Journal of Statistical Planning and Inference 232 (2024) 106152

Contents lists available at ScienceDirect

Journal of Statistical Planning and Inference

journal homepage: www.elsevier.com/locate/jspi

Scale tests for a multilevel step-stress model with exponential lifetimes under Type-II censoring

Maria Kateri *, Nikolay I. Nikolov [1]

Institute of Statistics, RWTH Aachen University, Aachen, 52056, Germany

ARTICLE INFO

Keywords:
Accelerated life-testing
Cumulative exposure model
Likelihood-ratio test
Normalized exponential spacings
Scale invariant statistics

ABSTRACT

Step-stress is a special type of accelerated life-testing procedure that allows the experimenter to test the units of interest under various stress conditions changed (usually increased) at different intermediate time points. In this paper, we study the problem of testing hypothesis for the scale parameter of a simple step-stress model with exponential lifetimes and under Type-II censoring. We consider several modifications of the log-likelihood ratio statistic and eliminate the distributional dependence on the unknown lifetime parameters by exploiting the scale invariant properties of the normalized failure spacings. The presented results and the ratio statistic are further generalized to the multilevel step-stress case under the log-link assumption. We compare the power performance of the proposed tests via Monte Carlo simulations. As an illustration, the described procedures are applied to a real data example from the literature.

1. Introduction

In many industrial life-testing experiments, the units under investigation are extremely reliable and have large mean times to failure under normal operating conditions (NOC). In such situations, accelerated life-testing (ALT) is an efficient tool to obtain more failures that leads to reduction in time and expenses of the examination. Step-stress ALT (SSALT) is a widely used testing scheme with applications in diverse fields, ranging from engineering sciences and quality control to medical and clinical studies. A product tested by SSALT is exposed to stress levels other than NOC at different intermediate stages of the experiment. In particular, the stress conditions may be increased over time (progressive SSALT) or decreased over time (regressive SSALT). An introduction to SSALT models can be found in the fundamental references for ALT, like the monographs by Nelson (1990), Meeker and Escobar (1998) and Bagdonavicius and Nikulin (2001).

The lifetime of the tested items in SSALT experiments are commonly considered to be exponentially distributed on each stress level. The impact of the experienced stress level change is usually modeled by assuming a cumulative exposure (CE) model under which the lifetime of a single unit has an absolute continuous cumulative distribution function (cdf). The main interest in a SSALT experiment lies mostly on inference for the mean lifetime (failure rate) at each stress level. Typically, these results are extrapolated to NOC by using a life-stress relationship, for example the log-link connection, see Nelson (1990, pp. 71–98). Alternatively, interest may lie on inference for a specific quantile of the lifetime distribution under NOC. There is a rich body of literature on inferential analysis of the exponential SSALT model in the context of various data frameworks, such as Type-II censoring (Xiong, 1998 and Balakrishnan et al., 2007), hybrid censoring (Balakrishnan and Xie, 2007a and Balakrishnan and Xie, 2007b), multi-sampling (Kateri et al., 2009 and Kateri et al., 2010) or under order restrictions (Balakrishnan et al., 2009). Extensive reviews on these developments are given

* Corresponding author.
E-mail address: maria.kateri@rwth-aachen.de (M. Kateri).
[1] Since January 2024 within the Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia, Bulgaria 1113.

https://doi.org/10.1016/j.jspi.2024.106152
Received 2 February 2022; Received in revised form 10 January 2024; Accepted 1 February 2024
Available online 3 February 2024

in Balakrishnan (2009) and Kundu and Ganguly (2017). However, most of the results are for the simple SSALT model, which involves just two stress levels, and there are only few studies on parameter testing and goodness-of-fit procedures, see Wang (2008, 2009) and Bedbur et al. (2015). More recently, Zhu et al. (2020) proposed an exact log-likelihood ratio test under Type-II censoring for the unknown failure rates in the simple SSALT.

In this study, we consider the problem of testing the scale parameter in SSALT model and follow the approach given in Xiong (1998) and Wang (2008, 2009) for constructing procedures based on exponential spacings. By making use of the independent and scale invariant properties of the normalized spacings, we express the log-likelihood ratio statistic, derived by Zhu et al. (2020), in terms of spacings and propose several modifications for the simple SSALT under Type-II censoring. As pointed in Zhu et al. (2020), the likelihood ratio test cannot be applied directly since its exact critical regions depend on the unknown lifetime parameters, whereas using their estimations could cause a substantial change of the test significance. The suggested methods in the current paper aim to eliminate this issue. The obtained results and the log-likelihood ratio statistic are further extended to the case of multilevel SSALT under the log-link assumption. Multilevel experiment designs may lead in some cases to a power advantage of the associated tests on the lifetime parameters. This is shown with the help of a simulation study.

The rest of this paper is organized as follows. In Section 2, we describe the multilevel SSALT model with exponential lifetimes under Type-II censoring. Furthermore, we define a transformation of the lifetimes and show an exponential property of the associated normalized spacings, which is important for the derivation of the statistical tests in the sequel. In Section 3, we consider statistical tests for the scale parameter in the life-stress relationship. Several alternatives of the log-likelihood ratio statistic for the simple SSALT are proposed in Section 3.1, while the general multilevel case is studied in Section 3.2. In Section 4, we compare the presented test statistics via Monte Carlo simulations. As an illustration, the described procedures are applied to a real experimental example in Section 5. Finally, concluding remarks are provided in Section 6.

2. Model description and spacings

Let $T_1, T_2, \ldots, T_n$ be the random lifetimes of n units that are tested on m (≥ 2) stress levels $x_1, x_2, \ldots, x_m$ that are changed at prefixed times $\tau_1, \tau_2, \ldots, \tau_{m-1}$. We denote the class of exponentially distributed random variables with mean θ by $Exp(1/\theta)$. We assume that the lifetime on level x_j has $Exp\left(1/\theta_j\right)$ distribution, for $j = 1, 2, \ldots, m$, and that the CE assumption holds. Furthermore, we consider that the log-link relation between the mean lifetimes and the stress levels holds, i.e., θ_j satisfy

$$\log\left(\theta_j\right) = \alpha + \beta h\left(x_j\right), \quad \text{for } j = 0, 1, \ldots, m, \tag{1}$$

where α and β are unknown parameters and $h(\cdot)$ is a known function of the stress levels $x_0, x_1, \ldots, x_m$, with x_0 and θ_0 being the stress level and the expected mean lifetime under NOC, respectively. Commonly used options for $h(\cdot)$ are the identity, logarithmic and reciprocal functions. The choice of $h(\cdot)$ should be motivated by the underlying physical phenomenon of the experiment. For instance, the Arrhenius law in physical chemistry is associated with the reciprocal function. Furthermore, in Section 4.2 we show that the standardized logarithmic and reciprocal functions behave approximately like the identity link in an equally-spaced setup with relatively small step of the stress increment. More details and examples of life-stress relationships can be found in Nelson (1990, pp. 71–98).

The lifetime T of a single unit under the SSALT model described above has the following cdf

$$F_T(t) = 1 - \exp\left(-\frac{t-\tau_{k-1}}{\theta_k}\right)\exp\left(-\sum_{j=1}^{k-1}\frac{\tau_j - \tau_{j-1}}{\theta_j}\right), \quad \text{for } \tau_{k-1} < t \leq \tau_k, \tag{2}$$

where $0 \equiv \tau_0 < \tau_1 < \cdots < \tau_{m-1} < \tau_m \equiv \infty$, $k = 1, 2 \ldots, m$ and $\sum_{j=1}^{0}(\cdot) \equiv 0$. Let D_k be the number of failures on level k, i.e.,

$$D_k = \sum_{i=1}^{r} \mathbb{1}\left\{\tau_{k-1} < T_i \leq \tau_k\right\}, \quad \text{with } \mathbb{1}\{\cdot\} \text{ being the indicator function and } k = 1, 2, \ldots, m. \tag{3}$$

Then, the level $\ell(i)$ on which the ith failure $T_{i:n}$ is observed can be written as

$$\ell(i) = \min\left\{1 \leq k \leq m : \sum_{j=1}^{k} D_j \geq i\right\}.$$

Motivated by the analysis for simple SSALT made in Xiong (1998), we consider the following transformation of the lifetimes $T_1, T_2, \ldots, T_n$:

$$T^*_{i:n} = \gamma_{\ell(i)}\left(T_{i:n} - \tau_{\ell(i)-1}\right) + \sum_{j=1}^{\ell(i)-1}\gamma_j\left(\tau_j - \tau_{j-1}\right), \quad \text{for } i = 1, 2, \ldots, n \text{ and } \gamma_j = \frac{\theta_1}{\theta_j},\ j = 1, 2 \ldots, m, \tag{4}$$

with $\sum_{j=1}^{0}(\cdot) \equiv 0$. The following theorem, proved in Appendix, extends the exponential property of this transformation, presented in Xiong (1998, Section 4) for a simple SSALT procedure ($m = 2$), to a multilevel SSALT setup ($m \geq 2$).

Theorem 1. *The transformed ordered lifetimes $T^*_{1:n}, T^*_{2:n}, \ldots, T^*_{n:n}$, given in (4), have the same joint distribution as the order statistics of independent and identically distributed (IID) random variables $T^*_1, T^*_2, \ldots, T^*_n$ with $Exp\left(1/\theta_1\right)$ marginal distribution.*

Theorem 1 and the well-known distributional properties of the exponential spacings, see Arnold et al. (1992, pp. 72–73), immediately imply the following corollary for the normalized spacings of $T^*_{1:n}, T^*_{2:n}, \dots, T^*_{n:n}$, defined by

$$S_i = (n-i+1)\left(T^*_{i:n} - T^*_{i-1:n}\right), \quad \text{for } i = 1, 2, \dots, n \text{ and } T^*_{0:n} \equiv 0. \tag{5}$$

Corollary 1. *The spacings $S_1, S_2, \dots, S_n$ in (5) are IID and have $Exp\left(1/\theta_1\right)$ marginal distribution.*

If the observed data is incomplete, for some statistical procedures the spacings (5) could be more useful than the original lifetimes or the transformation (4). In particular, let us consider Type-II censoring, i.e., the case when only the first r ordered statistics $0 < T_{1:n} < T_{2:n} < \cdots < T_{r:n}$ are observed with prefixed integer r such that $1 \le r \le n$. Then, according to Theorem 1, the transformed lifetimes $T^*_{1:n} < T^*_{2:n} < \cdots < T^*_{r:n}$ are the first r order statistics of a sample with $Exp\left(1/\theta_1\right)$ marginal distribution, whereas $S_1, S_2, \dots, S_r$ are IID with $Exp\left(1/\theta_1\right)$. Thus, in order to simplify the analysis for incomplete data we can make use of the independence and exponential properties of the spacings (5), while for the joint distribution of $T^*_{1:n}, T^*_{2:n}, \dots, T^*_{r:n}$ we need to consider also their dependence structure. Furthermore, notice that when $r = n$ there is no censoring, i.e., the complete sample can be considered as a special case of Type-II censoring.

For a simple SSALT experiment ($m = 2$), we can express $S_1, S_2, \dots, S_r$ by the original lifetimes in the following way,

$$S_i = \begin{cases} (n-i+1)\left(T_{i:n} - T_{i-1:n}\right), & \text{if } T_{i:n} \le \tau_1, \\ (n-i+1)\left(\gamma_2\left(T_{i:n} - \tau_1\right) + \tau_1 - T_{i-1:n}\right), & \text{if } T_{i-1:n} \le \tau_1 < T_{i:n}, \\ (n-i+1)\left(\gamma_2\left(T_{i:n} - T_{i-1:n}\right)\right), & \text{if } \tau_1 < T_{i-1:n}, \end{cases} \tag{6}$$

for $i = 1, 2, \dots, r$ and $T_{0:n} \equiv 0$.

In practice, the ratios $\gamma_2, \gamma_3, \dots, \gamma_m$ in (4) are not observed and the scaling factors of transformation (4) are unknown. However, under the log-link assumption (1) these scaling ratios depend only on the parameter β. Therefore, in the next section we consider statistics for testing the value of β in multilevel SSALT models ($m \ge 2$) and study in more detail the case when $m = 2$, corresponding to tests for γ_2 in (6).

3. Scale tests

Let us consider the problem of testing the values of the unknown scale parameter β in (1), i.e., testing

$$H_0 : \beta = \beta_0 \quad \text{against the alternative} \quad H_1 : \beta \ne \beta_0 \tag{7}$$

for a specified $\beta_0 \in [0, \infty)$. Notice that the hypothesis of no change in mean lifetime across the stress levels is included in (7) as the special case $\beta_0 = 0$, which was first studied in Xiong (1998) for the simple SSALT model.

