

AN EULERIAN FINITE ELEMENT METHOD FOR TANGENTIAL NAVIER-STOKES EQUATIONS ON EVOLVING SURFACES

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Abstract. The paper introduces a geometrically unfitted finite element method for the numerical solution of the tangential Navier–Stokes equations posed on a passively evolving smooth closed surface embedded in $\mathbb{R}^3$. The discrete formulation employs finite difference and finite elements methods to handle evolution in time and variation in space, respectively. A complete numerical analysis of the method is presented, including stability, optimal order convergence, and quantification of the geometric errors. Results of numerical experiments are also provided.

Key words. surface Navier–Stokes system, surface PDEs, evolving surfaces, TraceFEM

AMS subject classifications. 65M12, 65M15, 65M60

1. Introduction. Lipid membranes and liquid crystal shells are examples of deformable thin structures exhibiting lateral fluidity [6, 8, 21]. Continuum based modeling of such materials leads to systems of PDEs posed on evolving surfaces embedded in $\mathbb{R}^3$. For example, the motion of an inextensible viscous fluid layer represented by a material surface is governed by the evolving surface Navier–Stokes equations derived in e.g., [17, 22, 28, 44, 50]. While the evolving surface Navier–Stokes system was discussed in the literature, to the best of our knowledge there is no existing well-posedness or numerical analysis of this problem. The recent paper [37] addressed well-posedness of a simplified problem consisting of *tangential* surface Navier–Stokes equations (TSNSE) posed on a *passively* evolving surface embedded in $\mathbb{R}^3$. A weak variational formulation of TSNSE was shown to be well-posed for any finite final time and without smallness conditions on the data. For that variational formulation we introduce a discretization method and prove its stability and optimal order convergence. This is the first study addressing numerical analysis of a fluid PDE system posed on an evolving surface.

For discretization of the TSNSE we consider a geometrically unfitted finite element method known as TraceFEM [32, 35]. The TraceFEM applies to a fully Eulerian formulation of the problem and does not require a surface triangulation, which makes it convenient for deforming surfaces. In TraceFEM one uses standard (bulk) finite element spaces to approximate unknown quantities on the surface $\Gamma(t)$ which propagates through a given triangulation of an ambient volume Ω , i.e. $\Gamma(t) \subset \Omega$ for all times t . The discrete formulation does not need a surface parametrization and uses tangential calculus in the embedding space $\mathbb{R}^3$. For scalar PDEs on evolving surfaces, a space–time and a hybrid (finite difference in time – finite elements in space) variant of TraceFEM are known in the literature [26, 36]. For the TSNSE we choose the hybrid approach since it is more flexible in terms of implementation and the choice of elements.

A variant of the hybrid FEM that we consider in this paper has recently been applied to time-dependent Stokes equations in a moving volume domain $\Omega(t) \subset \mathbb{R}^3$ in [4, 49]. In [49] Taylor–Hood finite elements are used, whereas in [4] equal order finite element spaces combined with a continuous interior penalty pressure stabilization are applied for space discretization. Both papers present a complete discretization error analysis. The resulting error bounds in [49] are suboptimal both for velocity and pressure and in [4] the bound for the pressure error is suboptimal. In both papers it is mentioned that the suboptimality of the bounds is essentially due to the lack of a uniform discrete pressure stability bound, cf [49, Remark 5.11], [4, Section 4.1]. Related to this we mention a further new contribution of this paper. We introduce, in the surface case, a new argument which leads to a discrete pressure stability bound that is uniform in the parameter range $h^2 \lesssim \Delta t$, cf. Remark 4.2. Using this we derive error estimates for velocity and pressure that are optimal in the parameter

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range $h^2 \lesssim \Delta t \lesssim h$. These parameter range restrictions are reasonable if one considers BDF1 or BDF2 time discretization methods and low order finite element pairs, e.g. Taylor-Hood P_2 - P_1 .

Concerning related work on numerical analysis and development of computational methods for fluid equations posed on surfaces we note the following. In the past few years there was an increasing interest in this research field; see, e.g., [2, 3, 7, 10, 13, 18, 23, 31, 33, 34, 37, 39, 40, 42–44, 46, 48]. In many of these papers the main topic is the development of numerical methods. The papers that focus on numerical (error) analysis mostly address the “simple” case of a homogeneous viscous surface fluid flow on a *steady* smooth surface. For the scope of this paper, the most relevant results can be found in [2, 18, 23, 33], where first stability and error analyses of finite element discretizations for the Stokes problem on a steady surface are presented. The authors of [2, 23] analyzed $H(\text{div})$ -conforming finite elements for the surface Stokes equation. The surface Stokes problem has been discretized using unfitted P_2 - P_1 elements in [33], and with higher order Taylor-Hood bulk elements [18]. We also mention the work in [14, 16, 20], where the surface FEM of Dziuk [9] and TraceFEM are analyzed for the surface vector-Laplace problem, which is closely related to the surface Stokes problem. Here we will use results from these papers for the analysis of the geometric error.

The rest of the paper is organized in four sections. In section 2 we introduce necessary notations from tangential calculus and formulate the TSNSE as a well-posed variational problem. Section 3 presents the discretization method for this problem. We first explain the idea of how the system is integrated numerically in time and proceed to a fully discrete method. Section 4 contains the main results. The error analysis presented in this section is rather long and technical but its structure is canonical. We derive continuity and stability estimates for the discrete problem. Substituting the continuous solution in the discrete variational formulation results in consistency terms for which bounds are derived. A priori discretization error bounds are derived by using an established approach, based on discrete stability, consistency error bounds and interpolation error bounds. In Remark 4.1 we give more explanation concerning the structure and the key new ingredients of the error analysis. The main result of the paper is Theorem 4.14 which yields optimal order error estimates for the velocity in an energy norm and for pressure in a special H^1 -type norm. Results of a numerical experiment presented in section 5 illustrate the optimal order convergence of the method.

2. Problem formulation. In this section we explain the tangential surface Navier-Stokes equations that we treat in this paper. Consider, for $t \in [0, T]$, a *material* surface $\Gamma(t)$ embedded in $\mathbb{R}^3$ as defined in [15, 30], with a density distribution $\rho(t, \mathbf{x})$. By $\mathbf{u}(t, \mathbf{x})$, $\mathbf{x} \in \Gamma(t)$, we denote the velocity field of the density flow on $\Gamma(t)$, i.e. $\mathbf{u}(t, \mathbf{x})$ is the velocity of a material point $\mathbf{x} \in \Gamma(t)$. We further assume that the geometric evolution of $\Gamma(t)$ is determined by a given smooth velocity field $\mathbf{w} = \mathbf{w}(t, \mathbf{x})$, which passively advects the initial surface $\Gamma_0 := \Gamma(0)$:

$$\Gamma(t) = \{\mathbf{y} \in \mathbb{R}^3 \mid \mathbf{y} = \mathbf{x}(t, \mathbf{z}), \mathbf{z} \in \Gamma_0\}, \quad (2.1)$$

with the trajectories $\mathbf{x}(t, \mathbf{z})$ being the unique solutions of the Cauchy problem

$$\frac{d}{dt}\mathbf{x}(t, \mathbf{z}) = \mathbf{w}(t, \mathbf{x}(t, \mathbf{z})), \quad \mathbf{x}(0, \mathbf{z}) = \mathbf{z} \quad (2.2)$$

for all $\mathbf{z} \in \Gamma_0$. This induces the smooth space-time manifold

$$\mathcal{G} = \bigcup_{t \in [0, T]} \{t\} \times \Gamma(t) \subset \mathbb{R}^4.$$

