x} \to -\hat{\mathbf{e}}_3$ does not exist. Furthermore, we have employed $\theta(x) = \pi$, which holds for minimizing profiles. For non-minimizing axisymmetric $\mathbf{m}$, $\cos(\theta(\pi)) = 1$ is possible, resulting in a change of the sign.

After computing the scalar product, the constants $\phi_1(-\hat{\mathbf{e}}_3)$ and $\phi_2(-\hat{\mathbf{e}}_3)$ can be moved in front of the integral so that we are left with standard integrals of the type

$$\int_0^{2\pi} \cos(\varphi)\cos(k\varphi)\,\mathrm{d}\varphi \quad \text{and} \quad \int_0^{2\pi} \sin(\varphi)\cos(k\varphi)\,\mathrm{d}\varphi.$$

For $k > 1$ and $k = 0$, both of these integrals are 0. For $k = 1$, their values are π and 0, respectively. Therefore, $a_1^{(1)}(\pi) = -\phi_2(-\hat{\mathbf{e}}_3)$ and $a_1^{(2)}(\pi) = -\phi_1(-\hat{\mathbf{e}}_3)$. Similarly, the limits of $b_k^{(i)}$ at π exist and satisfy

$$\lim_{x\to\pi} b_k^{(1)}(x) = \frac{1}{\pi} \int_0^{2\pi} \phi(-\hat{\mathbf{e}}_3) \cdot \begin{pmatrix} \sin\varphi\sin(k\varphi) \\ -\cos\varphi\sin(k\varphi) \\ 0 \end{pmatrix} \mathrm{d}\varphi = \begin{cases} \phi_1(-\hat{\mathbf{e}}_3) & k = 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\lim_{x\to\pi} b_k^{(2)}(x) = \frac{1}{\pi} \int_0^{2\pi} \phi(-\hat{\mathbf{e}}_3) \cdot \begin{pmatrix} -\cos\varphi\sin(k\varphi) \\ -\sin\varphi\sin(k\varphi) \\ 0 \end{pmatrix} \mathrm{d}\varphi = \begin{cases} -\phi_2(-\hat{\mathbf{e}}_3) & k = 1 \\ 0 & \text{otherwise.} \end{cases}$$

At $x = 0$, the result is identical but ϕ is evaluated at $\hat{\mathbf{e}}_3$. $\square$

A.2. Analyticity of Functionals

In chapter 4 we have proven a Łojasiewicz-Simon type inequality for the pulled-back functional $F = \mathcal{E} \circ \psi$. One of the key ingredients is the analyticity of F because it allows to apply the finite-dimensional Łojasiewicz inequality to a reduced function Γ. In the following, we follow [56] to give a definition of analytic functionals. We then collect some useful results.

Definition 7. *Let A, B be Banach spaces. The functional $f\colon A \to B$ is called analytic in $a \in A$ if there exists a sequence of n-linear operators $\phi_n\colon A^n \to B$ and $r > 0$ such that*

$$\sum_{n=1}^{\infty} \|\phi_n\|_{\mathcal{L}^n(A,B)} r^n < \infty$$

and for all $h \in A$ with $\|h\| < r$ it holds that

$$f(a+h) - f(a) = \sum_{n=1}^{\infty} \phi_n(h, \ldots, h).$$

By the linearity of norms it immediately follows that the analytic functionals on A form a linear space. To have some first examples, consider n-linear operators taking n times the same argument. These are the equivalent of order n monomials in finite dimensions.

Lemma A.2. *Let $\mathcal{B}_1$, $\mathcal{B}_2$ be Banach spaces, $k \in \mathbb{N}$ and $T \in \mathcal{L}^k(\mathcal{B}_1, \mathcal{B}_2)$. Then the operator $\tilde{T}\colon \mathcal{B}_1 \to \mathcal{B}_2, x \mapsto T(x, \ldots x)$ is analytic.*

Proof. We show that $\tilde{T}$ is analytic at 0 with an infinite radius of convergence.