3.1. The case $m = 2$

In the case of simple SSALT model, Zhu et al. (2020) derived the log-likelihood ratio test statistic Λ for the general problem (7) conditioned on the random number of failures D_1 on the first stress level. Given that $D_1 = d$, we have

$$\Lambda = -2\log\left(\left(\frac{\gamma_0\hat{\theta}_1}{\hat{\theta}_2^{(0)}}\right)^d \left(\frac{\hat{\theta}_2}{\hat{\theta}_2^{(0)}}\right)^{r-d}\right), \tag{8}$$

where $\hat{\theta}_1$ and $\hat{\theta}_2$ are the maximum likelihood estimates (MLEs) of θ_1 and θ_2, $\gamma_0 = \exp\left(\beta_0\left(h(x_2) - h(x_1)\right)\right)$ is the ratio θ_2/θ_1 under the null hypothesis H_0 in (7), i.e., $\gamma_0 = \theta_2/\theta_1 = 1/\gamma_2$, and

$$\hat{\theta}_2^{(0)} = \frac{d\hat{\theta}_1\gamma_0 + (r-d)\hat{\theta}_2}{r} = \frac{d\hat{\theta}_1\exp\left(\beta_0\left(h(x_2) - h(x_1)\right)\right) + (r-d)\hat{\theta}_2}{r} \tag{9}$$

is the MLE of θ_2 under H_0. Balakrishnan et al. (2007) showed that $\hat{\theta}_1$ and $\hat{\theta}_2$ exist only when $1 \le D_1 \le r-1$ and are given by

$$\hat{\theta}_1 = \frac{\sum_{i=1}^{D_1} T_{i:n} + \left(n - D_1\right)\tau_1}{D_1} \quad \text{and} \quad \hat{\theta}_2 = \frac{\sum_{i=D_1+1}^{r}\left(T_{i:n} - \tau_1\right) + (n-r)\left(T_{r:n} - \tau_1\right)}{r - D_1}.$$

The critical region for an exact α-level test based on Λ is determined as

$$P\left(\Lambda > \lambda_\alpha \,\middle|\, 1 \le D_1 \le r-1,\ \theta_2 = \gamma_0\theta_1\right) = \alpha. \tag{10}$$

Procedures for evaluating λ_α in (10) and the exact conditional distribution of Λ under H_0 are given in Zhu et al. (2020). However, the exact values of these characteristics depend on the condition $1 \le D_1 \le r-1$ and the parameter θ_1. In practice, the value of D_1 is observed, but θ_1 is unknown and should be replaced with some estimate. Zhu et al. (2020) examined the effect of using $\hat{\theta}_1$ instead of the true value θ_1 via a simulation study for the null hypothesis $\theta_1 = \theta_2$. From the obtained results they concluded that the likelihood ratio test maintains close to the desired level of significance when the changing point τ_1 is large. Nevertheless, it is unclear if this is true for other null hypothesis and how large τ_1 should be in order to apply the test. Therefore, we propose several modifications of the statistic Λ in the next lines.

By using (6) the MLEs $\hat{\theta}_1$ and $\hat{\theta}_2$ can be rewritten as

$$\hat{\theta}_1 = \frac{\sum_{i=1}^{D_1} S_i + Z}{D_1} \quad \text{and} \quad \hat{\theta}_2 = \frac{\gamma_0 \left(\sum_{i=D_1+1}^{r} S_i - Z \right)}{r - D_1}, \tag{11}$$

where

$$Z = \left(n - D_1\right) \left(\tau_1 - \sum_{j=1}^{D_1} \frac{S_j}{n-j+1} \right) = \left(n - D_1\right) \left(\tau_1 - T_{D_1:n}\right), \tag{12}$$

is the normalized difference between τ_1, the time point for changing the stress from x_1 to x_2, and $T_{D_1:n}$, the time of the last failure on the first stress level. Hence, under H_0 and conditioned on $D_1 = d$, the test statistic Λ in (8) could be rewritten as

$$\Lambda = -2\log\left(\frac{r^r}{d^d (r-d)^{r-d}} \left(\frac{\sum_{i=1}^{d} S_i + Z}{\sum_{i=1}^{r} S_i} \right)^d \left(\frac{\sum_{i=d+1}^{r} S_i - Z}{\sum_{i=1}^{r} S_i} \right)^{r-d} \right).$$

Following the approach proposed in Xiong (1998) and Wang (2008, 2009), one could try to adjust Λ to a simpler statistic based on the spacings (6). Therefore, let us study in more detail the random variable Z, given in (12). The asymptotic behavior of Z is stated in Theorem 2, which is proved in Appendix.

Theorem 2. *Let the uncensored sample proportion ρ be fixed with $1 - \exp\left(-\tau_1/\theta_1\right) < \rho \leq 1$. Further let the number of observed failures r be the integer part of ρn, i.e., $r = [\rho n]$. Then, in accordance with the notations above and given that D_1 in (3) satisfies $1 \leq D_1 \leq r-1$, the asymptotic distribution of Z in (12) is $Exp\left(1/\theta_1\right)$ as $n \to \infty$.*

Remark 1. Notice that the condition on the uncensored sample proportion ρ in Theorem 2 is not restrictive in practice. By the definition of Type-II censored experiments, we have $0 < \rho \leq 1$. The sharper lower bound of the inequality in Theorem 2 ensures that, for $n \to \infty$, not all r failures are observed almost surely on the first stress level, since from (2) we have $F_T(\tau_1) = 1 - \exp\left(-\tau_1/\theta_1\right)$. For this, experimental designs with $\rho < F_T\left(\tau_1\right)$ make no practical sense for large values of n. Thus, the problem of fixing ρ such that $F_T\left(\tau_1\right) < \rho$ is related to the design of the experiment. For given stress levels (x_1, x_2), the changing point τ_1 is selected accordingly. However, $F_T\left(\tau_1\right)$ depends on the unknown parameter θ_1 and has to be estimated for the experimental design purposes. The estimation is based on prior knowledge of the product under investigation, e.g., results from previous experiments or a pilot study.

From Theorem 2 it is straightforward to derive the following property of Z, the proof of which is sketched in Appendix.

Corollary 2. *Given the conditions in Theorem 2, the ratios Z/D_1 and $Z/\left(r - D_1\right)$ converge in probability to 0 as $n \to \infty$.*

Since $\hat{\theta}_1$ is consistent estimator of θ_1, i.e., $\hat{\theta}_1$ converges in probability to θ_1 $\left(\hat{\theta}_1 \xrightarrow{\text{P}} \theta_1\right)$, see Bartholomew (1963), by combining the first statement of Corollary 2 and the expression of $\hat{\theta}_1$ in (11) it follows that

$$\tilde{\theta}_1 = \frac{\sum_{i=1}^{D_1} S_i}{D_1} \quad \text{is also consistent estimator of } \theta_1. \tag{13}$$

Similarly, from Corollary 2 and the consistency of $\hat{\theta}_2$, see Balakrishnan et al. (2007), it can be concluded that

$$\tilde{\theta}_2 = \frac{\gamma_0 \left(\sum_{i=D_1+1}^{r} S_i \right)}{r - D_1} \quad \text{is consistent estimator of } \theta_2. \tag{14}$$

Thus, we can construct a new test statistic similar to Λ by replacing $\hat{\theta}_1$ and $\hat{\theta}_2$ in (8) with $\tilde{\theta}_1$ and $\tilde{\theta}_2$, respectively. For $D_1 = d$, $1 \leq d \leq r-1$, the obtained statistic $\tilde{\Lambda}$ takes the following form

$$\tilde{\Lambda} = -2\log\left(\frac{r^r}{d^d (r-d)^{r-d}} \left(\frac{\sum_{i=1}^{d} S_i}{\sum_{i=1}^{r} S_i} \right)^d \left(\frac{\sum_{i=d+1}^{r} S_i}{\sum_{i=1}^{r} S_i} \right)^{r-d} \right). \tag{15}$$

Similar to the statistic Λ, the critical region for an α-level test based on $\tilde{\Lambda}$ is determined from

$$P\left(\tilde{\Lambda} > \tilde{\lambda}_\alpha \,\middle|\, 1 \leq D_1 \leq r-1,\, \theta_2 = \gamma_0 \theta_1 \right) = \alpha. \tag{16}$$

Since it is not easy to derive an expression for the exact conditional distribution function of $\tilde{\Lambda}$ under H_0 and given that $1 \leq D_1 \leq r-1$, the value of $\tilde{\lambda}_\alpha$ in (16) can be computed by using a Monte Carlo procedure for approximating the distribution $\tilde{\Lambda}$ for fixed model parameters. Additionally, similar to the test based on Λ, the true value of θ_1 is unknown even under H_0 and should be estimated in the computation of the critical value, which may change the significance level of the test. However, from Corollary 1 and expression (15) it follows that $\tilde{\Lambda}$ depends on θ_1 only through the conditioning $D_1 = d$. Thus, it is worth to further examine and adjust the statistic $\tilde{\Lambda}$. We prove the asymptotic equivalence of Λ and $\tilde{\Lambda}$ (Proposition 1) in Appendix.

Proposition 1. *For $r = [\rho n]$ with $1 - \exp\left(-\tau_1/\theta_1\right) < \rho \leq 1$ and given that $1 \leq D_1 \leq r-1$, the test statistics Λ and $\tilde{\Lambda}$ defined in (8) and (15) are asymptotically equivalent as $n \to \infty$.*

Remark 2. From the conditional moment generating function of $\hat{\theta}_1$, given by formula (29) in Balakrishnan et al. (2007, p. 44), it follows that

$$E\left(\hat{\theta}_1 \middle| D_1 = d\right) = \theta_1 + \tau_1 \left(\frac{n-d}{d} - \frac{1-p}{p}\right), \quad \text{where } p = 1 - \exp\left(-\frac{\tau_1}{\theta_1}\right) = F_T\left(\tau_1\right),$$

with $F_T(\cdot)$ being the cdf of the SSALT model provided in (2). Therefore, $\sum_{i=1}^{D_1} T_{i:n}/D_1$ has a negative bias of $\tau_1(1-p)/p$, which is compensated to some degree in $\hat{\theta}_1$ by the term $\left(n - D_1\right)\tau_1/D_1$. On the other hand, this correction is distributed to both $\tilde{\theta}_1$ and $\tilde{\theta}_2$ in a proportion that is associated with the random variable Z. The statement of Corollary 2 implies that these quantities are negligible for $n \to \infty$. Hence, $\tilde{\theta}_1 \leq \hat{\theta}_1$ and $\tilde{\theta}_2 \geq \hat{\theta}_2$ with $\tilde{\theta}_1 - \hat{\theta}_1 \xrightarrow{\text{p}} 0$ and $\tilde{\theta}_2 - \hat{\theta}_2 \xrightarrow{\text{p}} 0$ as $n \to \infty$, where $\xrightarrow{\text{p}}$ denotes convergence in probability. In practice, the value of $\tilde{\theta}_2$ can be computed only when the ratio γ_0 is known. However, in case γ_0 is given (corresponding to a null hypothesis in (7)) a better estimation of θ_2 is $\hat{\theta}_2^{(0)}$, given in (9). Thus, the statistic Λ is based on comparing $\gamma_0\hat{\theta}_1$ and $\hat{\theta}_2$ to $\hat{\theta}_2^{(0)}$, while $\tilde{\Lambda}$ is determined by the ratios $\gamma_0\tilde{\theta}_1/\hat{\theta}_2^{(0)}$ and $\tilde{\theta}_2/\hat{\theta}_2^{(0)}$. As the results from the simulation study for $\gamma_0 = 1$ in Section 4 suggest, the testing procedure derived by Λ is more powerful when $\theta_1 > \theta_2$, while $\tilde{\Lambda}$ is better for the alternative $\theta_1 < \theta_2$. The reason for this lies in the fact that $\hat{\theta}_2^{(0)}$ overestimates θ_2 when $\theta_1 > \theta_2$ and underestimates it when $\theta_1 < \theta_2$. Next, we use the form of $\tilde{\Lambda}$ in (15) as a basis for additional modifications of the log-likelihood ratio statistic Λ.