We need a few notations of geometric quantities and tangential differential operators. For a given $t \in [0, T]$ we write $\Gamma = \Gamma(t)$. The outward pointing normal vector on Γ is denoted by $\mathbf{n}$. The normal projector on the tangential space at $\mathbf{x} \in \Gamma$ is given by $\mathbf{P} = \mathbf{P}(\mathbf{x}) = \mathbf{I} - \mathbf{n}\mathbf{n}^T$. For a scalar function $p : \Gamma \rightarrow \mathbb{R}$ or a vector field $\mathbf{u} : \Gamma \rightarrow \mathbb{R}^3$ their smooth extensions to a neighborhood $\mathcal{O}(\Gamma)$ of Γ are denoted by p^e and $\mathbf{u}^e$, respectively. The surface gradient, covariant derivative and surface divergence on Γ can be defined through derivatives in $\mathbb{R}^3$ as $\nabla_\Gamma p = \mathbf{P}\nabla p^e$, $\nabla_\Gamma \mathbf{u} = \mathbf{P}\nabla \mathbf{u}^e \mathbf{P}$, and $\text{div}_\Gamma \mathbf{u} = \text{tr}(\nabla_\Gamma \mathbf{u})$. These definitions are independent of a particular smooth extension of p and $\mathbf{u}$ off Γ . The surface rate-of-strain tensor [15] is given by $E_s(\mathbf{u}) = \frac{1}{2}(\nabla_\Gamma \mathbf{u} + \nabla_\Gamma \mathbf{u}^T)$. By $\mathbf{H} = \nabla_\Gamma \mathbf{n} \in \mathbb{R}^{3 \times 3}$

we denote the Weingarten mapping and by $\kappa := \text{tr}(\mathbf{H})$ twice the mean curvature. For velocity fields on $\Gamma(t)$ we use a splitting into tangential and normal components

$$\mathbf{u} = \mathbf{u}_T + \mathbf{u}_N = \mathbf{u}_T + u_N \mathbf{n}, \quad \text{with } u_N = \mathbf{u} \cdot \mathbf{n}.$$

In our setting, the normal component of $\mathbf{u}$ is completely determined by the ambient flow $\mathbf{w}$, i.e. $\mathbf{u}_N = \mathbf{w}_N$ on Γ . For the normal component of the given smooth velocity field we have $\mathbf{w}_N = w_N \mathbf{n}$, with $w_N = \mathbf{w} \cdot \mathbf{n}$ the given smooth normal velocity of the surface Γ . Besides derivatives on Γ we also need the material derivative along $\mathcal{G}$, denoted by $\dot{g}$, which is the derivative of a surface quantity g along the trajectories $\mathbf{x}(t, \mathbf{z})$ of material points. It can be written as $\dot{g} = \frac{\partial g^e}{\partial t} + (\mathbf{u} \cdot \nabla)g^e$ on $\mathcal{G}$. We also introduce the so-called normal time derivative of g , denoted by $\partial^\circ g$, which describes the variation of g along normal trajectories of points on $\Gamma(t)$:

$$\partial^\circ g := \dot{g} - \mathbf{u} \cdot \nabla_\Gamma g = \frac{\partial g^e}{\partial t} + (\mathbf{w}_N \cdot \nabla)g^e \quad \text{on } \mathcal{G}. \quad (2.3)$$

For a vector valued quantity we use this definition componentwise. Conservation of momentum and an inextensibility condition lead to the following system governing the free lateral motion of the material viscous surface [17, 37]: For a given density distribution $\rho > 0$, viscosity coefficient $\mu > 0$, find the tangential velocity field $\mathbf{u}_T$ and surface pressure p satisfying the initial condition $\mathbf{u}_T(0) = \mathbf{u}_0$ and the system of equations, which we call the tangential surface Navier-Stokes equations (TSNSE):

$$\begin{cases} \rho(\mathbf{P} \partial^\circ \mathbf{u}_T + w_N \mathbf{H} \mathbf{u}_T + (\nabla_\Gamma \mathbf{u}_T) \mathbf{u}_T) - 2\mu \mathbf{P} \text{div}_\Gamma E_s(\mathbf{u}_T) + \nabla_\Gamma p = \mathbf{f} \\ \text{div}_\Gamma \mathbf{u}_T = f \end{cases} \quad \text{on } \Gamma(t), \quad (2.4)$$

with right-hand sides known in terms of geometric quantities, w_N and the tangential component of the external area force $\mathbf{b}$:

$$f = -w_N \kappa, \quad \mathbf{f} = \mathbf{b}_T + 2\mu \mathbf{P} \text{div}_\Gamma(w_N \mathbf{H}) + \frac{\rho}{2} \nabla_\Gamma w_N^2. \quad (2.5)$$

The system can be seen as an idealized model for the motion of a thin fluid layer embedded in bulk fluid, where one neglects friction forces between the surface and bulk fluids as well as any effect of the layer on the bulk flow (more precisely, one may assume that a lateral component of the normal bulk stress is given by $\mathbf{b}_T$). System (2.4)–(2.5) also appears as an auxiliary problem if one applies directional splitting to the full system of equations governing the evolution of a material inextensible fluidic surface; see [17]. We further set $\rho = 1$.

In this paper, we represent $\Gamma(t)$ as the zero level set of a smooth level-set function $\phi(t, \mathbf{x})$,

$$\Gamma(t) = \{\mathbf{x} \in \mathbb{R}^3 : \phi(t, \mathbf{x}) = 0\},$$

such that $|\nabla \phi| \geq c > 0$ in $\mathcal{O}(\mathcal{G})$, a neighborhood of $\mathcal{G}$. To simplify the presentation and analysis, we make the assumption that the *level set function has the signed distance property*. This assumption, however, is not essential. In Remark 3.2 we comment on the generalization of the method to the case where ϕ is not necessarily a signed distance function.

3. Discretization method. In this section we present a fully Eulerian finite element method for the TSNSE (2.4). The method is based on the same ideas as used in [26], namely a combination of an implicit time stepping scheme with a TraceFEM in space. We start with the discretization of the system's evolution in time.

3.1. Time-stepping scheme. Consider uniformly distributed time nodes $t_n = n\Delta t$, $n = 0, \dots, N$, with the time step $\Delta t = T/N$ and $I_n := [t_{n-1}, t_n]$, $1 \leq n \leq N$. We assume that the time step Δt is sufficiently small such that

$$\Gamma(t_n) \subset \mathcal{O}(\Gamma(t_{n-1})), \quad n = 1, \dots, N, \quad (3.1)$$

with $\mathcal{O}(\Gamma(t))$ a neighborhood of $\Gamma(t)$ where a smooth extension of surface quantities from $\Gamma(t)$ is well defined. Among smooth extensions of a (scalar or vector valued) function g we now choose the

constant extension in normal direction g^e , i.e., $\nabla d \cdot \nabla g^e = 0$ in $O(\Gamma(t))$, with $d = d(t, \cdot)$ the signed distance function to $\Gamma(t)$. For the normal extension the last term in (2.3) vanishes and thus we have

$$\partial^\circ g = \frac{\partial g^e}{\partial t} \quad \text{on } \mathcal{G}. \quad (3.2)$$

Based on this we introduce on $\Gamma(t_n)$ the normal time derivative approximation

$$\mathbf{P} \partial^\circ \mathbf{u}_T = \mathbf{P} \frac{\partial \mathbf{u}_T^e}{\partial t} \approx \frac{\mathbf{u}_T(t_n) - \mathbf{P}(t_n) \mathbf{u}_T(t_{n-1})^e}{\Delta t}. \quad (3.3)$$

Due to (3.1) $\mathbf{u}_T(t_{n-1})^e$ is defined on $\Gamma(t_n)$. For the normal $\mathbf{n}$ on $\Gamma(t_j)$ its constant extension in $\mathcal{O}(\Gamma(t_j))$ is also denoted by $\mathbf{n}$, i.e., $\mathbf{n} = \nabla d$. We further use the notation $\mathbf{u}_T^j$ and p^j for an approximation of $\mathbf{u}_T(t_j)^e$ and $p(t_j)$, respectively. Based on (3.3) and (3.1) we consider the following time discretization method for (2.4). Given $\mathbf{u}_T^0 = \mathbf{u}_T(0)^e$ in $\mathcal{O}(\Gamma_0)$, for $n = 1, \dots, N$, find $\mathbf{u}_T^n$, defined in $\mathcal{O}(\Gamma(t_n))$ and tangential to $\Gamma(t_n)$, i.e. $(\mathbf{u}_T^n \cdot \mathbf{n})|_{\Gamma(t_n)} = 0$, and p^n defined on $\Gamma(t_n)$ such that

$$\begin{cases} \frac{\mathbf{u}_T^n - \mathbf{P} \mathbf{u}_T^{n-1}}{\Delta t} + w_N^n \mathbf{H} \mathbf{u}_T^n + (\nabla_\Gamma \mathbf{u}_T^n) \mathbf{u}_T^{n-1} - 2\mu \mathbf{P} \operatorname{div}_\Gamma E_s(\mathbf{u}_T^n) + \nabla_\Gamma p^n = \mathbf{f}^n & \text{on } \Gamma(t_n), \\ \operatorname{div}_\Gamma \mathbf{u}_T^n = f^n \end{cases} \quad (3.4)$$