Set $\phi^\ell\colon \mathcal{B}_1^k \to \mathbb{R}$ to be

$$\phi^\ell(x^{(1)}, \ldots, x^{(\ell)}) = T(x^{(1)}, \ldots, x^{(k)}) \text{ if } \ell = k$$

and $\phi^\ell \equiv 0$ otherwise. Then all ϕ^ℓ are ℓ-linear mappings, $\tilde{T}(u) = \sum_{k=0}^{\infty} \phi^k(u, \ldots u)$ and

$$\sum_{\ell=0}^{\infty} \|\phi^\ell\|_{L^k(\mathcal{B}_1,\mathcal{B}_2)} r^\ell = \|T\|_{L^l} r^k < \infty \text{ for all } r > 0.$$

□

Corollary A.1. *The functional*

$$F_1\colon H^2(\mathbb{S}^2; \mathbb{R}^3) \to \mathbb{R}, \ m \mapsto \frac{1}{2} \int\limits_{\mathbb{S}^2} |\nabla m|^2 \, d\sigma$$

is analytic.

Proof. We can write $F_1(m) = \langle \nabla m, \nabla m\rangle_{L^2(\mathbb{S}^2;\mathbb{R}^3)}$. Since the L^2-scalar product is a bilinear, bounded operator on $H^1(\mathbb{S}^2;\mathbb{R}^3)$ while $\nabla\colon H^2 \to H^1$ is bounded and linear, analyticity follows from Lemma A.2 and the following result by Whittlesey. □

Theorem
– from [56]
The composition of analytic functionals is analytic.

Definition 7 extends the notion of analyticity from finite to infinite dimensions in a natural way. Hence, functionals that are defined via an analytic function can inherit this property if the norm $\|\cdot\|_A$ is sufficiently strong.

Lemma A.3. *Let A be a Banach space consisting of functions $u\colon \Omega \to \mathbb{R}^n$ where $\Omega \subset \mathbb{R}^m$ is open and bounded and assume $\|u\|_{L^\infty(\Omega)} \leq c\|u\|_A$ for some $c > 0$. If a function $f\colon \mathbb{R}^n \to \mathbb{R}$ is analytic on a neighborhood of 0, then so is*

$$F\colon A \to \mathbb{R},\ F(u) = \int\limits_\Omega f(u(x))\ dx.$$

Proof. We show that F is analytic in 0. For $u \in A$ close to 0, the proof is identical as long as f is analytic in $\|u\|_{L^\infty} \leq c\|u\|_A$.

Since f is analytic in 0, there are $r > 0$ and $a_\alpha \in \mathbb{R}$ for all multi-indices $\alpha \in \mathbb{N}^n$ such that

$$f(x) = f(0) + \sum_{k=1}^{\infty} \sum_{|\alpha|=k} a_\alpha x^\alpha \text{ for all } |x| < r,$$

where the series is absolutely convergent on $B_r(0)$. Given a multi-index $\alpha \in \mathbb{N}^n$ with length $|\alpha| = \sum_{i=1}^n \alpha_i = k$, set

$$\phi^\alpha\colon E^n \to \mathbb{R},\ \phi^\alpha(u^{(1)}, \dots, u^{(k)}) = \int\limits_\Omega a_\alpha \prod_{i=1}^{k} \prod_{j=1+\sum_{l=1}^{i-1}\alpha_l}^{\sum_{l=1}^{i}\alpha_l} u_i^{(i)}$$

and

$$\phi_k = \sum_{|\alpha|=k} \phi^\alpha.$$

Then ϕ^k is k-linear as the sum of k-linear functionals. Furthermore, ϕ^k is bounded due to $|u(x)| \leq \|u\|_{L^\infty(\Omega)} \leq c$ for almost all $x \in \Omega$ and $u \in A$ with $\|u\|_A = 1$:

$$\begin{aligned}
\|\phi^k\|_{L^k(E,\mathbb{R})} &= \sup_{\|u^{(i)}\|_A=1} |\phi^k(u^{(1)},\dots,u^{(k)})| \leq \sum_{|\alpha|=1} \sup_{\|u^{(i)}\|_A=1} |\phi^\alpha(u^{(1)},\dots,u^{(k)})| \\
&\leq \sum_{k=1}^{n} \sup_{\|u^{(i)}\|_A=1} \int_\Omega |a_\alpha| \prod_{i=1}^{k} \|u^{(i)}\|_{L^\infty(\Omega)}\ dx \\
&\leq \sum_{k=1}^{n} \sup_{\|u^{(i)}\|_A=1} \int_\Omega |a_\alpha| c^k\ dx \\
&= \sum_{k=1}^{n} |\Omega||a_\alpha|c^k.
\end{aligned}$$

For $|\rho| < \frac{r}{c}$, the series $\sum_{k=0}^{\infty} \|\phi^k\|_{L^k(E,\mathbb{R})}\rho^k$ is absolutely convergent since the terms are bounded by

$$\|\phi^k\|_{L^k(A,\mathbb{R})}\rho^k \leq |\Omega| \sum_{|\alpha|=k} |a_\alpha||c\rho|^k$$

and the series representation of f is absolutely convergent.

Furthermore, since absolute convergence and integrability of $\phi^k(u,\dots,u)$ due to the L^∞-embedding allow to change the order of summation and integration, it follows for $\|u\|_A < \frac{r}{c}$ that

$$\begin{aligned}
F(u) = \int_\Omega f(u(x))\ dx &= \int_\Omega f(0)\,\mathrm{d}x + \sum_{k=1}^{\infty} \sum_{|\alpha|=k} a_\alpha \prod_{i=1}^{k} u_i(x)^{\alpha_i}\ dx \\
&= F(0) + \sum_{k=0}^{\infty} \phi^k(u,\dots,u).
\end{aligned}$$

□

By a similar argument, the composition operator with an analytic function is analytic if it is defined on a Banach space that is an algebra. We prove this statement for the special case $H^2(\Omega)$ where $\Omega \subset \mathbb{R}^n$ and $n \in \{2,3\}$.

Lemma A.4. *Let $\Omega \in \mathbb{R}^n$ be bounded for $n \in \{2,3\}$ and $f\colon \mathbb{R}^n \to \mathbb{R}^n$ analytic in a neighborhood of 0 in the sense that there are $r > 0$ and $a_\alpha \in \mathbb{R}^n$ for every multi-index $\alpha \in \mathbb{N}^n$ such that*

$$f(x) = f(0) + \sum_{k=1}^{\infty} \sum_{|\alpha|=k} a_\alpha x^\alpha \text{ for all } |x| < r$$

and such that all n components of the sum converge absolutely for all $|x| < r$. Then the functional

$$F\colon H^2(\Omega;\mathbb{R}^n) \to H^2(\Omega,\mathbb{R}^n),\ u \mapsto f \circ u$$

is analytic in a neighborhood of 0.

Proof. Since $2 > \frac{n}{2} \in \{1, \frac{3}{2}\}$, there is $c_\infty > 0$ such that $\|u\|_\infty = \|u\|_{C^0} \le c_\infty \|u\|_{H^2}$. Consequently, if $u \in B_{\frac{r}{c_\infty}}(0) \subset H^2(\Omega; \mathbb{R}^n)$, $|u(x)| \le \|u\|_{L^\infty} < r$ for almost all $x \in \Omega$ and

$$F(u)(x) = f \circ u(x) = f(0) + \sum_{k=1}^{\infty} \sum_{|\alpha|=k} a_\alpha u(x)^\alpha$$

with absolute convergence for almost every $x \in \Omega$. For a multi-index $\alpha \in \mathbb{N}^n$ with length $k = \sum_{l=1}^{n} \alpha_l$, set