In order to construct a test statistic having distribution free of the parameter θ_1, we should remove the conditioning on D_1, since D_1 has a binomial distribution with parameters n and $F_T(\tau_1) = 1 - \exp\left(-\tau_1/\theta_1\right)$. We make use of the exponential properties of the spacings in (5) and take the sum of $\tilde{\Lambda}$ in (15) over all possible values of D_1:

$$T = -2\sum_{k=1}^{r-1} \log\left(\frac{r^r}{k^k(r-k)^{r-k}} \left(\frac{\sum_{i=1}^k S_i}{\sum_{i=1}^r S_i}\right)^k \left(\frac{\sum_{i=k+1}^r S_i}{\sum_{i=1}^r S_i}\right)^{r-k}\right). \tag{17}$$

In D'Agostino and Stephens (1986, p. 430), it is shown that for

$$V_k = \frac{\sum_{i=1}^k S_i}{\sum_{i=1}^r S_i}, \quad k = 1, 2, \ldots, r-1,$$

with $S_1, S_2, \ldots, S_r$ being IID and exponentially distributed under H_0, the joint distribution of $V_1, V_2, \ldots, V_{r-1}$ is the same as the joint distribution of the order statistics obtained from a random sample with size $r-1$ and standard uniform marginal distribution. Since the statistic T in (17) can be rewritten as

$$T = -2\sum_{k=1}^{r-1} \log\left(\frac{r^r}{k^k(r-k)^{r-k}} \left(V_k\right)^k \left(1 - V_k\right)^{r-k}\right), \tag{18}$$

it is obvious that the distribution of T under H_0 is scale invariant and does not depend on θ_1 when there are no restrictions for D_1, i.e., for $0 \leq D_1 \leq r$. Like in the case of $\tilde{\Lambda}$, the exact critical values for the test based on T can be evaluated via Monte Carlo simulation. Since these calculations can be computationally intensive, we seek to further simplify the statistic T. By combining the uniform properties of $V_1, V_2, \ldots, V_{r-1}$ under the null hypothesis and the fact that T is equivalent to

$$\tilde{T} = -2\sum_{k=1}^{r-1} k\log\left(V_k\right) - 2\sum_{k=1}^{r-1}(r-k)\log\left(1 - V_k\right),$$

it is natural to consider the test based on the following statistic

$$\chi^2 = -2\sum_{k=1}^{r-1} \log\left(V_k\right). \tag{19}$$

In case $0 \leq D_1 \leq r$, clearly (19) is scale invariant and rejects H_0 in (7) for either small or large values of χ^2. The statistic χ^2 is also proposed in Wang (2008, 2009) for testing the exponential assumption in models with Type-II censoring scheme. Since Wang (2008) showed that χ^2 in (19) has chi-square distribution with $2r-2$ degrees of freedom under H_0, the exact critical region for a two-sided α-level test based on χ^2 is given by

$$P\left(\chi^2 < q_{\frac{\alpha}{2}} \middle| H_0\right) + P\left(\chi^2 > q_{1-\frac{\alpha}{2}} \middle| H_0\right) = \alpha,$$

where $q_{\frac{\alpha}{2}}$ and $q_{1-\frac{\alpha}{2}}$ are the $\frac{\alpha}{2}$ and $\left(1 - \frac{\alpha}{2}\right)$ quantiles of chi-square distribution with $2r-2$ degrees of freedom. Furthermore, as pointed by Wang (2009), from the simulation studies in Henze and Meintanis (2005) and Wang (2008) it can be concluded that the procedure based on χ^2 is one of the most powerful tests for exponentiality under various non-exponential alternatives. It is also important to note that, Λ and $\tilde{\Lambda}$ exist only when $1 \leq D_1 \leq r-1$, whereas T in (17) and χ^2 in (19), not depending on D_1, are well defined for $0 \leq D_1 \leq r$. However, in the case of $D_1 = 0$ or $D_1 = r$ (i.e., $D_2 = 0$), all failures are observed on a single stress level and the associated tests degenerate to tests for exponentiality under this stress level; a test for the SSALT setup, i.e., of (7), is not possible. In order to ensure that the test based on T or χ^2 tests (7), its distribution under H_0 has to be simulated under the condition that $1 \leq D_1 \leq r-1$. As we shall see in the simulation studies in Section 4, the corresponding unconditional distribution approximates the conditional one quite well.

Remark 3. Since the spacings S_i in (6) depend on β_0 through (9), similar to Λ in (8), the test statistics $\tilde{\Lambda}$, T and χ^2, given in (15), (17) and (19), respectively, can be calculated only under fixed null hypothesis H_0 in (7), i.e., only for given value of $\beta_0 = \log\left(\gamma_0\right) / \left(h\left(x_2\right) - h\left(x_1\right)\right)$.

3.2. The general case $m > 2$

In order to derive the log-likelihood ratio statistic for (7) when $m > 2$, we first consider transformation (4) under $H_0 : \beta = \beta_0$ and under $H_1 : \beta = \beta_1$ for a fixed value $\beta_1 \neq \beta_0$. For $i = 1, 2, \ldots, r$, let $T^{(0)}_{i:n}$ be (4) with $\gamma_j = \gamma^{(0)}_j$, where $\gamma^{(0)}_j$ is the ratio $\frac{\theta_1}{\theta_j}$ under H_0 and the log-link assumption (1), i.e., $\gamma^{(0)}_j = \exp\left(\beta_0\left(h(x_1) - h(x_j)\right)\right)$ for $j = 1, 2, \ldots, m$. Similarly, we denote transformation (4) for $\gamma_j = \gamma^{(1)}_j = \exp\left(\beta_1\left(h(x_1) - h(x_j)\right)\right)$ by $T^{(1)}_{i:n}$. Then, the normalized spacings under H_0 and H_1, denoted by $\left\{S^{(0)}_i\right\}_{i=1}^r$ and $\left\{S^{(1)}_i\right\}_{i=1}^r$, respectively, are given by

$$S^{(k)}_i = (n - i + 1)\left(T^{(k)}_{i:n} - T^{(k)}_{i-1:n}\right), \quad \text{for } i = 1, 2, \ldots, r,\ T^{(k)}_{0:n} \equiv 0 \text{ and } k = 0, 1. \tag{20}$$

From the independent and exponential properties of $S^{(0)}_1, S^{(0)}_2, \ldots, S^{(0)}_r$ under H_0 (Corollary 1) it follows that the maximized value of the log-likelihood is, upon a constant, given by

$$\sup_{\theta_1 > 0} \ell_0\left(\theta_1, \beta_0\right) = \ell_0\left(\tilde{\theta}_1, \beta_0\right) = -r\log\left(\tilde{\theta}_1\right) - \frac{1}{\tilde{\theta}_1}\sum_{i=1}^r S^{(0)}_i, \quad \text{where } \tilde{\theta}_1 = \frac{\sum_{i=1}^r S^{(0)}_i}{r}. \tag{21}$$

By taking definition (20) and the expressions of $T^{(0)}_{i:n}$ and $T^{(1)}_{i:n}$ in (4), we obtain

$$T^{(0)}_{i:n} = \sum_{j=1}^{i} \frac{S^{(0)}_j}{n - j + 1} \quad \text{and} \quad T^{(1)}_{i:n} = \kappa^{a_{\ell(i)}} T^{(0)}_{i:n} + \sum_{j=1}^{\ell(i)-1} \gamma^{(0)}_j \left(\kappa^{a_j} - \kappa^{a_{\ell(i)}}\right)\left(\tau_j - \tau_{j-1}\right), \quad \text{for } i = 1, 2, \ldots, r,$$

where $a_j = h(x_j) - h(x_1)$, $j = 1, 2, \ldots, m$ and

$$\kappa = \exp\left(\beta_0 - \beta_1\right) \tag{22}$$

is a reparametrization of β_1. Therefore, the spacing $S^{(1)}_i$ depends on $\left\{S^{(0)}_j\right\}_{j=1}^i$ in the following way,

$$\begin{aligned} S^{(1)}_i = \kappa^{a_{\ell(i)}} S^{(0)}_i + (n - i + 1)\Bigg(&\left(\kappa^{a_{\ell(i)}} - \kappa^{a_{\ell(i-1)}}\right)\sum_{j=1}^{i-1}\frac{S^{(0)}_j}{n - j + 1} + \sum_{j=1}^{\ell(i)-1}\gamma^{(0)}_j\left(\kappa^{a_j} - \kappa^{a_{\ell(i)}}\right)\left(\tau_j - \tau_{j-1}\right) \\ &- \sum_{j=1}^{\ell(i-1)-1}\gamma^{(0)}_j\left(\kappa^{a_j} - \kappa^{a_{\ell(i-1)}}\right)\left(\tau_j - \tau_{j-1}\right)\Bigg), \end{aligned} \tag{23}$$

with $\sum_{j=1}^0 (\cdot) \equiv 0$ and $\ell(0) = 1$. Thus, using parametrization (22), the distributional properties of $S^{(1)}_1, S^{(1)}_2, \ldots, S^{(1)}_r$ under H_1 (Corollary 1) and (23), we can express the log-likelihood under H_1, upon a constant, similarly as in (21), as

$$\ell_1\left(\theta_1, \kappa\right) = \log\left(L\left(S^{(1)}_1, S^{(1)}_2, \ldots, S^{(1)}_r \,\middle|\, H_1\right)\right) = -r\log\left(\theta_1\right) + b\log(\kappa) - \frac{1}{\theta_1}G(\kappa), \tag{24}$$

with

$$b = \sum_{i=1}^r a_{\ell(i)} \quad \text{and} \quad G(\kappa) = \sum_{i=1}^r \left\{\kappa^{a_{\ell(i)}} S^{(0)}_i + \left(\kappa^{a_{\ell(i)}} - \kappa^{a_{\ell(i-1)}}\right) Z_i + \sum_{j=\ell(i-1)}^{\ell(i)-1}\left(\kappa^{a_j} - \kappa^{a_{\ell(i)}}\right) c_{i,j}\right\}, \tag{25}$$

where $\sum_{j=j_l}^{j_u} (\cdot) \equiv 0$, for $j_l > j_u$,

$$Z_i = (n - i + 1)\sum_{j=1}^{i-1}\frac{S^{(0)}_j}{n - j + 1} - \sum_{j=1}^{\ell(i-1)-1} c_{i,j} \quad \text{and} \quad c_{i,j} = (n - i + 1)\gamma^{(0)}_j\left(\tau_j - \tau_{j-1}\right), \tag{26}$$

for $i = 1, \ldots, r$ and $j = 1, \ldots, m - 1$.

Notice that if all observed failures are on the same stress level, i.e., $\ell(1) = \ell(2) = \cdots = \ell(r)$, the function $G(\cdot)$ in (25) is simplified to

$$G(\kappa) = \sum_{i=1}^r \kappa^{a_{\ell(i)}} S^{(0)}_i = \kappa^{a_{\ell(1)}} \sum_{i=1}^r S^{(0)}_i,$$

with (24) depending on θ_1 and κ only through the ratio $\kappa^{a_{\ell(1)}}/\theta_1$. Furthermore, in this case (24) coincide with the log-likelihood under H_0. Hence, we can derive the log-likelihood ratio statistic only if

$$\ell(1) \leq \ell(2) \leq \cdots \leq \ell(r), \quad \text{with at least one strict inequality, i.e.,} \tag{27}$$

only when there are failures on at least two stress levels. It is not hard to check that the log-likelihood in (24) is maximized for $\theta_1 = G(\hat{\kappa})/r$ and $\kappa = \hat{\kappa}$, where $\hat{\kappa}$ is a solution of

$$\frac{\kappa G'(\kappa)}{G(\kappa)} - \frac{b}{r} = 0, \quad \text{for } \kappa \in (0, \infty), \tag{28}$$

with $G'(\cdot)$ being the derivative of $G(\cdot)$ with respect to κ. Then, combining (21) and (24) results to the following log-likelihood ratio statistic

$$\Lambda_m = -2\left(\ell_0\left(\tilde{\theta}_1, \beta_0\right) - \ell_1\left(\frac{G(\hat{\kappa})}{r}, \hat{\kappa}\right)\right) = -2\log\left(\left(\frac{1}{\hat{\kappa}}\right)^b \left(\frac{G(\hat{\kappa})}{\sum_{i=1}^r S_i^{(0)}}\right)^r\right), \tag{29}$$

where $\hat{\kappa}$ is the unique maximum-likelihood solution of Eq. (28), as shown in Theorem 3 below. The optimal value $\hat{\kappa}$ can be evaluated by using some numerical procedure for solving non-linear equations, for example the Newton–Raphson method. In addition, notice that the case $\beta_0 < \beta_1$ is associated with $0 < \kappa < 1$, whereas $\kappa > 1$ corresponds to $\beta_0 > \beta_1$. Thus, if H_0 is rejected, the case $0 < \hat{\kappa} < 1$ would suggest $\beta > \beta_0$, while $\hat{\kappa} > 1$ would point to $\beta < \beta_0$. In next theorem, proved in Appendix, we show that $\hat{\kappa}$ is the unique solution of (28) under certain assumptions about the stress factor.