$$\mathbf{n} \cdot \nabla \mathbf{u}_T^n = 0 \quad \text{in } \mathcal{O}(\Gamma(t_n)), \quad (3.5)$$

with $w_N^n := w_N(t_n)$, $\mathbf{f}^n := \mathbf{f}(t_n)$, $f^n := f(t_n)$. Note that in the inertia term we also use $\mathbf{P} \mathbf{u}_T^{n-1}$, since $(\nabla_\Gamma \mathbf{u}_T^n) \mathbf{u}_T^{n-1} = (\nabla_\Gamma \mathbf{u}_T^n) \mathbf{P} \mathbf{u}_T^{n-1}$ holds. Geometric information in (3.4) is taken for $\Gamma(t_n)$, i.e. $\mathbf{n} = \mathbf{n}(t_n)$, $\mathbf{P} = \mathbf{P}(t_n)$, $\mathbf{H} = \mathbf{H}(t_n)$. In (3.4) we use a BDF1 (i.e., Euler implicit) type time discretization, which has local truncation error $\mathcal{O}(\Delta t^2)$. Related to that we use a simple, first order in Δt accurate, linearization of the quadratic nonlinearity, i.e., $(\nabla_\Gamma \mathbf{u}_T^n) \mathbf{u}_T^n$ is replaced by $(\nabla_\Gamma \mathbf{u}_T^n) \mathbf{u}_T^{n-1}$. This approach has a straightforward extension to higher order in Δt schemes, cf. the discussion in Remark 3.2 below. The space discretization of (3.4)–(3.5) is presented in the next section.

3.2. Space discretization method. Consider a fixed polygonal domain $\Omega \subset \mathbb{R}^3$ that strictly contains $\Gamma(t)$ for all $t \in [0, T]$. Let $\{\mathcal{T}_h\}_{h>0}$ be a family of shape-regular consistent triangulations of Ω , with $\max_{K \in \mathcal{T}_h} \operatorname{diam}(K) \leq h$. Corresponding to the bulk triangulation we define a standard finite element space of piecewise polynomial continuous functions of a fixed degree $k \geq 1$:

$$V_{h,k} = \{v_h \in C(\Omega) : v_h \in P_k(K), \quad \forall K \in \mathcal{T}_h\}. \quad (3.6)$$

The bulk velocity and pressure finite element spaces are standard Taylor–Hood spaces:

$$\mathbf{U}_h = (V_{h,m+1})^3, \quad Q_h = V_{h,m}, \quad \text{with } m \geq 1. \quad (3.7)$$

We want to avoid the assumption that all $\Gamma(t_n)$ are given in an explicit parametric form or that the exact level set functions $\phi(t_n, \cdot)$ are known. If, for example, (2.4) is part of a system, where tangential surface motions are coupled to the normal ones, then finding $\Gamma(t_n)$ is part of the problem, and knowledge of only a (finite element) approximation to $\phi(t_n, \cdot)$ would be a more realistic assumption. In turn, this results in approximation of all geometric quantities involved in (2.4). This issue of geometry approximation is explained in section 3.2.1 below.

If not specified otherwise, all constants C , c , c_0 , c_1 , etc. appearing later in the text are generic positive constants which are *independent* of h , Δt , other discretization parameters, time instance t_n , and the position of Γ in the background mesh, but *may depend* on w_N , $\mathcal{G}$, $\mathbf{u}$, and the shape regularity of $\mathcal{T}_h$. In order to reduce the repeated use of such constants, we often write $x \lesssim y$ to state that the inequality $x \leq cy$ holds for quantities x, y with such generic constant c . Similarly for $x \gtrsim y$, and $x \simeq y$ will mean that both $x \lesssim y$ and $x \gtrsim y$ hold.

3.2.1. Geometry approximation. For any fixed $t \in [0, T]$, $\phi_h(\cdot) = \phi_h(t, \cdot)$ is a given continuous piecewise polynomial approximation (with respect to $\mathcal{T}_h$) of $\phi(\cdot) = \phi(t, \cdot)$, which satisfies

$$\|\phi - \phi_h\|_{L^\infty(\Omega)} + h \|\nabla(\phi - \phi_h)\|_{L^\infty(\Omega)} \lesssim h^{q+1}, \quad (3.8)$$

with some $q \geq 1$. For this estimate to hold, we assume that the level set function has the smoothness property $\phi(t, \cdot) \in C^{q+1}(\Omega)$. Moreover, we assume that $|\nabla \phi_h| \geq C > 0$ in $\mathcal{O}(\Gamma(t))$, $t \in [0, T]$, and that ϕ_h is sufficiently regular in time such that with $\phi_h^n(\mathbf{x}) = \phi_h(t_n, \mathbf{x})$, $n = 0, \dots, N$, there holds

$$\|\phi_h^{n-1} - \phi_h^n\|_{L^\infty(\Omega)} \lesssim \Delta t \|w_N\|_{L^\infty(I_n \times \Omega)}, \quad (3.9a)$$

$$\|\nabla \phi_h^{n-1} - \nabla \phi_h^n\|_{L^\infty(\Omega)} \lesssim \Delta t (\|w_N\|_{L^\infty(I_n \times \Omega)} + \|\nabla w_N\|_{L^\infty(I_n \times \Omega)}), \text{ for } n = 1, \dots, N. \quad (3.9b)$$

We define the discrete surfaces $\Gamma_h \approx \Gamma$ as the zero level of ϕ_h :

$$\Gamma_h(t) := \{\mathbf{x} \in \mathbb{R}^3 : \phi_h(t, \mathbf{x}) = 0\}. \quad (3.10)$$

From the property (3.8) it follows that the Lipschitz surface Γ_h is an approximation to Γ with

$$\text{dist}(\Gamma_h^n, \Gamma) = \max_{x \in \Gamma_h^n} |\phi(\mathbf{x})| = \max_{x \in \Gamma_h^n} |\phi(\mathbf{x}) - \phi_h(\mathbf{x})| \leq \|\phi - \phi_h\|_{\infty, \Omega} \lesssim h^{q+1}. \quad (3.11)$$

Furthermore, it also follows that the vector field $\mathbf{n}_h = \nabla \phi_h / |\nabla \phi_h|$ ($\mathbf{n}_h$ is the normal to Γ_h), and the extended normal vector to Γ satisfy

$$|\mathbf{n}_h(\mathbf{x}) - \mathbf{n}(\mathbf{x})| \leq c |\nabla \phi_h(\mathbf{x}) - \nabla \phi(\mathbf{x})| \lesssim h^q, \text{ in } \mathcal{O}(\Gamma(t)). \quad (3.12)$$

On Γ_h we have the tangential projection operator $\mathbf{P}_h = \mathbf{I} - \mathbf{n}_h \mathbf{n}_h^T$. Besides the geometric quantity $\mathbf{n}_h$ we also need a discrete approximation of the Weingarten mapping $\mathbf{H}_h \approx \mathbf{H}$. We assume that this approximation is of the form

$$\mathbf{H}_h = \nabla_{\Gamma_h} \bar{\mathbf{n}}_h, \text{ with } \bar{\mathbf{n}}_h \in H^1(\Gamma_h)^3 \text{ that satisfies } |\bar{\mathbf{n}}_h(\mathbf{x}) - \mathbf{n}(\mathbf{x})| \lesssim h^q, \text{ } \mathbf{x} \in \Gamma_h. \quad (3.13)$$

Here ∇_{Γ_h} is defined (a.e. on Γ_h) analogous to ∇_Γ . The normal approximation $\bar{\mathbf{n}}_h$ may be chosen as a suitable interpolation operator in a finite element space applied to $\mathbf{n}_h$, e.g., $\bar{\mathbf{n}}_h = I_h \mathbf{n}_h$, with I_h the (componentwise) Oswald averaging operator that maps into the finite element space $(V_{h,q-1})^3$.

We emphasize that all bounds in (3.8)–(3.13) are uniform in t (as well as in h , Δt , n and position of Γ or Γ_h in the mesh).

REMARK 3.1. In the method that we present below integrals over Γ_h occur. We *assume that these integrals can be computed accurately*. In practice, this is straightforward for piecewise linear $\phi_h(\cdot, t)$. The higher order case $q > 1$ is more involved and requires special approaches for the construction of quadrature rules or the use of an parametric FEM technique [11, 12, 24, 29, 38, 45, 47].