$$\phi^\alpha(u_1, \dots u_k) = a_\alpha \prod_{i=1}^{k} \prod_{j=(\sum_{\ell=1}^{i-1}\alpha_\ell)+1}^{\sum_{\ell=1}^{i}\alpha_\ell} (u_j)_i$$

and define

$$\phi^k \colon \left(H^2(\Omega; \mathbb{R}^n)\right)^k \to H^2(\Omega; \mathbb{R}^n) \text{ by } \phi^k(u^{(1)}, \dots, u^{(k)}) = \sum_{|\alpha|=k} \phi^\alpha(u^{(1)}, \dots, u^{(k)}).$$

Since $H^2(\Omega)$ is an algebra, each component of ϕ^k is well-defined for all $k \in \mathbb{N}$ and as in Lemma A.3, we have found a series representation of F. It remains to prove the convergence by finding suitable bounds for the ϕ^k as n-linear forms. Proceeding as in the proof of the algebraic property, we can estimate $\|\phi^k\|_{\mathcal{L}^k(H^2;H^2)}$. To do so, choose $u^{(1)}, \dots, u^{(k)} \in H^2(\Omega; \mathbb{R}^3)$ arbitrarily with $\|u_i\|_{H^2} = 1$ for all $1 \le i \le k$ where $|\alpha| = k \ge 2$ and fix a component $1 \le m \le n$. Then it holds for the mth entry of $\phi^\alpha = \sum \phi^\alpha_m \hat{\mathbf{e}}_m$ that

$$\begin{aligned}
\|(\phi^\alpha(u^{(1)}, \dots, u^{(k)}))_m\|_{L^2}^2 &= \int_\Omega |(a_\alpha)_m|^2 \prod_{i=1}^{k} \prod_{\dots} |(u^{(j)}(x)_i|^2 \, dx \\
&\le \int_\Omega |(a_\alpha)_m|^2 \prod_{i=1}^{k} \|u^{(i)}\|_\infty^2 \, dx \\
&\le |\Omega| |(a_\alpha)_m|^2 c_\infty^{2k},
\end{aligned}$$

$$\begin{aligned}
\|\partial_p \phi^\alpha(u^{(1)}, \dots, u^{(k)})_m\|_{L^2}^2 &\le 2 \int_\Omega |a_\alpha|^2 \sum_{\ell=1}^{k} |\partial_p(u^{(l)})|^2 \prod_{i=1}^{k} \prod_{\dots} |(u^{(j)}(x))_i|^2 \, dx \\
&\le 2 \sum_{\ell=1}^{k} \int_\Omega (a_\alpha)_m^2 |\partial_p u^{(\ell)}(x)|^2 c_\infty^{2(k-1)} \, dx \\
&\le 2 \sum_{\ell=1}^{k} ((a_\alpha)_m)^2 \|u^{(\ell)}\|_{H^2}^2 = k((a_\alpha)_m c_\infty^{k-1})^2,
\end{aligned}$$

and finally, with $\|\partial_p u\|_{L^4} \le \|u\|_{W^{1,4}} \le c_1 \|u\|_{H^2}$ due to choice of n and the implied continuous embedding $H^2 \subset W^{1,4}$ as well as $2 \le 4 < \min\{\infty, 6\}$),