Theorem 3. *Let the function $h(\cdot)$ in the life-stress relation (1) be monotonic on the stress levels $x_1, x_2, \ldots, x_m$, i.e., $h(x_1) < h(x_2) < \cdots < h(x_m)$ or $h(x_1) > h(x_2) > \cdots > h(x_m)$. Then, Eq. (28) has an unique solution $\hat{\kappa}$ such that*

$$\sup_{\substack{\theta_1 > 0 \\ \kappa > 0, \kappa \neq 1}} \ell_1\left(\theta_1, \kappa\right) = \ell_1\left(\frac{G(\hat{\kappa})}{r}, \hat{\kappa}\right).$$

If the assumption for $h(\cdot)$ in Theorem 3 holds, then the log-likelihood (24) has only one critical point with respect to κ and the Newton–Raphson algorithm for solving (28) converges to the true optimal value $\hat{\kappa}$. Hence, if $h(\cdot)$ is monotonic, we can calculate the statistic Λ_m with just one initial point in the numerical procedure for computing $\hat{\kappa}$.

Remark 4. As shown in the proof of Theorem 3, the monotonic assumption of $h(\cdot)$ can be relaxed. In particular, $h(x_1) < h(x_2) < \cdots < h(x_m)$ could be replaced by a partial ordering condition $h(x_1) < h(x_j)$, for $j = 2, \ldots, m$, and $h(x_j) \neq h(x_k)$, for $j \neq k$. However, in most SSALT experimental designs the stress factor is increased on each step (progressive SSALT), which commonly leads to monotonicity of the function $h(\cdot)$. Therefore, Theorem 3 allow us to find the maximum of (24) and evaluate Λ_m in many practical problems.

Similar to the simple SSALT ($m = 2$), we determine the critical region for an α-level test based on Λ_m under $H_0 : \beta = \beta_0$ and (27) by

$$P\left(\Lambda_m > \lambda_\alpha^{(m)} \,\middle|\, \beta = \beta_0,\ \ell(i) < \ell(j) \text{ for some } i < j\right) = \alpha. \tag{30}$$

Since we do not have a closed expression for the exact conditional distribution of Λ_m, we can compute the value of $\lambda_\alpha^{(m)}$ in (30) by approximation the null distribution of Λ_m via Monte Carlo simulation. Notice also that this distribution depends on the parameter α in the log-link relationship (1).

For $m = 2$ and given that $D_1 = d$, we have

$$b = a_2(r-d), \quad \hat{\kappa}^{a_2} = \frac{r-d}{d}\frac{\sum_{i=1}^d S_i^{(0)} + Z}{\sum_{i=d+1}^r S_i^{(0)} - Z} \quad \text{and} \quad G(\hat{\kappa}) = \frac{r}{d}\left(\sum_{i=1}^d S_i^{(0)} + Z\right), \quad \text{where } Z = c_{d+1,1} - Z_{d+1}, \text{ i.e.,}$$

the statistic Λ_2 in (29) coincide with Λ. Thus, the statistic $\tilde{\Lambda}$ in (15) for the simple SSALT ($m = 2$) can be extended to the multilevel case ($m > 2$) by replacing the function $G(\cdot)$ in (28) and (29) with $\tilde{G}(\kappa) = \sum_{i=1}^r \kappa^{a_{\ell(i)}} S_i^{(0)}$. Then, for testing (7) when $m > 2$ we define the following statistic

$$\tilde{\Lambda}_m = -2\log\left(\left(\frac{1}{\hat{\kappa}^*}\right)^b \left(\frac{\tilde{G}(\hat{\kappa}^*)}{\sum_{i=1}^r S_i^{(0)}}\right)^r\right), \quad \text{where } \hat{\kappa}^* \text{ is the solution of } \quad \frac{\kappa\left(\tilde{G}(\kappa)\right)'}{\tilde{G}(\kappa)} - \frac{b}{r} = 0, \quad \text{for } \kappa \in (0, \infty). \tag{31}$$

The uniqueness of the solution $\hat{\kappa}^*$ in (31) follows directly from the definition of $\tilde{G}(\cdot)$ and the arguments presented in the proof of Theorem 3. Like in the case $m = 2$, the critical region based on $\tilde{\Lambda}_m$ for $m > 2$ can be evaluated by Monte Carlo approximation.

Since (17) and (19) depend only on the normalized spacings, the statistics T and χ^2 are easily generalized for $m \geq 2$ by considering the spacings $S_1^{(0)}, \ldots, S_r^{(0)}$ in (20) under H_0. The corresponding statistics are

$$T_m = -2\sum_{k=1}^{r-1}\log\left(\frac{r^r}{k^k(r-k)^{r-k}}\left(\frac{\sum_{i=1}^k S_i^{(0)}}{\sum_{i=1}^r S_i^{(0)}}\right)^k \left(\frac{\sum_{i=k+1}^r S_i^{(0)}}{\sum_{i=1}^r S_i^{(0)}}\right)^{r-k}\right) \quad \text{and} \quad \chi_m^2 = -2\sum_{k=1}^{r-1}\log\left(\frac{\sum_{i=1}^k S_i^{(0)}}{\sum_{i=1}^r S_i^{(0)}}\right).$$

In contrast to Λ_m and $\tilde{\Lambda}_m$, the unconditional null distributions of T_m and χ_m^2 do not depend on the parameter α in log-link (1). Furthermore, the tests based on T_m and χ_m^2 are applicable even if (27) is not satisfied, i.e., when all failures are observed on the same stress level. However, in this case, analogously to the tests based on T and χ^2 when $m = 2$, only the exponentiality assumption, and not the specific SSALT structure, is tested. Similar to the simple SSALT tests, the critical regions associated with T_m or χ_m^2 can be computed via Monte Carlo simulations under condition (27). In addition, as the simulation results in Section 4 suggest, the

Table 1
Simulated powers for all considered conditional tests in case of $m = 2$, $n = 20$, and $r \in \{8, 12, 16\}$, with $F(\tau_1) = 0.20$. The expected number of failures per stress-level are given in parentheses. For T and χ^2, the approximated power values based on their unconditional distributions are given in a separate line, below the corresponding exact powers.

θ_2	$r = 8$ (4, 4)				$r = 12$ (4, 8)				$r = 16$ (4, 12)			
	Λ	$\tilde{\Lambda}$	T	χ^2	Λ	$\tilde{\Lambda}$	T	χ^2	Λ	$\tilde{\Lambda}$	T	χ^2
0.20	0.6688	0.2071	0.3719	0.2583	0.8343	0.3808	0.5991	0.4754	0.9053	0.5130	0.7160	0.6263
			0.3624	0.2323			0.5929	0.4411			0.7081	0.5969
0.25	0.5668	0.1615	0.2933	0.2239	0.7231	0.2856	0.4676	0.4001	0.8042	0.3863	0.5551	0.5254
			0.2839	0.1993			0.4613	0.3661			0.5457	0.4928
0.50	0.2317	0.0558	0.1069	0.1112	0.2700	0.0664	0.1301	0.1581	0.2991	0.0760	0.1362	0.1855
			0.1018	0.0944			0.1264	0.1357			0.1313	0.1626
2.00	0.0781	0.1768	0.1173	0.0859	0.1472	0.2332	0.1598	0.1275	0.1827	0.2642	0.1693	0.1446
			0.1119	0.0878			0.1557	0.1277			0.1633	0.1487
4.00	0.3061	0.4560	0.3218	0.2435	0.4657	0.5699	0.4036	0.3651	0.5131	0.6047	0.3944	0.3908
			0.3123	0.2489			0.3975	0.3667			0.3864	0.3979
5.00	0.4087	0.5557	0.4040	0.3188	0.5725	0.6681	0.4832	0.4575	0.6166	0.6976	0.4697	0.4791
			0.3939	0.3255			0.4768	0.4591			0.4616	0.4864

corresponding conditional distributions under H_0 can be well approximated by the unconditional ones. Thus, the critical values for χ^2_m are similar to those obtained from chi-square distribution with $2r - 2$ degrees of freedom.

In the next section, we compare the described scale tests via simulation studies under various experimental settings. It is worth pointing out that the unconditional (scale invariant) distribution of χ^2_m under H_0 makes its implementation much easier and with less computational resources.

4. Simulation study

In this section, the powers of the tests presented in Section 3 are compared under various SSALT parameter configurations. We focus only on the problem associated with the special null hypothesis $H_0 : \beta = 0$, i.e., testing if the mean lifetime varies over different stress levels. Without loss of generality we assume that $\theta_1 = 1$ which is equivalent to set $\alpha = 0$ in (1). Since the exact distributions of Λ, $\tilde{\Lambda}$, T and χ^2 under H_0, conditional on the fact that there are failures on at least two stress levels, are not in a closed form, for each choice of the model parameters in this study we determine the corresponding critical regions via 100,000 Monte Carlo simulations, conditioned on (27). The exact powers of the tests based on Λ, $\tilde{\Lambda}$, T and χ^2 are then evaluated by generating 100,000 simulations for each alternative. In order to compare the conditional and unconditional tests based on T and χ^2, beyond the exact power of T and χ^2, we also report the power based on their unconditional distributions. For the χ^2 test, it is an easily obtained approximation of the associated conditional on (27) distribution. All powers given in this section are computed at 5%-level of significance. It is important to note that the critical values of Λ and $\tilde{\Lambda}$, as well as the conditional ones of T and χ^2, depend on the fixed value of θ_1 and therefore the exact corresponding powers when θ_1 is unknown differ from the presented ones. However, from similar analysis in Zhu et al. (2020) it can be assumed that for the used parameter settings the powers based on Λ when θ_1 is unknown are close to the powers in the case of given θ_1. For most of the parameter settings used in the sequel, the probability that inequalities (27) hold is relatively close to 1, i.e., under most settings, the number of simulated samples for which the statistics Λ and $\tilde{\Lambda}$ cannot be evaluated is very small. These samples are discarded (and resampled) in all simulation results that follow. For the simulations summarized in Tables 1 to 4 (Section 4.1), the highest proportions of samples that needed to be resampled were 32.72%, 14.91% and 6.08% corresponding to the simulations reported for the cases $r = 16$ of Table 3, $r = 20$ of Table 4 and $r = 12$ of Table 2, respectively. The largest proportions of resampling for Table 1 were 4.56% ($r = 8$) and 1.18% ($r = 10$, $r = 16$), while in all other simulations the required resampling was less than 1%. For the simulations summarized in Tables 5 to 8 (Section 4.2), with the exception of the case $r = 20$ in Table 7, for which the highest proportion of resampling was 2.69%, all resampling proportions lied below 1%.

4.1. Powers for $m = 2$

For the simple SSALT ($m = 2$) the hypothesis $H_0 : \beta = 0$ takes the form $H_0 : \theta_1 = \theta_2$. Since we have fixed $\theta_1 = 1$ the alternative is set by the value of θ_2. We choose the time τ_1 for changing the stress factor to be such that either $F(\tau_1) = 0.20$ or $F(\tau_1) = 0.40$. The results of the described simulation study when $m = 2$, $n = 20, 40$ and for various values of r and θ_2 are given in Tables 1–4.

The parameter designs in Tables 1 and 3 suggest more failures on the second level and are associated with progressive SSALT $(\theta_1 > \theta_2)$, while Tables 2 and 4 match the regressive case $(\theta_1 < \theta_2)$. The expected number of failures on the two stress levels are given in brackets next to the values of r for each parameter configuration. For example, when $n = 20$, $F(\tau_1) = 0.20$ and $r = 16$ we would expect $4\ (= nF(\tau_1))$ observations on the first level and 12 on the second. We need to point out that the simulation set-ups in Tables 1 and 2 for $r = 12$ and $r = 16$ are chosen to coincide with those in Zhu et al. (2020) and therefore we obtain similar power results for the test based on Λ in these cases.

From the performed simulations we can conclude that the tests corresponding to T and χ^2 lack of power when $m = 2$. Moreover, the power results based on their conditional and unconditional tests are very close, which suggests that the critical regions of T

Table 2
Simulated powers for all considered conditional tests in case of $m = 2$, $n = 20$, and $r \in \{8, 12, 16\}$, with $F(\tau_1) = 0.40$. The expected number of failures per stress-level are given in parentheses. For T and χ^2, the approximated power values based on their unconditional distributions are given in a separate line, below the corresponding exact powers.