3.2.2. Fully discrete method. For computational efficiency reasons, we use an extension not in the given (h and Δt -independent) neighborhood $\mathcal{O}(\Gamma(t_n))$ of $\Gamma(t_n)$ but in a smaller (Δt -dependent) *narrow band* around $\Gamma_h^n = \Gamma_h(t_n)$. This narrow band consists of all tetrahedra within a δ_n distance from the surface, with $\delta_n \simeq \Delta t$. More precisely, we define the mesh-dependent narrow bands

$$\begin{aligned} U_\delta(\Gamma_h^n) &:= \{\mathbf{x} \in \Omega : \text{dist}(\mathbf{x}, \Gamma_h^n) \leq \delta_n\}, \\ \mathcal{O}_\delta(\Gamma_h^n) &:= \bigcup \{\bar{K} : K \in \mathcal{T}_h \text{ and } \text{dist}(\mathbf{x}, \Gamma_h^n) \leq \delta_n \text{ for some } \mathbf{x} \in K\} \supset U_\delta(\Gamma_h^n). \end{aligned} \quad (3.14)$$

We also need a subdomain of $\mathcal{O}_\delta(\Gamma_h^n)$ consisting of tetrahedra intersected by Γ_h^n ,

$$\omega_\Gamma^n := \bigcup \{\bar{K} \in \mathcal{T}_h : K \cap \Gamma_h^n \neq \emptyset\}.$$

Note that the subdomains $\mathcal{O}_\delta(\Gamma_h^n)$ and ω_Γ^n consist of unions of tetrahedra $K \in \mathcal{T}_h$. The finite element spaces for velocity and pressure are *restrictions to these narrow bands* $\mathcal{O}_\delta(\Gamma_h^n)$ and ω_Γ^n of the time-independent bulk spaces $\mathbf{U}_h$ and Q_h :

$$\mathbf{U}_h^n := \{\mathbf{v}|_{\mathcal{O}_\delta(\Gamma_h^n)} \mid \mathbf{v} \in \mathbf{U}_h\}, \quad Q_h^n := \{q|_{\omega_\Gamma^n} \mid q \in Q_h\}. \quad (3.15)$$

We also use the notation $V_{h,m}^n := \{v|_{\mathcal{O}_\delta(\Gamma_h^n)} \mid v \in V_{h,m}\}$. In the derivation of a finite element formulation based on the discrete-in-time system (3.4)–(3.5), we need to address three important aspects: tangentiality of $\mathbf{u}_T^n$, extension of the velocity along normal directions as in (3.5) and a

handling of the inertia term. First, we relax the condition for the solution to be tangential to $\Gamma(t_n)$ to allow for $\mathbf{U}_h^n$ as trial and test velocity space. The tangentiality condition is weakly enforced using a penalty approach, which is often used in finite element methods for vector-valued surface PDEs [16, 19, 34, 39]. The constraint in (3.5) is also relaxed. For this we use a penalty (or stabilization) approach that is standard in trace finite element method and based on adding a volume normal derivative term to the discrete bilinear form. The treatment of inertia also follows an established approach. For this we rewrite the corresponding trilinear form, where we use integrals over the exact surface $\Gamma = \Gamma(t_n)$ and $\mathbf{u}_T \cdot \nabla_\Gamma \mathbf{v}_T = (\nabla_\Gamma \mathbf{v}_T) \mathbf{u}_T$:

$$\int_\Gamma (\mathbf{u}_T \cdot \nabla_\Gamma \mathbf{u}_T) \cdot \mathbf{v}_T ds = \frac{1}{2} \int_\Gamma (\mathbf{u}_T \cdot \nabla_\Gamma \mathbf{u}_T) \cdot \mathbf{v}_T - (\mathbf{u}_T \cdot \nabla_\Gamma \mathbf{v}_T) \cdot \mathbf{u}_T ds - \frac{1}{2} \int_\Gamma \operatorname{div}_\Gamma \mathbf{u}_T (\mathbf{u}_T \cdot \mathbf{v}_T) ds.$$

Using $\operatorname{div}_\Gamma \mathbf{u}_T = f$ on Γ , the second term on the right-hand side becomes linear. The first term is linearized using the $\mathbf{u}_T$ approximation from the previous time step leading to an antisymmetric bilinear form, which is convenient for the stability analysis of the discretization.

Putting these components together leads to the following *fully discrete problem* (one time step), where we use (sufficiently accurate) extensions $w_N^e, f^e, \mathbf{f}^e$ of the data $w_N, f, \mathbf{f}$: For given $\mathbf{u}_h^{n-1} \in \mathbf{U}_h^{n-1}$ find $\mathbf{u}_h^n \in \mathbf{U}_h^n, p_h^n \in Q_h^n$, satisfying

$$\begin{aligned} & \int_{\Gamma_h^n} \left(\frac{\mathbf{u}_h^n - \mathbf{u}_h^{n-1}}{\Delta t} + w_N^{e,n} \mathbf{H}_h \mathbf{u}_h^n \right) \cdot \mathbf{P}_h \mathbf{v}_h ds_h \\ & + \frac{1}{2} \int_{\Gamma_h^n} (\mathbf{u}_h^{n-1} \cdot \nabla_{\Gamma_h} \mathbf{P}_h \mathbf{u}_h^n) \cdot \mathbf{v}_h - (\mathbf{u}_h^{n-1} \cdot \nabla_{\Gamma_h} \mathbf{P}_h \mathbf{v}_h) \cdot \mathbf{u}_h^n ds_h - \frac{1}{2} \int_{\Gamma_h^n} f^{e,n} \mathbf{u}_h^n \cdot \mathbf{P}_h \mathbf{v}_h ds_h \\ & + 2\mu \int_{\Gamma_h^n} E_{s,h}(\mathbf{P}_h \mathbf{u}_h^n) : E_{s,h}(\mathbf{P}_h \mathbf{v}_h) ds_h + \underbrace{\tau \int_{\Gamma_h^n} (\tilde{\mathbf{n}}_h \cdot \mathbf{u}_h^n)(\tilde{\mathbf{n}}_h \cdot \mathbf{v}_h) ds_h}_{\text{penalty for } \mathbf{n} \cdot \mathbf{u}^n = 0} \end{aligned} \quad (3.16)$$

$$\begin{aligned} & + \int_{\Gamma_h^n} \nabla_{\Gamma_h} p_h^n \cdot \mathbf{v}_h ds_h + \underbrace{\rho_u \int_{\mathcal{O}_\delta(\Gamma_h^n)} (\mathbf{n}_h \cdot \nabla \mathbf{u}_h^n)(\mathbf{n}_h \cdot \nabla \mathbf{v}_h) d\mathbf{x}}_{\text{velocity stabilization and extension}} = \int_{\Gamma_h^n} \mathbf{f}^{e,n} \cdot \mathbf{v}_h ds_h \quad \forall \mathbf{v}_h \in \mathbf{U}_h^n, \\ & - \int_{\Gamma_h^n} \nabla_{\Gamma_h} q_h \cdot \mathbf{u}_h^n ds_h + \underbrace{\rho_p \int_{\omega_\Gamma^n} (\mathbf{n}_h \cdot \nabla p_h^n)(\mathbf{n}_h \cdot \nabla q_h) d\mathbf{x}}_{\text{pressure stabilization}} = \int_{\Gamma_h^n} f^{e,n} q_h ds_h \quad \forall q_h \in Q_h^n. \end{aligned} \quad (3.17)$$

Here we mimic the notation used in the PDE system on Γ . For example, $\nabla_{\Gamma_h} \mathbf{v}_h = \mathbf{P}_h \nabla \mathbf{v}_h \mathbf{P}_h$, $E_{s,h}(\mathbf{w}_h) = \frac{1}{2}(\nabla_{\Gamma_h} \mathbf{w}_h + \nabla_{\Gamma_h}^T \mathbf{w}_h)$. The additional term $\tau \int_{\Gamma_h^n} (\tilde{\mathbf{n}}_h \cdot \mathbf{u}_h^n)(\tilde{\mathbf{n}}_h \cdot \mathbf{v}_h) ds$ with penalty parameter $\tau > 0$ is included to weakly enforce the tangentiality condition $\tilde{\mathbf{n}}_h \cdot \mathbf{u}_h^n = 0$ on Γ_h^n . In this term we use an “improved normal”, denoted by $\tilde{\mathbf{n}}_h$, which has one order better accuracy (as approximation of $\mathbf{n}$) than the discrete surface normal $\mathbf{n}_h$. We assume (compare to (3.12)):

$$|\tilde{\mathbf{n}}_h(\mathbf{x}) - \mathbf{n}(t_n, \mathbf{x})| \lesssim h^{q+1}, \quad \mathbf{x} \in \Gamma_h^n. \quad (3.18)$$