$$
\begin{aligned}
&\|\partial_p \partial_q \phi^\alpha(u^{(1)}, \dots, u^{(k)})_m\|_{L^2}^2 \\
&\quad \le \int_\Omega (a_\alpha)_m^2 \left(\sum_{\ell=1}^k \sum_{\substack{o=1 \\ o\neq\ell}} |\partial_p u^{(\ell)}||\partial_q u^{(o)}| \prod_{\substack{i=1 \\ i\neq \ell,o}} |u^{(i)}| + \sum_{\ell=1}^k |\partial_p \partial_q u^{(\ell)}| \prod_{i=1}^k |u^{(i)}| \right)^2 \, dx \\
&\quad \le 4 \int_\Omega (a_\alpha)_m^2 \sum_{\ell=1}^k \sum_{\substack{o=1 \\ o\neq\ell}} |\partial_p u^{(\ell)}||\partial_q u^{(o)}| c_\infty^{2(k-2)} \, dx + 2 \int_\Omega (a_\alpha)_m^2 \sum_{\ell=1}^k |\partial_p \partial_q u^{(\ell)}|^2 c_\infty^{2(k-1)} \, dx \\
&\quad \le 4 |a_\alpha|^2 c_\infty^{2(k-2)} \sum_{\ell=1}^k \sum_{\substack{o=1 \\ o\neq\ell}} \|\partial_p u^{(\ell)}\|_{L^4}^2 \|\partial_q u^{(o)}\|_{L^4}^2 + 2(a_\alpha)_m^2 c_\infty^{2(k-1)} \sum_{\ell=1}^k \|u^{(\ell)}\|_{H^2}^2 \\
&\quad \le 4 (a_\alpha)_m^2 c_\infty^{2(k-2)} \sum_{\ell=1}^k \sum_{\substack{o=1 \\ o\neq\ell}}^k c_1^4 + 2k \left((a_\alpha)_m c_\infty^{k-1} \right)^2 \\
&\quad = 4k(k-1) \left((a_\alpha)_m c_\infty^{k-2} c_1^2 \right)^2 + 2k \left((a_\alpha)_m c_\infty^{k-1} \right)^2 .
\end{aligned}
$$

For $k \le 2$, similar estimates hold. They are not important for the convergence of the series, though.

Set $c = \max\{1, c_\infty, c_1\}$. Then,

$$
\|(\phi^\alpha)_m\|_{L^k(H^2, H^2)}^2 \le (|\Omega| + 4k + 4k^2)((a_\alpha)_m c^k)^2
$$

and since the first factor does not affect the radius of convergence, the series

$$
\begin{aligned}
\sum_{k=1}^\infty \|\phi^k\|_{\mathcal{L}^k(H^2;H^2)} r^k &\le \sum_{k=1}^\infty \sum_{|\alpha=k|} \|\phi^\alpha\|_{\mathcal{L}^k(H^2;H^2)} r^k \\
&\le \text{const} + \sum_{k=2}^\infty \sum_{|\alpha|=k} \sqrt{|\Omega| + 4k + 4k^2} |(a_\alpha)_m| |cr|^k
\end{aligned}
$$

converges for $|z| < \frac{r}{c}$. □

Note: In Lemma A.3 and A.4, the set Ω could be replaced by a Riemannian manifold such as $\mathbb{S}^2$. For Lemma A.4, the manifold should have dimension 2 or 3 to ensure that $H^2(M)$ is an algebra.

We end this section by citing another theorem from [56].

Theorem
– from [56]
If $f\colon D \to D$ is an analytic diffeomorphism, then $f^{-1}\colon f(D) \to D$ is analytic.

This theorem is employed in [56] to prove that in the inverse as well as in the implicit function theorem on Banach spaces, analyticity of the original map implies analyticity of the inverse or implicit map.

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Eidesstattliche Erklärung

Ich, Helene Schroeder, erkläre hiermit, dass diese Dissertation und die darin dargelegten Inhalte die eigenen sind und selbstständig, als Ergebnis der eigenen originären Forschung, generiert wurden.

Hiermit erkläre ich an Eides statt

(1) Diese Arbeit wurde vollständig oder größtenteils in der Phase als Doktorand dieser Fakultät und Universität angefertigt;

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(6) Wenn immer ein Teil dieser Dissertation auf der Zusammenarbeit mit anderen basiert, wurde von mir klar gekennzeichnet, was von anderen und was von mir selbst erarbeitet wurde;

(7) Kein Teil dieser Arbeit wurde vor deren Einreichung veröffentlicht.

Aachen, 27. Januar 2024