θ_2	$r = 10$ (8, 2)				$r = 12$ (8, 4)				$r = 16$ (8, 8)			
	Λ	$\tilde{\Lambda}$	T	χ^2	Λ	$\tilde{\Lambda}$	T	χ^2	Λ	$\tilde{\Lambda}$	T	χ^2
0.20	0.5623	0.1998	0.2533	0.1219	0.6936	0.3096	0.3832	0.1883	0.8808	0.5628	0.6747	0.3776
			0.2478	0.1136			0.3829	0.1797			0.6780	0.3772
0.25	0.4752	0.1668	0.2130	0.1126	0.5982	0.2531	0.3181	0.1655	0.7881	0.4580	0.5600	0.3259
			0.2076	0.1045			0.3179	0.1580			0.5637	0.3254
0.50	0.2012	0.0722	0.0982	0.0740	0.2496	0.0902	0.1247	0.0955	0.3266	0.1363	0.1904	0.1463
			0.0945	0.0685			0.1245	0.0900			0.1932	0.1460
2.00	0.0453	0.1599	0.1007	0.0723	0.0988	0.2155	0.1307	0.0954	0.1964	0.2996	0.1992	0.1496
			0.0973	0.0783			0.1306	0.0977			0.2018	0.1517
4.00	0.2487	0.4619	0.3106	0.2031	0.4390	0.6006	0.4299	0.3075	0.6661	0.7595	0.5969	0.4986
			0.3043	0.2182			0.4296	0.3133			0.6004	0.5021
5.00	0.3622	0.5748	0.4084	0.2744	0.5798	0.7214	0.5502	0.4143	0.7899	0.8572	0.7156	0.6325
			0.4021	0.2928			0.5499	0.4207			0.7188	0.6361

Table 3
Simulated powers for all considered conditional tests in case of $m = 2$, $n = 40$, and $r \in \{16, 24, 32\}$, with $F(\tau_1) = 0.20$. The expected number of failures per stress-level are given in parentheses. For T and χ^2, the approximated power values based on their unconditional distributions are given in a separate line, below the corresponding exact powers.

θ_2	$r = 16$ (8, 8)				$r = 24$ (8, 16)				$r = 32$ (8, 24)			
	Λ	$\tilde{\Lambda}$	T	χ^2	Λ	$\tilde{\Lambda}$	T	χ^2	Λ	$\tilde{\Lambda}$	T	χ^2
0.20	0.8742	0.5587	0.6674	0.3773	0.9785	0.8365	0.9081	0.7053	0.9944	0.9239	0.9632	0.8661
			0.6624	0.3768			0.9063	0.7057			0.9633	0.8648
0.25	0.7903	0.4651	0.5628	0.3287	0.9304	0.7255	0.8062	0.6122	0.9686	0.8390	0.8843	0.7758
			0.5576	0.3282			0.8031	0.6127			0.8845	0.7735
0.50	0.3327	0.1404	0.1928	0.1496	0.4190	0.2098	0.2589	0.2386	0.4650	0.2528	0.2780	0.2984
			0.1891	0.1494			0.2555	0.2391			0.2784	0.2962
2.00	0.1806	0.2980	0.1965	0.1452	0.2823	0.3870	0.2551	0.2252	0.3131	0.4223	0.2548	0.2531
			0.1924	0.1459			0.2517	0.2215			0.2554	0.2505
4.00	0.6441	0.7523	0.5878	0.4931	0.7581	0.8309	0.6386	0.6352	0.7832	0.8501	0.6124	0.6566
			0.5825	0.4940			0.6349	0.6304			0.6129	0.6539
5.00	0.7702	0.8505	0.7027	0.6191	0.8483	0.8991	0.7304	0.7412	0.8648	0.9125	0.6992	0.7496
			0.6984	0.6203			0.7272	0.7370			0.6997	0.7474

Table 4
Simulated powers for all considered conditional tests in case of $m = 2$, $n = 40$, and $r \in \{20, 24, 32\}$, with $F(\tau_1) = 0.40$. The expected number of failures per stress-level are given in parentheses. For T and χ^2, the approximated power values based on their unconditional distributions are given in a separate line, below the corresponding exact powers.

θ_2	$r = 20$ (16, 4)				$r = 24$ (16, 8)				$r = 32$ (16, 16)			
	Λ	$\tilde{\Lambda}$	T	χ^2	Λ	$\tilde{\Lambda}$	T	χ^2	Λ	$\tilde{\Lambda}$	T	χ^2
0.20	0.7432	0.4032	0.3858	0.1518	0.8984	0.6523	0.6558	0.2778	0.9913	0.9286	0.9434	0.6229
			0.3854	0.1442			0.6560	0.2776			0.9438	0.6215
0.25	0.6577	0.3406	0.3253	0.1361	0.8322	0.5638	0.5652	0.2468	0.9689	0.8653	0.8763	0.5492
			0.3248	0.1291			0.5656	0.2465			0.8772	0.5480
0.50	0.2907	0.1324	0.1385	0.0864	0.3898	0.2013	0.2187	0.1244	0.5292	0.3396	0.3609	0.2383
			0.1382	0.0824			0.2189	0.1243			0.3625	0.2376
2.00	0.1366	0.2599	0.1569	0.1018	0.2631	0.3729	0.2447	0.1614	0.4230	0.5201	0.3743	0.2705
			0.1566	0.1076			0.2448	0.1621			0.3760	0.2793
4.00	0.5974	0.7202	0.5461	0.3545	0.8328	0.8842	0.7565	0.5845	0.9448	0.9646	0.8884	0.8172
			0.5456	0.3673			0.7567	0.5856			0.8892	0.8242
5.00	0.7356	0.8243	0.6736	0.4774	0.9226	0.9491	0.8673	0.7289	0.9832	0.9901	0.9535	0.9189
			0.6732	0.4905			0.8674	0.7297			0.9540	0.9228

and χ^2 can be well approximated from their easy to evaluate scale invariant null distributions. The test based on Λ is the most powerful in the case $\theta_1 > \theta_2$, whereas $\tilde{\Lambda}$ has more power when $\theta_1 < \theta_2$. This can be explained by the properties of $\hat{\theta}_1$ and $\tilde{\theta}_1$ described in Remark 2. Moreover, under the alternative hypothesis the order-restricted MLE of θ_1 coincide with $\hat{\theta}_1$ when $\hat{\theta}_2 \leq \gamma_0 \hat{\theta}_1$,

Table 5
Simulated powers for all considered conditional tests in case of $m = 3$, $n = 20$, and $r \in \{12, 16\}$, with $F(\tau_1) = 0.40$, $F(\tau_2) = 0.60$ (left) and $F(\tau_1) = 0.20$, $F(\tau_2) = 0.40$ (middle and right). The expected number of failures per stress-level are given in parentheses. For T and χ^2, the approximated power values based on their unconditional distributions are given in a separate line, below the corresponding exact powers.

θ_2	$r = 16\ (8,4,4)$				$r = 12\ (4,4,4)$				$r = 16\ (4,4,8)$			
	Λ	$\tilde{\Lambda}$	T	χ^2	Λ	$\tilde{\Lambda}$	T	χ^2	Λ	$\tilde{\Lambda}$	T	χ^2
0.20	0.8715	0.5546	0.6773	0.3787	0.8063	0.3904	0.5936	0.4440	0.6551	0.5476	0.7742	0.6236
			0.6723	0.3790			0.5928	0.4413			0.7696	0.6240
0.25	0.7516	0.4544	0.5717	0.3282	0.6699	0.2960	0.4672	0.3738	0.4107	0.4889	0.7081	0.5659
			0.5668	0.3285			0.4662	0.3708			0.7030	0.5664
0.50	0.2183	0.1715	0.2648	0.1733	0.1428	0.0998	0.1960	0.1841	0.1355	0.2479	0.4060	0.3307
			0.2601	0.1737			0.1954	0.1820			0.4006	0.3311

Table 6
Simulated powers for all considered conditional tests in case of $m = 3$, $n = 40$, and $r \in \{24, 32\}$, with $F(\tau_1) = 0.40$, $F(\tau_2) = 0.60$ (left) and $F(\tau_1) = 0.20$, $F(\tau_2) = 0.40$ (middle and right). The expected number of failures per stress-level are given in parentheses. For T and χ^2, the approximated power values based on their unconditional distributions are given in a separate line, below the corresponding exact powers.

θ_2	$r = 32\ (16,8,8)$				$r = 24\ (8,8,8)$				$r = 32\ (8,8,16)$			
	Λ	$\tilde{\Lambda}$	T	χ^2	Λ	$\tilde{\Lambda}$	T	χ^2	Λ	$\tilde{\Lambda}$	T	χ^2
0.20	0.9890	0.9255	0.9416	0.6242	0.9769	0.8358	0.9075	0.7030	0.7862	0.9371	0.9790	0.8782
			0.9426	0.6221			0.9065	0.7055			0.9796	0.8771
0.25	0.9408	0.8614	0.8783	0.5522	0.9187	0.7282	0.8058	0.6104	0.5366	0.9171	0.9635	0.8340
			0.8800	0.5501			0.8044	0.6132			0.9643	0.8325
0.50	0.2875	0.4528	0.5029	0.2937	0.1843	0.2944	0.3822	0.2980	0.2967	0.6681	0.7471	0.5753
			0.5064	0.2913			0.3802	0.3009			0.7501	0.5730

see formula (15) in Zhu et al. (2020). Thus, in the next section we compare the power of the four tests under multilevel ($m > 2$) experiment design only for progressive SSALT.

4.2. Powers for $m > 2$

Before setting the parameters for the multilevel simulations ($m > 2$), we make the following remark. In numerous practical problems and theoretical analysis it is assumed that the stress factor levels are equally-spaced, see for example Lu et al. (2006) and Wu et al. (2008). In this case the experiment stress levels form an arithmetic sequence, i.e.,

$$x_j = x_1 + (j-1)q, \quad \text{for } j = 1, 2, \ldots, m, \tag{32}$$

with $q > 0$ in progressive SSALT and $q < 0$ in regressive SSALT. Additionally, if the function $h(\cdot)$ in the log-link relation (1) is monotonic, it can be standardized. When $h(\cdot)$ is increasing and $q > 0$ in (32), we can define

$$h^*(x_j) = \frac{h(x_j) - h(x_1)}{h(x_m) - h(x_1)}, \quad \text{such that } 0 \le h^*(x_j) \le 1 \text{ for } j = 1, 2, \ldots, m. \tag{33}$$

Then the life-stress relationship (1) can be expressed as

$$\log(\theta_j) = \alpha^* + \beta^* h^*(x_j), \quad \text{where } \alpha^* = \alpha + \beta h(x_1) \text{ and } \beta^* = \beta\left(h(x_m) - h(x_1)\right).$$

If $h(\cdot)$ is the identity function, then $h^*(\cdot)$ in (33) is simplified to $h^*(x_j) = (j-1)/(m-1)$. Furthermore, when $h(x_j) = \log(x_j)$, we get

$$h^*(x_j) = \frac{\log\left(1 + (j-1)\frac{q}{x_1}\right)}{\log\left(1 + (m-1)\frac{q}{x_1}\right)} \quad \text{and} \quad h^*(x_j) \longrightarrow \frac{j-1}{m-1} \text{ as } \frac{q}{x_1} \to 0.$$

In a similar way, it is easy to check that

$$h^*(x_j) = \frac{j-1}{m-1}\,\frac{1-(m-1)\frac{q}{x_1}}{1-(j-1)\frac{q}{x_1}} \quad \text{and} \quad h^*(x_j) \longrightarrow \frac{j-1}{m-1} \text{ as } \frac{q}{x_1} \to 0,$$

when $q < 0$ in (32), $-q/x_1 < 1/(m-1)$ and $h(x_j) = 1/x_j$ (corresponding to the Arrhenius law). Hence, if the absolute value of q is relatively small compared to x_1, the standardized stress loading $h(\cdot)$ does not depend on the stress increment q and (1) is close to the log-linear relationship. Further, notice that both the standardized logarithmic and reciprocal link functions approximate the identity relation from above, with the logarithmic having a smaller convergence rate. Therefore, we can conclude that the Arrhenius life-stress relation is less sensitive than the logarithmic link in terms of relatively small changes of the stress increment step. However,

Table 7
Simulated powers for all considered conditional tests in case of $m = 4$, $n = 20$, and $r \in \{12, 16\}$, with $F(\tau_1) = 0.35$, $F(\tau_2) = 0.50$, $F(\tau_3) = 0.65$ (left) and $F(\tau_1) = 0.15$, $F(\tau_2) = 0.30$, $F(\tau_3) = 0.45$ (middle and right). The expected number of failures per stress-level are given in parentheses. For T and χ^2, the approximated power values based on their unconditional distributions are given in a separate line, below the corresponding exact powers.