From the literature it is known that such a more accurate normal in this penalty term is needed for optimal order discretization errors, cf. [16, 20]. In section 5 we explain how $\tilde{\mathbf{n}}_h$ is determined in our implementation of the method. The volumetric term $\int_{\mathcal{O}_\delta(\Gamma_h^n)} (\mathbf{n}_h \cdot \nabla \mathbf{u}_h^n)(\mathbf{n}_h \cdot \nabla \mathbf{v}_h) d\mathbf{x}$, scaled by the parameter ρ_u , plays a twofold role. Firstly, due to this term instabilities caused by “small cuts” are damped, resulting in satisfactory conditioning of the stiffness matrix for velocity, cf. e.g. [5]. Secondly, this term weakly enforces the extension condition (3.5) with $\mathcal{O}(\Gamma(t_n))$ replaced by $\mathcal{O}_\delta(\Gamma_h^n)$. The volumetric term $\rho_p \int_{\omega_\Gamma^n} (\mathbf{n}_h \cdot \nabla p_h^n)(\mathbf{n}_h \cdot \nabla q_h) d\mathbf{x}$ is added for the purpose of numerical stabilization of pressure, both with respect to finite element (LBB) stability and conditioning of the resulting matrix, cf. [33]. Due to these stabilizations the algebraic system (in each time step) is well-posed and has conditioning properties comparable to those of a discretized linearized Navier-Stokes system in Euclidean domains.

The formulation (3.16)–(3.17) is *consistent* up to geometric errors: If Γ_h^n is replaced by $\Gamma^n = \Gamma(t_n)$ and all geometric quantities by the corresponding exact ones (e.g., $\mathbf{H}_h$ by $\mathbf{H}$), then the equations in (3.16)–(3.17) are satisfied with $\mathbf{u}_h^n$, p_h^n , $\mathbf{u}_h^{n-1}$ replaced by the solution $\mathbf{u}^n$, p^n and $\mathbf{u}^{n-1}$ of (3.4), extended along normal directions.

3.3. Discussion of the method. In addition to the mesh size parameter h and time step Δt , the finite element method involves several other discretization parameters summarized below.

- ρ_u : a normal stabilization and extension parameter for velocity;
- ρ_p : a normal stabilization parameter for pressure;
- τ : a penalty parameter for $\mathbf{n}_h \cdot \mathbf{u}_h = 0$ constraint;
- δ_n : a narrow band parameter at time t_n .

We assume that these parameters satisfy the conditions

$$h \leq c_1, \quad \Delta t \leq c_2, \quad (3.19a)$$

$$c_\rho h^{-1} \leq \rho_u \lesssim h^{-1}, \quad (3.19b)$$

$$\rho_p \simeq h, \quad (3.19c)$$

$$\tau \simeq h^{-2}, \quad (3.19d)$$

$$\Delta t \geq 2\tau^{-1}, \quad (3.19e)$$

$$c_\delta \Delta t \leq \delta_n \lesssim h, \quad (3.19f)$$

where c_1, c_2 are sufficiently small $O(1)$ and c_δ, c_ρ are sufficiently large $O(1)$ constants independent of the parameters, time, and position of Γ_h in the mesh. The parameter conditions (3.19b) and (3.19c) are known from the literature on trace finite element methods. The condition “ c_ρ sufficiently large” is needed to obtain sufficient control of a velocity extension in the narrow band (it is used to prove a key estimate (4.10c)). Parameter τ as in (3.19d) guarantees accurate enough fulfillment of the tangentiality condition for the discrete velocity. Taking τ even larger may lead to a ‘locking’ phenomenon. Eq. (3.19e) is a technical condition that we need to prove the stability estimate in Theorem 4.7. The upper bound in (3.19f) on the narrow band parameter δ_n keeps the complexity of the method optimal, i.e. the number of active degrees of freedom is $O(h^{-2})$ on each time step. Concerning the lower bound on δ_n we note the following. In a time step from t_{n-1} to t_n , the surface $\Gamma(t)$ can move at most $\Delta t \sup_{t \in I_n} \|w_N\|_{L^\infty(\Gamma(t))}$ distance in normal direction. Thanks to assumption (3.9a) the maximum distance from Γ_h^n to Γ_h^{n-1} is also proportional to $\Delta t \sup_{t \in I_n} \|w_N\|_{L^\infty(\Gamma(t))}$. Therefore, c_δ in (3.19f) can be taken such that

$$\omega_\Gamma^n \subset \mathcal{O}_\delta(\Gamma_h^{n-1}). \quad (3.20)$$

This condition is the discrete analog of (3.1) and is essential for the well-posedness of the finite element problem. The assumptions (3.19d)–(3.19f) can be satisfied if we take Δt and h such that the scaling conditions

$$h^2 \lesssim \Delta t \lesssim h \quad (3.21)$$

hold. These scalings are reasonable. Consider Taylor–Hood elements with $m = 1$. For optimal $O(h^2)$ -convergence in the energy norm of (3.16) one needs $\Delta t \simeq h^2$. For the more time-accurate BDF2 scheme (this is our practical choice, cf. Remark 3.3) $\Delta t \simeq h$ leads to $O(h^2)$ -convergence in the energy norm, and $\Delta t \simeq h^{3/2}$ is the choice consistent with the best possible $O(h^3)$ -order in the velocity L^2 -norm. In the remarks below we briefly address a few further aspects of the method.

REMARK 3.2. In practice we typically do not use a level set function that has the signed distance property. Then the extension does not satisfy (3.5), but is approximately constant along the normals to the level lines and instead of $\mathbf{P} \partial^\circ \mathbf{u}_T = \mathbf{P} \frac{\partial \mathbf{u}_T}{\partial t}$, cf. (3.2), one uses the general relation

$$\mathbf{P} \partial^\circ \mathbf{u}_T = \mathbf{P} \left(\frac{\partial \mathbf{u}_T^e}{\partial t} + (\nabla \mathbf{u}_T^e) \mathbf{w}_N \right),$$

as basis for the discretization. Hence, for the case of a general level set function we include in (3.16) the term $\int_{\Gamma_h^n} (\mathbf{w}_N \cdot \nabla \mathbf{P}_h \mathbf{u}_h^n) \cdot \mathbf{P}_h \mathbf{v}_h \, ds_h$. Apart from this, the discretization method stays the same.

REMARK 3.3. The BDF2 variant is very similar to the method introduced above. The time difference $\mathbf{u}_h^n - \mathbf{u}_h^{n-1}$ in the first line in (3.16) is then replaced by $\frac{3}{2}\mathbf{u}_h^n - 2\mathbf{u}_h^{n-1} + \frac{1}{2}\mathbf{u}_h^{n-2}$ and in the linearization of the inertia term we use $2\mathbf{u}_h^{n-1} - \mathbf{u}_h^{n-2}$ instead of $\mathbf{u}_h^{n-1}$. All other terms in (3.16)–(3.17) remain the same. For BDF2, in addition to (3.20) one has to guarantee $\omega_\Gamma^n \subset \mathcal{O}_\delta(\Gamma_h^{n-2})$, which means that the narrow bands $\mathcal{O}_\delta(\Gamma_h^n)$ have to be taken thicker than in the BDF1 method.

REMARK 3.4. In the surface gradient operators ∇_{Γ_h} in the inertia term and the surface rate-of-strain tensor $E_{s,h}(\cdot)$ we use projected vector fields $\mathbf{P}_h \mathbf{v}$. Differentiating projector $\mathbf{P}_h$ (which in general is discontinuous across element faces) we loose H^1 conformity, cf. [20]. Based on the formula $\nabla_\Gamma \mathbf{v}_T = \nabla_\Gamma \mathbf{v} - v_N \mathbf{H}$, in order to avoid differentiation of $\mathbf{P}_h$, in the implementation of the method we replace $\nabla_{\Gamma_h} \mathbf{P}_h \mathbf{v}_h$ by $\nabla_{\Gamma_h} \mathbf{v}_h - (\mathbf{n}_h \cdot \mathbf{v}_h) \mathbf{H}_h$.

REMARK 3.5. For convenience, in the error analysis below we assume that the data extensions $w_N^e, f^e, \mathbf{f}^e$ are constant extensions along the normals $\mathbf{n}$. In practice one typically uses other (sufficiently accurate) extensions.