θ_2	$r = 16$ (7, 3, 3, 3)				$r = 12$ (3, 3, 3, 3)				$r = 16$ (3, 3, 3, 7)			
	Λ	$\tilde{\Lambda}$	T	χ^2	Λ	$\tilde{\Lambda}$	T	χ^2	Λ	$\tilde{\Lambda}$	T	χ^2
0.20	0.7780	0.5990	0.7531	0.4594	0.6840	0.3506	0.5838	0.4973	0.3695	0.6566	0.8868	0.7301
			0.7488	0.4598			0.5832	0.4945			0.8840	0.7305
0.25	0.6158	0.5388	0.6886	0.4160	0.4551	0.2855	0.4987	0.4395	0.3091	0.6587	0.8618	0.6972
			0.6842	0.4164			0.4981	0.4367			0.8586	0.6976
0.50	0.2636	0.2766	0.4015	0.2500	0.0769	0.1438	0.2842	0.2553	0.1371	0.3168	0.5162	0.4344
			0.3965	0.2504			0.2835	0.2532			0.5102	0.4349

Table 8
Simulated powers for all considered conditional tests in case of $m = 4$, $n = 40$, and $r \in \{24, 32\}$, with $F(\tau_1) = 0.35$, $F(\tau_2) = 0.50$, $F(\tau_3) = 0.65$ (left) and $F(\tau_1) = 0.15$, $F(\tau_2) = 0.30$, $F(\tau_3) = 0.45$ (middle and right). The expected number of failures per stress-level are given in parentheses. For T and χ^2, the approximated power values based on their unconditional distributions are given in a separate line, below the corresponding exact powers.

θ_2	$r = 32$ (14, 6, 6, 6)				$r = 24$ (6, 6, 6, 6)				$r = 32$ (6, 6, 6, 14)			
	Λ	$\tilde{\Lambda}$	T	χ^2	Λ	$\tilde{\Lambda}$	T	χ^2	Λ	$\tilde{\Lambda}$	T	χ^2
0.20	0.9224	0.9474	0.9690	0.7191	0.8716	0.8030	0.9062	0.7660	0.7294	0.9801	0.9972	0.9440
			0.9696	0.7175			0.9054	0.7683			0.9972	0.9435
0.25	0.8015	0.9203	0.9448	0.6674	0.5916	0.7214	0.8343	0.6869	0.7088	0.9821	0.9958	0.9283
			0.9458	0.6657			0.8329	0.6895			0.9959	0.9278
0.50	0.4767	0.6754	0.7249	0.4336	0.1518	0.4574	0.5737	0.4400	0.2588	0.7717	0.8540	0.7067
			0.7277	0.4312			0.5716	0.4427			0.8559	0.7048

in order to make our simulation study independent of the stress values x_j, for $j = 1, \ldots, m$, we assume equally-spaced stress levels with $h(\cdot)$ being the identity function. The results of the performed simulations for $m = 3, 4$, $n = 20, 40$ and different choices of r, τ_1, τ_2 and τ_3 are given in Tables 5–8. Similar to the tables for simple SSALT, the expected numbers of failures on each stress level are given next to the value of r. In addition, for the multilevel SSALT settings the reported powers based on the unconditional distributions of T and χ^2 are even closer to conditional ones compared to the simple SSALT results in Tables 1–4.

By comparing the simulation results for $n = 20$, $r = 16$ in Tables 5 and 7 and for $n = 40$, $r = 32$ in Tables 6 and Table 8, we can conclude that in contrast to the tests based on $\tilde{\Lambda}$, T and χ^2 the power of Λ is not strictly increasing when we increase the number of stress levels. Therefore, in some multilevel scenarios the test procedure associated with the log-likelihood statistic Λ is not the most powerful. In general, the test induced by T seems to overpower the others in most multilevel parameter configurations. Furthermore, if we consider the case $n = 20$ and $r = 16$, the greatest power for $\theta_2 = 0.50$ is 0.5162, obtained by T with $m = 4$, $F(\tau_1) = 0.15$, $F(\tau_2) = 0.30$, $F(\tau_3) = 0.45$ and exceeds all Λ powers in the corresponding simple SSALT designs ($m = 2$). Similarly, for $\theta_2 = 0.25$ the best power 0.8618 is also for $m = 4$ and T. When $n = 20$, $r = 16$ and $\theta_2 = 0.20$ the largest power of T for $m = 4$ (0.8868) is close to the best power of Λ for $m = 2$ (0.9053). If we fix $n = 20$, $r = 16$, $m = 4$, $F(\tau_1) = 0.10$, $F(\tau_2) = 0.20$ and $F(\tau_3) = 0.40$, further simulation experiments (not presented here) show that the powers of T are increased to 0.9739 and 0.9513 for $\theta_2 = 0.50$ and $\theta_2 = 0.25$, respectively, while for $\theta_2 = 0.20$ the power is slightly decreased to 0.4759. Hence, the test statistic T and the multilevel settings should be included in the procedure of constructing an optimal design based on test powers. Additionally, since the simulated power results indicate that χ^2 and T have similar design sensitivity, in order to find the best experiment design we could use the χ^2 test based on its unconditional (scale invariant) distribution as an easy to implement approximation that requires fewer computational resources. We should remark as well that under the multilevel design we use the log-link assumption which allows us to extrapolate the model to NOC. Therefore, the statistics T and χ^2 are appropriate not only in testing if there is a change of the mean lifetime over the stress levels but also in revealing the structure of this relation.

5. Illustrative example

In this section, we apply the derived scale tests to a real data example in Lu et al. (2006). The study aimed to use a multilevel SSALT procedure to estimate the reliability of a rear suspension aft lateral link. Lu et al. (2006) made the following assumptions:

(a) under stress level x_j (load, in lbs), the unit lifetime (in cycles) has a two-parameter Weibull distribution with cdf $F(t) = 1 - \exp\left(-(t/\theta_j)^\delta\right)$, where the shape parameter δ is constant on all stress levels,

(b) $h(x_j) = \log(x_j)$ in the log-link relation, i.e.,

$$\log\left(\theta_j\right) = \alpha + \beta \log\left(x_j\right),$$

(c) the cumulative exposure (CE) assumption holds.

Table 9
Experiment design and estimations based on the pilot and main studies by Lu et al. (2006).

Stress load (lbs)	Stress level interval	τ_j	$F_T^*(\tau_j)$	θ_j^*	D_j	$\hat{F}_T(\tau_j)$	$\hat{\theta}_j$
2000	$0 \le t < 6123$	6123	0.058	102 062	0	0.014	447 051
2400	$6123 \le t < 9184$	9184	0.148	30 606	1	0.088	38 809
2800	$9184 \le t < 10715$	10 715	0.258	11 055	3	0.332	4915
3200	$10715 \le t < 11480$	11 480	0.372	4576	5	0.737	821
3600	$11480 \le t < 11863$	11 863	0.477	2102	3	0.973	169
4000	$11863 \le t < 12054$	12 054	0.564	1048	0	1.000	41
4400	$12054 \le t < 12150$	12 150	0.633	558	0	1.000	11
4800	$12150 \le t < 12198$	12 198	0.685	314	0	1.000	4

Table 10
Observed values of test statistics with associated critical values and p-values for testing $H_0 : \beta = \hat{\beta} = -15.5920$ in case of complete data ($r = n = 12$) and under 17% and 33% censoring ($r = 10$ and $r = 8$). The critical values for the two-sided χ^2 tests are given as vectors, i.e., the corresponding acceptance regions are provided. For T and χ^2, the approximated critical values and p-values based on their unconditional distributions are given in a separate line, below the conditional ones.

	$r = 12$				$r = 10$				$r = 8$			
	Λ	$\tilde{\Lambda}$	T	χ^2	Λ	$\tilde{\Lambda}$	T	χ^2	Λ	$\tilde{\Lambda}$	T	χ^2
Obs. statistic	77.09	0.01	1.06	19.97	56.43	0.02	1.20	16.18	40.47	0.01	1.25	12.08
Crit. value(s)	118.86	5.03	31.01	$(11.04, 36.82)$	93.30	5.02	25.77	$(8.26, 31.54)$	69.47	4.94	20.52	$(5.80, 26.07)$
			30.84	$(10.98, 36.78)$			26.03	$(8.23, 31.53)$			20.62	$(5.63, 26.12)$
p-value	0.698	0.933	0.997	0.822	0.724	0.907	0.985	0.837	0.658	0.934	0.954	0.799
			0.996	0.830			0.985	0.840			0.953	0.800

By conducting a pilot study on constant stress levels, Lu et al. (2006) constructed a SSALT design with $m = 8$ levels and Type-I censoring scheme. The selected stress loads x_j and the changing times τ_j are given in Table 9. As pointed in Hu et al. (2013), all confident intervals for δ from the pilot study results contain the value 1, i.e., we may assumed that the unit lifetimes are exponentially distributed. Furthermore, after standardizing the stress function by

$$h^*(x_j) = \frac{\log(x_j) - \log(1500)}{\log(4800) - \log(1500)}, \quad \text{for } j = 1, 2, \ldots, 8, \text{ where 1500 is the stress load under NOC,}$$

Hu et al. (2013) obtained that the least square estimates of α and β are 13.4337 and -7.6836, respectively. The corresponding single unit cdf $F_T^*(\cdot)$ at τ_j and mean lifetimes θ_j^* are presented in Table 9.

In their main SSALT experiment, Lu et al. (2006) tested $n = 12$ units, all of which broke on the first 5 stress levels. The number of failures on each level D_j are given as the sixth column of Table 9, whereas the exact failure times can be found in Lu et al. (2006). From the obtained results we find the least square estimates to be $\hat{\alpha} = 16.8668$ and $\hat{\beta} = -15.5920$. The associated cdf $\hat{F}_T(\cdot)$ at τ_j and mean lifetimes $\hat{\theta}_j$ are also listed in Table 9. Since the observed values D_j do not agree with the pilot study estimates $F_T^*(\tau_j)$, it is evident that -7.6836 cannot be accepted as the value of β. However, we can use the scale tests in Section 3 for testing $H_0 : \beta = \beta_0 = \hat{\beta}$. As there is no censoring observed in the data, we apply the four test for the complete sample ($r = 12$) and also for Type-II censoring with $r = 10$ and $r = 8$. The observed statistics, critical values and p-values are given in Table 10. All p-values are larger than 0.05, so we cannot reject the hypothesis that $\beta = -15.5920$. Moreover, it seems that the introduced Type-II censoring has a very small effect on the four testing procedures. Further, notice that the critical regions corresponding to the unconditional (scale invariant) tests of T and χ^2 are very similar to the conditional ones, leading thus to the same conclusions.

6. Conclusions

In this paper, we have considered three modifications of the log-likelihood ratio statistic Λ suggested by Zhu et al. (2020) for testing the scale parameter in the simple SSALT model with exponential lifetimes and under Type-II censoring. We have proved the independent and exponential properties of the normalized spacings in SSALT framework and used them to define the statistic $\tilde{\Lambda}$ which is asymptotically equivalent to Λ and more appropriate for regressive SSALT. Following the spacings based approach of other studies on exponential testing procedures we have proposed two scale invariant alternatives of Λ, namely T and χ^2. The derived statistics together with the log-likelihood method have been extended to the multilevel SSALT under the assumption of log-link relation between the mean lifetimes and the stress levels. Based on simulation results we have concluded that T and χ^2 are more helpful in multilevel scenarios which in some cases lead to more powerful tests compared to simple step-stress designs. As a future work it would be interesting to develop algorithms for finding an optimal SSALT design in terms of scale test powers and analyze simple versus multilevel designs. Furthermore, a generalization of the obtained results could be considered when the data sample is censored by interval monitoring of the experiment (see, Bobotas and Kateri, 2015 and Lee and Pan, 2012).

Acknowledgments

We would like to express our thanks to the associate editor and the reviewers for their useful and constructive comments which helped us improve the paper. The second author was funded by the Federal Ministry of Education and Research (BMBF), within the project BALd in the Cluster BattNutzung, Grant Nr. 03XP0320A.