4. Error analysis. We continue with an error analysis of the finite element scheme (3.16)–(3.17). We assume the solution of (2.4) is sufficiently smooth, at least $\mathbf{u} \in W^{m+2,\infty}(\mathcal{G})^3$, $p \in W^{m+1,\infty}(\mathcal{G})$. In the remainder we assume that the parameter conditions (3.19a)–(3.19f) are satisfied. As common in analysis of incompressible fluid problems, we assume a homogeneous divergence condition in (2.4):

$$f = 0.$$

See the Remark 4.3. We use the notations $(\cdot, \cdot)_S$ and $\|\cdot\|_S$ to denote the L^2 -scalar product and L^2 -norm over a surface or volumetric domain S . To represent the discrete problem in a compact form, we introduce

$$\begin{aligned} a_n(\mathbf{z}; \mathbf{u}, \mathbf{v}) &:= \widehat{a}_n(\mathbf{u}, \mathbf{v}) + c_n(\mathbf{z}; \mathbf{u}, \mathbf{v}), \quad \text{with} \\ \widehat{a}_n(\mathbf{u}, \mathbf{v}) &:= \int_{\Gamma_h^n} w_N^{e,n} \mathbf{v} \cdot \mathbf{H}_h \mathbf{u} \, ds_h + 2\mu \int_{\Gamma_h^n} (E_{s,h}(\mathbf{P}_h \mathbf{u}) : (E_{s,h}(\mathbf{P}_h \mathbf{v}))) \, ds_h \\ &\quad + \tau \int_{\Gamma_h^n} (\widetilde{\mathbf{n}}_h \cdot \mathbf{u})(\widetilde{\mathbf{n}}_h \cdot \mathbf{v}) \, ds_h + \rho_u \int_{\mathcal{O}_\delta(\Gamma_h^n)} (\mathbf{n}_h \cdot \nabla \mathbf{u})(\mathbf{n}_h \cdot \nabla \mathbf{v}) \, d\mathbf{x} \\ c_n(\mathbf{z}; \mathbf{u}, \mathbf{v}) &:= \frac{1}{2} \int_{\Gamma_h^n} (\mathbf{z} \cdot \nabla_{\Gamma_h} \mathbf{P}_h \mathbf{u}) \cdot \mathbf{v} - (\mathbf{z} \cdot \nabla_{\Gamma_h} \mathbf{P}_h \mathbf{v}) \cdot \mathbf{u} \, ds_h \\ b_n(p, \mathbf{v}) &:= \int_{\Gamma_h^n} \nabla_{\Gamma_h} p \cdot \mathbf{v} \, ds_h. \end{aligned} \tag{4.1}$$

Using this, the discrete problem can be rewritten as follows (recall that $f = 0$): given $\mathbf{u}_h^{n-1} \in \mathbf{U}_h^{n-1}$ find $\mathbf{u}_h^n \in \mathbf{U}_h^n$, $p_h^n \in Q_h^n$ such that for all $(\mathbf{v}_h, q_h) \in \mathbf{U}_h^n \times Q_h^n$:

$$\begin{aligned} (\mathbf{u}_h^n, \mathbf{P}_h \mathbf{v}_h)_{\Gamma_h^n} + \Delta t a_n(\mathbf{u}_h^{n-1}; \mathbf{u}_h^n, \mathbf{v}_h) + \Delta t b_n(p_h^n, \mathbf{v}_h) &= (\mathbf{u}_h^{n-1}, \mathbf{P}_h \mathbf{v}_h)_{\Gamma_h^n} + \Delta t (\mathbf{f}^{e,n}, \mathbf{P}_h \mathbf{v}_h)_{\Gamma_h^n} \\ b_n(q_h, \mathbf{u}_h^n) &= \rho_p (\mathbf{n}_h \cdot \nabla p_h^n, \mathbf{n}_h \cdot \nabla q_h)_{\omega_\Gamma^n}. \end{aligned} \tag{4.2}$$

We introduce the following natural energy norm

$$\|\mathbf{v}\|_{\tilde{U}^n}^2 := \frac{1}{2} \|\mathbf{v}\|_{\Gamma_h^n}^2 + 2\mu \|E_{s,h}(\mathbf{P}_h \mathbf{v})\|_{\Gamma_h^n}^2 + \frac{\tau}{2} \|\widetilde{\mathbf{n}}_h \cdot \mathbf{v}\|_{\Gamma_h^n}^2 + \rho_u \|\mathbf{n}_h \cdot \nabla \mathbf{v}\|_{\mathcal{O}_\delta(\Gamma_h^n)}^2. \tag{4.3}$$

The norm that we use for the pressure will be introduced later in (4.27).

REMARK 4.1. We outline the structure of the error analysis. We first collect some preliminary results and prove a few helpful estimates, with a particular emphasis on having them uniform in discretization parameters, time and the surface position in the bulk mesh. Using these results it is easy to derive a suitable coercivity and continuity estimates for the trilinear form $a_n(\cdot; \cdot, \cdot)$ used in (4.2). Based on rather straightforward arguments we derive in Theorem 4.7 a stability result for the discrete velocity solution $\mathbf{u}_h^n$ of (4.2) in a natural energy norm. The analysis continues with addressing the more delicate question of the finite element pressure stability. Here we introduce new arguments, namely the use of a specific (non-standard) pressure norm and of a special discrete

velocity function that gives control of the pressure via the discrete inf-sup stability condition, cf. Remark 4.2. Thus we obtain a uniform discrete stability estimate (Theorem 4.8) in the parameter range (3.21). In Section 4.4 we present a consistency analysis that quantifies geometric errors resulting from the approximation of surface and geometric quantities (normal vectors, tangential projectors, curvatures) in the finite element formulation. In that analysis we use results already available in the literature on surface vector-Laplace and Stokes problems. Finally, in Section 4.5 we apply a standard technique for proving discretization error bounds. The discretization error is split into an interpolation error and an error component that lies in the finite element space. Bounds for the latter can be derived using the discrete stability, consistency error bounds, and bounds for the linearization error. With respect to the interpolation error a key point is that in the Euclidean setting of this trace method we can use a (nodal) interpolation operator on a fixed bulk mesh. Due to this the time differentiation and interpolation operator commute, which then leads to optimal bounds for the difference of interpolation errors at $t = t_n$ and $t = t_{n-1}$, cf. Lemma 4.12.

4.1. Preliminaries. We start with an equivalence result between norms on Γ and Γ_h , which is uniform in time and discretization parameters. We use a lifting v^ℓ of functions v defined on Γ_h^n to a neighborhood $\mathcal{O}(\Gamma_h^n)$, defined as follows:

$$\text{for } \mathbf{y} \in \mathcal{O}(\Gamma_h^n) : \quad v^\ell(\mathbf{y}) := v(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Gamma_h^n \text{ such that } \mathbf{p}^n(\mathbf{y}) = \mathbf{p}^n(\mathbf{x}), \quad (4.4)$$

where $\mathbf{p}^n$ is the closest point projection on Γ^n . Differentiating the identities $\mathbf{v}^\ell(\mathbf{y}) = \mathbf{v}^\ell(\mathbf{p}^n(\mathbf{y}))$ one obtains the useful transformation relation

$$\nabla \mathbf{v}^\ell(\mathbf{y}) = \nabla \mathbf{v}^\ell(\mathbf{p}^n(\mathbf{y}))(\mathbf{P}(\mathbf{y}) - d(\mathbf{y})\mathbf{H}(\mathbf{y})), \quad \mathbf{v} \in H^1(\Gamma_h^n)^3, \quad (4.5)$$

with d the signed distance function for Γ^n . For vector-valued functions we define componentwise H^1 norms, namely $\|\mathbf{v}\|_{H^1(\Gamma(t_n))}^2 := \sum_{i=1}^3 \|v_i\|_{H^1(\Gamma(t_n))}^2$ for $\mathbf{v} \in H^1(\Gamma(t_n))^3$, and similarly on Γ_h^n for $\mathbf{v} \in H^1(\Gamma_h^n)^3$. For any $v \in H^1(\Gamma_h^n)$, $\mathbf{v} \in H^1(\Gamma_h^n)^3$ it holds

$$\|v\|_{L^p(\Gamma_h^n)} \simeq \|v^\ell\|_{L^p(\Gamma(t_n))}, \quad p \in [1, \infty], \quad \|\mathbf{v}\|_{H^1(\Gamma_h^n)} \simeq \|\mathbf{v}^\ell\|_{H^1(\Gamma(t_n))}. \quad (4.6)$$