Appendix. Proofs

Proof of Theorem 1. We derive the probability density function (pdf) of T by taking the derivative of the cdf in (2):

$$f_T(t) = \frac{1}{\theta_k}\exp\left(-\frac{t-\tau_{k-1}}{\theta_k}\right)\exp\left(-\sum_{j=1}^{k-1}\frac{\tau_j-\tau_{j-1}}{\theta_j}\right), \text{ for } \tau_{k-1} < t \le \tau_k,$$

where $0 \equiv \tau_0 < \tau_1 < \cdots < \tau_m \equiv \infty$, $k = 1, 2\ldots, m$ and $\sum_{j=1}^{0}(\cdot) \equiv 0$. Then, the joint density of $T_{1:n}, \ldots, T_{n:n}$ is

$$f(t_{1:n}, \ldots, t_{n:n}) = n!\prod_{i=1}^{n}\frac{1}{\theta_{\ell(i)}}\exp\left(-\frac{t_{i:n}-\tau_{\ell(i)-1}}{\theta_{\ell(i)}}\right)\exp\left(-\sum_{j=1}^{\ell(i)-1}\frac{\tau_j-\tau_{j-1}}{\theta_j}\right). \tag{34}$$

From (4) it follows that

$$T_{i:n} = \frac{\theta_{\ell(i)}}{\theta_1}T^*_{i:n} + \tau_{\ell(i)-1} - \theta_{\ell(i)}\sum_{j=1}^{\ell(i)-1}\frac{\tau_j-\tau_{j-1}}{\theta_j}, \quad \text{for } i = 1, 2, \ldots, n, \text{ i.e.,} \tag{35}$$

the determinant of the Jacobian transformation matrix is $\prod_{i=1}^{n}\left(\theta_{\ell(i)}/\theta_1\right)$. By combining this fact with (34) and (35), we obtain that the joint density of $T^*_{1:n}, \ldots, T^*_{n:n}$ is

$$f^*(t^*_{1:n}, \ldots, t^*_{n:n}) = n!\prod_{i=1}^{n}\frac{1}{\theta_1}\exp\left(-\frac{t^*_{i:n}}{\theta_1}\right),$$

which completes the proof. □

Proof of Theorem 2. Since $T_{D_1:n}$ is the largest (maximal) observation on the first level, by formula (25) in Balakrishnan et al. (2007), the conditional cdf of $T_{D_1:n}$, given that $D_1 = d$, can be expressed as

$$F_{T_{D_1:n}|D_1=d}(t) = \left(\frac{F(t)}{F(\tau_1)}\right)^d, \quad \text{for } 0 \le t \le \tau_1, \tag{36}$$

with $F(t) = 1 - \exp\left(-t/\theta_1\right)$ being the cdf of $Exp(1/\theta_1)$.

Let us denote by p the probability of a failure on the first stress level. Then, D_1 has binomial distribution with parameters n and p, where $p = F(\tau_1) = 1 - \exp\left(-\tau_1/\theta_1\right)$ and $0 < p < \rho \le 1$. Furthermore, by using (12) and (36), for $0 \le t \le \tau_1\,(n-r+1)$ we obtain

$$P\left(Z > t \mid 1 \le D_1 \le r-1\right) = F_{T_{D_1:n}|1\le D_1\le r-1}\left(\tau_1 - \frac{t}{n-D_1}\right) = \frac{\sum_{d=1}^{r-1}\binom{n}{d}\left(F\left(\tau_1-\frac{t}{n-d}\right)\right)^d(1-p)^{n-d}}{P\left(1 \le D_1 \le r-1\right)}. \tag{37}$$

Since the binomial properties of D_1 and the relation $r = [\rho n] > pn$, the number of failures on the first level is almost surely observed for large sample sizes, i.e., $P\left(1 \le D_1 \le r-1\right) \longrightarrow 1$ as $n \to \infty$. Hence, in order to find the asymptotic distribution of Z it is sufficient to study only the numerator in (37).

Let us fix a real constant δ, such that $1/2 < \delta < 1$, and denote the terms of the sum in (37) by $A(d,t)$ for short. Then, for large values of n we have

$$P\left(Z > t \mid 1 \le D_1 \le r-1\right) \approx \sum_{d=1}^{d^*-1}A(d,t) + \sum_{d=d^*}^{d^{**}}A(d,t) + \sum_{d=d^{**}+1}^{r-1}A(d,t), \tag{38}$$

where $d^* = \left[p\left(n-n^\delta\right)\right]$ and $d^{**} = \left[p\left(n+n^\delta\right)\right]$ with $[x]$ being the integer part of x and "$\approx$" denoting asymptotic equivalence. From the fact that $A\left(d,t\right) \le P\left(D_1 = d\right)$ for $t \ge 0$ and $0 \le d < n$, it follows that the first sum in (38) is less than or equal to $P\left(D_1 < d^*\right)$. However, it is easy to check, for example by applying a normal approximation, that this probability converges to 0 for large values of n. Similarly, the third sum in (38) converges to 0 as $n \to \infty$. For the second sum, we have

$$\sum_{d=d^*}^{d^{**}}\binom{n}{d}\left(F\left(\tau_1-\frac{t}{n-d^{**}}\right)\right)^d(1-p)^{n-d} \le \sum_{d=d^*}^{d^{**}}A(d,t) \le \sum_{d=d^*}^{d^{**}}\binom{n}{d}\left(F\left(\tau_1-\frac{t}{n-d^*}\right)\right)^d(1-p)^{n-d}.$$

By using similar arguments as above, it can be further shown that

$$\left(1-p+F\left(\tau_1-\frac{t}{n-d^{**}}\right)\right)^n - B_1 - B_2 \le \sum_{d=d^*}^{d^{**}}A(d,t) \le \left(1-p+F\left(\tau_1-\frac{t}{n-d^*}\right)\right)^n, \tag{39}$$

where $B_1 = \sum_{d=0}^{d^*-1} \binom{n}{d} \left(F\left(\tau_1 - \frac{t}{n-d^{**}}\right)\right)^d (1-p)^{n-d} \to 0$ and $B_2 = \sum_{d=d^{**}+1}^{n} \binom{n}{d} \left(F\left(\tau_1 - \frac{t}{n-d^{**}}\right)\right)^d (1-p)^{n-d} \to 0$, as $n \to \infty$, since $B_1 \le P\left(D_1 < d^*\right)$, $B_2 \le P\left(D_1 > d^{**}\right)$, $\left(d^* - np\right)/\sqrt{n} \to -\infty$ and $\left(d^{**} - np\right)/\sqrt{n} \to \infty$. In addition, Taylor expansion of the most left term in (39) gives

$$\left(1 - p + F\left(\tau_1 - \frac{t}{n-d^{**}}\right)\right)^n = \exp\left(n\left(-\frac{(1-p)t}{\theta_1 \left(n - d^{**}\right)} + O\left(\frac{1}{n^2}\right)\right)\right) \xrightarrow[n\to\infty]{} \exp\left(-\frac{t}{\theta_1}\right).$$

Analogously, the most right term in (39) converges to $\exp\left(-t/\theta_1\right)$, which implies that $\sum_{d=d^*}^{d^{**}} A(d,t)$ also goes to $\exp\left(-t/\theta_1\right)$. Therefore, by combining the asymptotic results for the three sums in (38), it follows that

$$P\left(Z > t \middle| 1 \le D_1 \le r-1\right) \xrightarrow[n\to\infty]{} \exp\left(-\frac{t}{\theta_1}\right),$$

which completes the proof. □

Proof of Corollary 2. Let us fix a constant δ such that $0 < \delta < 1$ and rewrite the ratio Z/D_1 as $\frac{Z}{D_1} = \frac{Z}{n^\delta}\frac{n^\delta}{D_1}$. Then, from Theorem 2 it follows that

$$P\left(\frac{Z}{n^\delta} > t \middle| 1 \le D_1 \le r-1\right) \xrightarrow[n\to\infty]{} 0, \quad \text{for any } t > 0,$$

while from the normal approximation of D_1 we have

$$P\left(\frac{n^\delta}{D_1} > t \middle| 1 \le D_1 \le r-1\right) \le \frac{P\left(\frac{n^\delta}{D_1} > t\right)}{P\left(1 \le D_1 \le r-1\right)} = \frac{P\left(\frac{D_1 - np}{\sqrt{np(1-p)}} < \frac{n^\delta/t - np}{\sqrt{np(1-p)}}\right)}{P\left(1 \le D_1 \le r-1\right)} \xrightarrow[n\to\infty]{} \frac{\Phi(-\infty)}{1} = 0,$$

for any $t > 0$, with $\Phi(\cdot)$ being the cdf of a standard normal variable. Thus, Z/n^δ and n^δ/D_1 converge in probability to 0 as $n \to \infty$, which implies that Z/D_1 converge to 0 as well. The proof for $Z/\left(r - D_1\right)$ is done in a similar manner. □

Proof of Proposition 1. In order to prove the asymptotic equivalence of Λ and $\tilde{\Lambda}$ let us first write their difference as

$$\Lambda - \tilde{\Lambda} = -2D_1 \log\left(1 + \frac{Z}{D_1\tilde{\theta}_1}\right) - 2\left(r - D_1\right)\log\left(1 - \frac{Z\gamma_0}{\left(r - D_1\right)\tilde{\theta}_2}\right),$$

where $\tilde{\theta}_1$ and $\tilde{\theta}_2$ are defined in (13) and (14), respectively. From Corollary 2 and the consistency of $\tilde{\theta}_1$ and $\tilde{\theta}_2$, we obtain

$$\frac{D_1}{Z}\log\left(1 + \frac{Z}{D_1\tilde{\theta}_1}\right) \xrightarrow{\text{p}} \frac{1}{\theta_1} \quad \text{and} \quad \frac{r - D_1}{Z}\log\left(1 - \frac{Z\gamma_0}{\left(r - D_1\right)\tilde{\theta}_2}\right) \xrightarrow{\text{p}} -\frac{1}{\theta_1}, \quad \text{as } n \to \infty,$$

where $\xrightarrow{\text{p}}$ denotes convergence in probability. Hence,

$$\frac{\Lambda - \tilde{\Lambda}}{Z} \xrightarrow{\text{p}} 0, \quad \text{as } n \to \infty.$$

Since from Theorem 2 we have that the asymptotic distribution of Z is $Exp\left(1/\theta_1\right)$, it follows that $\Lambda - \tilde{\Lambda} \xrightarrow{\text{p}} 0$. □

Proof of Theorem 3. Since $\kappa^{a_j} = \left(\kappa^*\right)^{a_j^*}$, where $\kappa^* = \kappa^{-1}$ and $a_j^* = -a_j$, the log-likelihood in (24) can be reparameterized with κ^* and a_j^* such that $G(\kappa)$ for $a_1, \ldots, a_m$ coincide with $G(\kappa^*)$ for $a_1^*, \ldots, a_m^*$. Thus, without loss of generality we can assume that the function $h(\cdot)$ is monotonically increasing on $x_1, x_2, \ldots, x_m$, i.e.,

$$0 = a_1 < a_2 < \cdots < a_m. \tag{40}$$

We can rewrite $G(\kappa)$ in (25) as $G(\kappa) = \sum_{j=1}^m w_j \kappa^{a_j}$, where $\left\{w_j\right\}_{j=1}^m$ are given by

$$w_j = \sum_{i\in\mathcal{I}_j} S_i^{(0)} + \sum_{i\in\mathcal{I}_j^{(=,<)}} Z_i - \sum_{i\in\mathcal{I}_j^{(>,=)}} Z_i - \sum_{i\in\mathcal{I}_j^{(=,<)}} \left(\sum_{k=\ell(i-1)}^{\ell(i)-1} c_{i,k}\right) + \sum_{i\in\mathcal{I}_j^{(>,\le)}} c_{i,j}, \tag{41}$$

with index sets

$$\mathcal{I}_j = \left\{1 \le i \le r : \ell(i) = j\right\}, \quad \mathcal{I}_j^{(=,<)} = \left\{1 \le i \le r : \ell(i) = j \textit{ and } \ell(i-1) < j\right\}$$
$$\mathcal{I}_j^{(>,=)} = \left\{1 \le i \le r : \ell(i) > j \textit{ and } \ell(i-1) = j\right\}, \quad \mathcal{I}_j^{(>,\le)} = \left\{1 \le i \le r : \ell(i) > j \textit{ and } \ell(i-1) \le j\right\}.$$

It is obvious that $\mathcal{I}_j^{(=,<)} \subseteq \mathcal{I}_j$ and $\mathcal{I}_j^{(>,=)} \subseteq \mathcal{I}_j^{(>,\le)}$. Then, we can reorder (41) in the following way

$$w_j = \sum_{i\in\mathcal{I}_j\backslash\mathcal{I}_j^{(=,<)}} S_i^{(0)} + \sum_{i\in\mathcal{I}_j^{(=,<)}} \left(S_i^{(0)} + Z_i - \sum_{k=\ell(i-1)}^{\ell(i)-1} c_{i,k}\right) + \sum_{i\in\mathcal{I}_j^{(>,=)}} \left(c_{i,j} - Z_i\right) + \sum_{i\in\mathcal{I}_j^{(>,\le)}\backslash\mathcal{I}_j^{(>,=)}} c_{i,j}. \tag{42}$$