The first equivalence in (4.6) is straightforward and follows from the uniform equivalence of the surface measures ds (on $\Gamma(t_n)$) and ds_h (on Γ_h^n). A proof of the second equivalence in (4.6) is given in [20, Lemma 5.14]. We also need uniform interpolation and Korn-type inequalities:

$$\|v\|_{L^4(\Gamma_h^n)} \lesssim \|v\|_{\Gamma_h^n}^{\frac{1}{2}} \|v\|_{H^1(\Gamma_h^n)}^{\frac{1}{2}} \quad \text{for all } v \in H^1(\Gamma_h^n) \quad (4.7)$$

$$\|\mathbf{v}\|_{L^4(\Gamma_h^n)} \lesssim \|\mathbf{v}\|_{\Gamma_h^n}^{\frac{1}{2}} \|\mathbf{v}\|_{H^1(\Gamma_h^n)}^{\frac{1}{2}} \quad \text{for all } \mathbf{v} \in H^1(\Gamma_h^n)^3 \quad (4.8)$$

$$\|\mathbf{v}\|_{H^1(\Gamma_h^n)} \lesssim \|\mathbf{v}\|_{U^n} \quad \text{for all } \mathbf{v} \in \mathbf{U}_h^n. \quad (4.9)$$

The results (4.6), (4.7) and (4.9) are derived in Appendix, Section A. Componentwise application of (4.7) yields (4.8).

In [26] a finite-difference in time – trace finite element method in space was analyzed, but for a *scalar* parabolic type problem on an evolving surface. The following lemma collects some results from that paper, useful for our error analysis.

LEMMA 4.1. *The following uniform estimates hold*

$$\|v_h\|_{U_\delta(\Gamma_h^n)}^2 \lesssim \delta_n \|v_h\|_{\Gamma_h^n}^2 + \delta_n^2 \|\mathbf{n}_h \cdot \nabla v_h\|_{\mathcal{O}_\delta(\Gamma_h^n)}^2 \quad \forall v_h \in V_{h,m}^n, \quad (4.10a)$$

$$\|v_h\|_{\mathcal{O}_\delta(\Gamma_h^n)}^2 \lesssim (\delta_n + h) \|v_h\|_{\Gamma_h^n}^2 + (\delta_n + h)^2 \|\mathbf{n}_h \cdot \nabla v_h\|_{\mathcal{O}_\delta(\Gamma_h^n)}^2 \quad \forall v_h \in V_{h,m}^n, \quad (4.10b)$$

$$\|v_h\|_{\Gamma_h^n}^2 \leq (1 + c\Delta t) \|v_h\|_{\Gamma_h^{n-1}}^2 + \frac{1}{2} \rho_u \Delta t \|\mathbf{n}_h^{n-1} \cdot \nabla v_h\|_{\mathcal{O}_\delta(\Gamma_h^{n-1})}^2 \quad \forall v_h \in V_{h,m}^{n-1}. \quad (4.10c)$$

The results in (4.10a)–(4.10b) give control of the L^2 -norm of a finite element function in a narrow band volume based on a combination of the L^2 -norm on the surface and the normal derivative in the volume that is provided by the stabilization. Eq. (4.10c) provides a bound for the trace of a function on Γ_h^n through its trace on Γ_h^{n-1} and the volume normal derivative. We also need a slightly modified version of the bound in (4.10c), given in the following lemma.

LEMMA 4.2. *The following uniform estimate holds for all $v_h \in V_{h,m}^{n-1}$,*

$$\|v_h\|_{\Gamma_h^n}^2 \lesssim \|v_h\|_{\Gamma_h^{n-1}}^2 + \delta_{n-1} \|\mathbf{n}_h^{n-1} \cdot \nabla v_h\|_{\mathcal{O}_\delta(\Gamma_h^{n-1})}^2. \quad (4.11)$$

A proof is given in Appendix B. It uses arguments similar to those found in [26].

In the analysis of two-dimensional Navier-Stokes equations one typically needs bounds for $\|\mathbf{u}\|_{L^4}$ to control the quadratic nonlinearity. We will derive such a bound in Lemma 4.4. As a preparation, in the next lemma we give an analog of the result (4.11), with the norm $\|\cdot\|_{L^2}$ replaced by $\|\cdot\|_{L^4}$.

LEMMA 4.3. *The following uniform estimate holds*

$$\|v_h\|_{L^4(\Gamma_h^n)}^2 \lesssim \|v_h\|_{L^4(\Gamma_h^{n-1})}^2 + \|\mathbf{n}_h \cdot \nabla v_h\|_{\mathcal{O}_\delta(\Gamma_h^{n-1})}^2, \quad \text{for all } v_h \in V_{h,m}^{n-1}. \quad (4.12)$$

Proof. We use the notation $\mathcal{O}_\delta = \mathcal{O}_\delta(\Gamma_h^{n-1})$. First we consider an inverse estimate for $\|\mathbf{n}_h \cdot \nabla v_h\|_{L^4(\mathcal{O}_\delta)}$. Using $\mathbf{n}_h = |\nabla \phi_h|^{-1} \nabla \phi_h$, where ϕ_h is a finite element function with $|\nabla \phi_h| \simeq 1$ in $\mathcal{O}_\delta$, and finite element inverse estimates we get

$$\begin{aligned} \|\mathbf{n}_h \cdot \nabla v_h\|_{L^4(\mathcal{O}_\delta)} &\leq \| |\nabla \phi_h|^{-1} \|_{L^\infty(\mathcal{O}_\delta)} \|\nabla \phi_h \cdot \nabla v_h\|_{L^4(\mathcal{O}_\delta)} \lesssim \|\nabla \phi_h \cdot \nabla v_h\|_{L^4(\mathcal{O}_\delta)} \\ &\lesssim h^{-\frac{3}{4}} \|\nabla \phi_h \cdot \nabla v_h\|_{L^2(\mathcal{O}_\delta)} \lesssim h^{-\frac{3}{4}} \| |\nabla \phi_h| \|_{L^\infty(\mathcal{O}_\delta)} \|\mathbf{n}_h \cdot \nabla v_h\|_{L^2(\mathcal{O}_\delta)} \\ &\lesssim h^{-\frac{3}{4}} \|\mathbf{n}_h \cdot \nabla v_h\|_{L^2(\mathcal{O}_\delta)}. \end{aligned} \quad (4.13)$$

Using this and (4.10b) applied to v_h^2 we get

$$\begin{aligned} \|v_h\|_{L^4(\mathcal{O}_\delta)}^2 &= \|v_h^2\|_{L^2(\mathcal{O}_\delta)} \lesssim (\delta_{n-1} + h)^{\frac{1}{2}} \|v_h^2\|_{L^2(\Gamma_h^{n-1})} + (\delta_{n-1} + h) \|\mathbf{n}_h \cdot \nabla v_h^2\|_{L^2(\mathcal{O}_\delta)} \\ &\lesssim (\delta_{n-1} + h)^{\frac{1}{2}} \|v_h\|_{L^4(\Gamma_h^{n-1})}^2 + (\delta_{n-1} + h) \|v_h\|_{L^4(\mathcal{O}_\delta)} \|\mathbf{n}_h \cdot \nabla v_h\|_{L^4(\mathcal{O}_\delta)} \\ &\leq c_0 (\delta_{n-1} + h)^{\frac{1}{2}} \|v_h\|_{L^4(\Gamma_h^{n-1})}^2 + \frac{1}{2} \|v_h\|_{L^4(\mathcal{O}_\delta)}^2 + c_1 h^{-\frac{3}{2}} (\delta_{n-1} + h)^2 \|\mathbf{n}_h \cdot \nabla v_h\|_{L^2(\mathcal{O}_\delta)}^2. \end{aligned}$$

Shifting the term $\|v_h\|_{L^4(\mathcal{O}_\delta)}^2$ to the left-hand side we obtain

$$\|v_h\|_{L^4(\mathcal{O}_\delta)}^2 \lesssim (\delta_{n-1} + h)^{\frac{1}{2}} \|v_h\|_{L^4(\Gamma_h^{n-1})}^2 + h^{-\frac{3}{2}} (\delta_{n-1} + h)^2 \|\mathbf{n}_h \cdot \nabla v_h\|_{L^2(\mathcal{O}_\delta)}^2. \quad (4.14)$$

We use (4.11) with v_h^2 , which is a finite element function of degree $2m$ and so (4.11) applies:

$$\|v_h\|_{L^4(\Gamma_h^n)}^2 = \|v_h^2\|_{L^2(\Gamma_h^n)} \lesssim \|v_h^2\|_{L^2(\Gamma_h^{n-1})} + \delta_{n-1}^{\frac{1}{2}} \|\mathbf{n}_h \cdot \nabla v_h^2\|_{L^2(\mathcal{O}_\delta)}. \quad (4.15)$$