From definitions (20) and (26) it is clear that $S_i^{(0)} \geq 0$ and $c_{i,j} > 0$, for $i = 1, \dots, r$ and $j = 1, \dots, m-1$. By (4), (20) and (26), we can show that

$$S_i^{(0)} + Z_i - \sum_{k=\ell(i-1)}^{\ell(i)-1} c_{i,k} = (n-i+1)\left[T_{i:n}^{(0)} - \sum_{k=1}^{\ell(i)-1} \gamma_k^{(0)}\left(\tau_k - \tau_{k-1}\right)\right] = (n-i+1)\gamma_{\ell(i)}^{(0)}\left(T_{i:n} - \tau_{\ell(i)-1}\right) \geq 0, \quad \text{if } i \in \mathcal{I}_j^{(=,<)},$$

and

$$c_{i,j} - Z_i = (n-i+1)\left[\sum_{k=1}^{\ell(i-1)} \gamma_k^{(0)}\left(\tau_k - \tau_{k-1}\right) - T_{i-1:n}^{(0)}\right] = (n-i+1)\gamma_{\ell(i-1)}^{(0)}\left(\tau_{\ell(i-1)} - T_{i-1:n}\right) \geq 0, \quad \text{if } i \in \mathcal{I}_j^{(>,=)}.$$

Therefore, we conclude that each term in (42) is non-negative. Thus, the index sets in (41) allow us to express the function $G(\kappa)$ in (25) as

$$G(\kappa) = \sum_{j=1}^{m} w_j \kappa^{a_j}, \quad \text{where } w_j \geq 0, \text{ for } j = 1, \dots, m, \text{ and } w_j > 0, \text{ if there is a failure on level } j. \tag{43}$$

In (43) it is possible $w_j > 0$ even if there are no failures on level j, for $1 \leq j < \ell(r)$. For instance, since $a_1 = 0$ and the union $\mathcal{I}_j \cup \mathcal{I}_j^{(>,\leq)}$ is non-empty for $j = 1$, we have that $w_1 > 0$. Hence, $G(\kappa)$ is strictly positive for $\kappa \geq 0$. It is easy to check that

$$\kappa G'(\kappa) = \sum_{j=1}^{m} w_j a_j \kappa^{a_j} \xrightarrow[\kappa \to 0^+]{} 0. \tag{44}$$

Furthermore, from inequalities (27) and (40) we conclude that $b = \sum_{i=1}^{r} a_{\ell(i)} > 0$. Thus, for the LHS of (28), denoted by $H(\kappa)$, we obtain

$$H(\kappa) = \frac{\kappa G'(\kappa)}{G(\kappa)} - \frac{b}{r} \xrightarrow[\kappa \to 0^+]{} -\frac{b}{r} < 0. \tag{45}$$

From (27) and (40) it follows that $a_{\ell(1)} \leq a_{\ell(2)} \leq \cdots \leq a_{\ell(r)}$ with at least one strict inequality. Therefore, by (43), (44) and the fact that $w_j = 0$ for $j > \ell(r)$,

$$\frac{\kappa G'(\kappa)}{G(\kappa)} = \frac{\kappa G'(\kappa)/\kappa^{a_{\ell(r)}}}{G(\kappa)/\kappa^{a_{\ell(r)}}} \xrightarrow[\kappa \to \infty]{} a_{\ell(r)}, \text{ which implies } H(\kappa) \xrightarrow[\kappa \to \infty]{} a_{\ell(r)} - \frac{b}{r} > 0. \tag{46}$$

The function $H(\kappa)$ is continuous for $\kappa \geq 0$, since $G(\kappa)$ and $\kappa G'(\kappa)$ are continuous for $\kappa \geq 0$ and $G(0) > 0$. Thus, from (45) and (46) it is obvious that $H(\kappa)$ has at least one root in the interval $(0, \infty)$. Hence, to prove that Eq. (28) has an unique solution, we need additionally to show that $H(\kappa)$ is strictly increasing. Since $G(\kappa) > 0$ and

$$H'(\kappa) = \frac{\left[\kappa G'(\kappa)\right]' G(\kappa) - \kappa \left[G'(\kappa)\right]^2}{G^2(\kappa)}, \quad \text{for } \kappa \geq 0,$$

it is sufficient to prove that

$$\kappa \left[\kappa G'(\kappa)\right]' G(\kappa) - \left[\kappa G'(\kappa)\right]^2 > 0, \quad \text{for } \kappa > 0. \tag{47}$$

From $\kappa \left[\kappa G'(\kappa)\right]' = \sum_{j=1}^{m} w_j a_j^2 \kappa^{a_j}$, it follows that (47) is equivalent to

$$\left(\sum_{j=1}^{m} w_j a_j^2 \kappa^{a_j}\right)\left(\sum_{j=1}^{m} w_j \kappa^{a_j}\right) - \left(\sum_{j=1}^{m} w_j a_j \kappa^{a_j}\right)^2 > 0, \quad \text{for } \kappa > 0. \tag{48}$$

By combining the factors of $\kappa^{a_i + a_j}$ in (48) for $i = 1, \dots, m$ and $j = i, \dots, m$, we obtain that (48) is equivalent to

$$\sum_{i=1}^{m} \sum_{j=i}^{m} w_i w_j \left(a_i - a_j\right)^2 \kappa^{a_i + a_j} > 0, \quad \text{for } \kappa > 0. \tag{49}$$

However, (49) is implied by the fact that $w_j \geq 0$, for $j = 1, \dots, m$, and $w_j > 0$ for at least two indices. Thus, the proof that $\hat{\kappa}$ is the unique solution of (28) is completed.

To show that $\ell_1\left(\theta_1, \kappa\right)$ is maximized at $\theta_1 = G(\hat{\kappa})/r$ and $\kappa = \hat{\kappa}$, we write the partial derivatives of ℓ_1 in (24) as

$$\frac{\partial^2 \ell_1}{\partial \theta_1^2} = \frac{r}{\theta_1} - \frac{2G(\kappa)}{\theta_1^3}, \quad \frac{\partial^2 \ell_1}{\partial \kappa^2} = -\frac{b}{\kappa^2} - \frac{G''(\kappa)}{\theta_1} \quad \text{and} \quad \frac{\partial^2 \ell_1}{\partial \theta_1 \partial \kappa} = \frac{\partial^2 \ell_1}{\partial \kappa \partial \theta_1} = \frac{G'(\kappa)}{\theta_1^2}.$$

Then, it is easy to check that for $\theta_1 = G(\hat{\kappa})/r$ and $\kappa = \hat{\kappa}$

$$\frac{\partial^2 \ell_1}{\partial \theta_1^2}\frac{\partial^2 \ell_1}{\partial \kappa^2} - \left(\frac{\partial^2 \ell_1}{\partial \theta_1 \partial \kappa}\right)^2 = \frac{r^4}{\hat{\kappa}^2 (G(\hat{\kappa}))^4}\left[\frac{b}{r}(G(\hat{\kappa}))^2 + \hat{\kappa}^2 G''(\hat{\kappa}) G(\hat{\kappa}) - \left(\hat{\kappa} G'(\hat{\kappa})\right)^2\right]. \tag{50}$$

From (28) and the fact that $H(\hat{\kappa}) = 0$, we have that

$$\frac{b}{r} = \frac{\hat{\kappa} G'(\hat{\kappa})}{G(\hat{\kappa})}.$$

Substituting the expression above in (50) results in

$$\frac{\partial^2 \ell_1}{\partial \theta_1^2}\frac{\partial^2 \ell_1}{\partial \kappa^2} - \left(\frac{\partial^2 \ell_1}{\partial \theta_1 \partial \kappa}\right)^2 = \frac{r^4}{\hat{\kappa}^2 (G(\hat{\kappa}))^4}\left[\hat{\kappa} G'(\hat{\kappa}) G(\hat{\kappa}) + \hat{\kappa}^2 G''(\hat{\kappa}) G(\hat{\kappa}) - \left(\hat{\kappa} G'(\hat{\kappa})\right)^2\right] > 0, \tag{51}$$

where the last inequality follows by (47). The proof of the theorem is completed by combining (51) with $\partial^2 \ell_1/\partial\theta_1^2 < 0$ and $\partial^2 \ell_1/\partial\kappa^2 < 0$ for $\theta_1 = G(\hat{\kappa})/r$ and $\kappa = \hat{\kappa}$.

Notice that the order of $a_2, \dots, a_m$ in the monotonicity assumption (40) is used only for the derivation of (46). Thus, (40) can be relaxed to $0 = a_1 < a_j$, for $j = 2, \dots, m$, and $a_j \neq a_k$, for $j \neq k$. Then, the limit in (46) follows by substituting $a_{\ell(r)}$ with $a_{max} = \max_{1 \le i \le r}\left\{a_{\ell(i)}\right\}$. □

References

Arnold, B.C., Balakrishnan, N., Nagaraja, H.N., 1992. A First Course in Order Statistics. Wiley, New York.

Bagdonavicius, V., Nikulin, M., 2001. Accelerated Life Models: Modeling and Statistical Analysis. Chapman & Hall, CRC Press, Boca Raton, FL.

Balakrishnan, N., 2009. A synthesis of exact inferential results for exponential step-stress models and associated optimal accelerated life-tests. Metrika 69 (2–3), 351–396.

Balakrishnan, N., Beutner, E., Kateri, M., 2009. Order restricted inference for exponential step-stress models. IEEE Trans. Reliab. 58 (1), 132–142.

Balakrishnan, N., Kundu, D., Ng, K.T., Kannan, N., 2007. Point and interval estimation for a simple step-stress model with Type-II censoring. J. Qual. Technol. 39 (1), 35–47.

Balakrishnan, N., Xie, Q., 2007a. Exact inference for a simple step-stress model with Type-I hybrid censored data from the exponential distribution. J. Statist. Plann. Inference 137 (11), 3268–3290.

Balakrishnan, N., Xie, Q., 2007b. Exact inference for a simple step-stress model with Type-II hybrid censored data from the exponential distribution. J. Statist. Plann. Inference 137 (8), 2543–2563.

Bartholomew, D., 1963. The sampling distribution of an estimate arising in life testing. Technometrics 5 (3), 361–374.

Bedbur, S., Kamps, U., Kateri, M., 2015. Meta-analysis of general step-stress experiments under repeated Type-II censoring. Appl. Math. Model. 39 (8), 2261–2275.

Bobotas, P., Kateri, M., 2015. The step-stress tampered failure rate model under interval monitoring. Stat. Methodol. 27, 100–122.

D'Agostino, R.B., Stephens, M.A., 1986. Goodness-of-Fit Techniques. Marcel Dekker, Inc., New York.

Henze, N., Meintanis, S.G., 2005. Recent and classical tests for exponentiality: a partial review with comparisons. Metrika 61 (1), 29–45.

Hu, C.-H., Plante, R.D., Tang, J., 2013. Statistical equivalency and optimality of simple step-stress accelerated test plans for the exponential distribution. Nav. Res. Logist. 60 (1), 19–30.

Kateri, M., Kamps, U., Balakrishnan, N., 2009. A meta-analysis approach for step-stress experiments. J. Statist. Plann. Inference 139 (9), 2907–2919.

Kateri, M., Kamps, U., Balakrishnan, N., 2010. Multi-sample simple step-stress experiment under time constraints. Stat. Neerl. 64 (1), 77–96.

Kundu, D., Ganguly, A., 2017. Analysis of Step-Stress Models: Existing Results and Some Recent Developments. Academic Press, London.

Lee, J., Pan, R., 2012. A GLM approach to step-stress accelerated life testing with interval censoring. J. Statist. Plann. Inference 142 (4), 810–819.

Lu, M.-W., Leiman, J.E., Rudy, R.J., Lee, Y.-L., 2006. Step-stress accelerated test. Int. J. Mater. Prod. Technol. 25 (1–3), 231–239.

Meeker, W.Q., Escobar, L.A., 1998. Statistical Methods for Reliability Data. Wiley, New York.

Nelson, W.B., 1990. Accelerated Life Testing: Statistical Models, Test Plans and Data Analyses. Wiley, New York.

Wang, B., 2008. Goodness-of-fit test for the exponential distribution based on progressively Type-II censored sample. J. Stat. Comput. Simul. 78 (2), 125–132.

Wang, B.X., 2009. Testing for the validity of the assumptions in the exponential step-stress accelerated life-testing model. Comput. Statist. Data Anal. 53 (7), 2702–2709.

Wu, S.-J., Lin, Y.-P., Chen, S.-T., 2008. Optimal step-stress test under type I progressive group-censoring with random removals. J. Statist. Plann. Inference 138 (4), 817–826.

Xiong, C., 1998. Inferences on a simple step-stress model with Type-II censored exponential data. IEEE Trans. Reliab. 47 (2), 142–146.

Zhu, X., Balakrishnan, N., Zhou, Y., 2020. Exact likelihood-ratio tests for a simple step-stress cumulative exposure model with censored exponential data. Methodol. Comput. Appl. Probab. 22, 497–509.