From the chain rule and the result (4.13) above we obtain

$$\|\mathbf{n}_h \cdot \nabla v_h^2\|_{L^2(\mathcal{O}_\delta)} \leq \|v_h\|_{L^4(\mathcal{O}_\delta)} \|\mathbf{n}_h \cdot \nabla v_h\|_{L^4(\mathcal{O}_\delta)} \lesssim h^{-\frac{3}{4}} \|v_h\|_{L^4(\mathcal{O}_\delta)} \|\mathbf{n}_h \cdot \nabla v_h\|_{L^2(\mathcal{O}_\delta)}$$

Substituting this in (4.15) and using (4.14) we get

$$\begin{aligned} \|v_h\|_{L^4(\Gamma_h^n)}^2 &\lesssim \|v_h\|_{L^4(\Gamma_h^{n-1})}^2 + \delta_{n-1}^{\frac{1}{2}} h^{-\frac{3}{4}} \|v_h\|_{L^4(\mathcal{O}_\delta)} \|\mathbf{n}_h \cdot \nabla v_h\|_{L^2(\mathcal{O}_\delta)} \\ &\lesssim \|v_h\|_{L^4(\Gamma_h^{n-1})}^2 + \delta_{n-1} h^{-\frac{3}{2}} \|v_h\|_{L^4(\mathcal{O}_\delta)}^2 + \|\mathbf{n}_h \cdot \nabla v_h\|_{L^2(\mathcal{O}_\delta)}^2 \\ &\lesssim (1 + \delta_{n-1} h^{-\frac{3}{2}} (\delta_{n-1} + h)^{\frac{1}{2}}) \|v_h\|_{L^4(\Gamma_h^{n-1})}^2 + (\delta_{n-1} h^{-3} (\delta_{n-1} + h)^2 + 1) \|\mathbf{n}_h \cdot \nabla v_h\|_{L^2(\mathcal{O}_\delta)}^2. \end{aligned}$$

Due to (3.19f) (i.e. $\delta_{n-1} \lesssim h$) we get $\delta_{n-1} h^{-\frac{3}{2}} (\delta_{n-1} + h)^{\frac{1}{2}} \lesssim 1$ and $\delta_{n-1} h^{-3} (\delta_{n-1} + h)^2 \lesssim 1$, which completes the proof. $\square$

Lemma 4.3 is used in the proof of the next lemma.

LEMMA 4.4. *The following uniform estimate holds*

$$\|\mathbf{v}_h\|_{L^4(\Gamma_h^n)}^2 \lesssim \|\mathbf{v}_h\|_{\Gamma_h^{n-1}} \|\mathbf{v}_h\|_{U^{n-1}} + h \|\mathbf{v}_h\|_{U^{n-1}}^2 \quad \text{for all } \mathbf{v}_h \in \mathbf{U}_h^{n-1}. \quad (4.16)$$

Proof. Take $\mathbf{v}_h \in \mathbf{U}_h^{n-1}$. Component-wise application of (4.12) yields

$$\|\mathbf{v}_h\|_{L^4(\Gamma_h^n)} \lesssim \|\mathbf{v}_h\|_{L^4(\Gamma_h^{n-1})} + \|\mathbf{n}_h \cdot \nabla \mathbf{v}_h\|_{\mathcal{O}_\delta(\Gamma_h^{n-1})} \lesssim \|\mathbf{v}_h\|_{L^4(\Gamma_h^{n-1})} + h^{\frac{1}{2}} \|\mathbf{v}_h\|_{U^{n-1}}. \quad (4.17)$$

Using (4.8) and (4.9) we get

$$\|\mathbf{v}_h\|_{L^4(\Gamma_h^{n-1})} \lesssim \|\mathbf{v}_h\|_{\Gamma_h^{n-1}}^{\frac{1}{2}} \|\mathbf{v}_h\|_{U^{n-1}}^{\frac{1}{2}}.$$

Substituting the latter estimate in (4.17) completes the proof. $\square$

4.2. Coercivity and continuity estimates. We derive coercivity and continuity estimates for the trilinear form $a_n(\cdot; \cdot, \cdot)$.

LEMMA 4.5. *For arbitrary $\mathbf{z} \in L^2(\Gamma_h^n)$ and $\mathbf{v}_h \in \mathbf{U}_h^n$ it holds*

$$a_n(\mathbf{z}; \mathbf{v}_h, \mathbf{v}_h) \geq \|\mathbf{v}_h\|_{U^n}^2 - \xi_h \|\mathbf{P}_h \mathbf{v}_h\|_{\Gamma_h^n}^2 \quad \text{with } \xi_h := 1 + \max_{n=0, \dots, N} \|w_N^{e,n} \mathbf{H}_h(t_n)\|_{L^\infty}. \quad (4.18)$$

Proof. From (3.12) and (3.18) it follows for sufficiently small h that

$$|\mathbf{v}|^2 = |\mathbf{n}_h \cdot \mathbf{v}|^2 + |\mathbf{P}_h \mathbf{v}|^2 \leq (1 + ch^q)(|\tilde{\mathbf{n}}_h \cdot \mathbf{v}|^2 + |\mathbf{P}_h \mathbf{v}|^2) \quad \forall \mathbf{v} \in \mathbb{R}^3.$$

Using this, $c_n(\mathbf{z}; \mathbf{v}_h, \mathbf{v}_h) = 0$, $\mathbf{H}_h = \mathbf{P}_h \mathbf{H}_h \mathbf{P}_h$, which follows from the definition of $\mathbf{H}_h$, cf. (3.13), and $\tau \geq 1 + ch^q$ (for h, τ satisfying (3.19a) and (3.19d)), we obtain the following lower estimate for $a_n(\mathbf{z}; \cdot, \cdot)$:

$$\begin{aligned} a_n(\mathbf{z}; \mathbf{v}_h, \mathbf{v}_h) &= \hat{a}_n(\mathbf{v}_h, \mathbf{v}_h) = (w_N^{e,n} \mathbf{H}_h \mathbf{v}_h, \mathbf{v}_h)_{\Gamma_h^n} + \|\mathbf{v}_h\|_{U^n}^2 + \frac{\tau}{2} \|\tilde{\mathbf{n}}_h \cdot \mathbf{v}\|_{\Gamma_h^n}^2 - \frac{1}{2} \|\mathbf{v}_h\|_{\Gamma_h^n}^2 \\ &\geq \|\mathbf{v}_h\|_{U^n}^2 - (\|w_N^{e,n} \mathbf{H}_h(t_n)\|_{L^\infty} + \frac{1}{2}(1 + ch^q)) \|\mathbf{P}_h \mathbf{v}_h\|_{\Gamma_h^n}^2 \quad \forall \mathbf{v}_h \in \mathbf{U}_h^n, \end{aligned}$$

from which the result follows. $\square$

LEMMA 4.6. *The following uniform estimate holds:*

$$a_n(\mathbf{z}; \mathbf{u}, \mathbf{v}) \lesssim (\|\mathbf{P}_h \mathbf{z}\|_{L^4(\Gamma_h^n)} + 1) \|\mathbf{u}\|_{U^n} \|\mathbf{v}\|_{U^n}, \quad \mathbf{z} \in L^4(\Gamma_h^n), \quad \mathbf{u}, \mathbf{v} \in \mathbf{U}_h^n. \quad (4.19)$$

Proof. For the terms in the trilinear form $a(\cdot; \cdot, \cdot)$ that do not depend on the first argument, cf. (4.1), we apply Cauchy-Schwarz inequality:

$$|\hat{a}_n(\mathbf{u}, \mathbf{v})| \lesssim \|\mathbf{u}\|_{U^n} \|\mathbf{v}\|_{U^n}. \quad (4.20)$$

We now consider the term $\int_{\Gamma_h^n} (\mathbf{z} \cdot \nabla_{\Gamma_h} \mathbf{P}_h \mathbf{u}) \cdot \mathbf{v} ds_h$. The term $\int_{\Gamma_h^n} (\mathbf{z} \cdot \nabla_{\Gamma_h} \mathbf{P}_h \mathbf{v}) \cdot \mathbf{u} ds_h$ can be treated similarly. Using (4.8) and (4.9) we obtain

$$\begin{aligned} \left| \int_{\Gamma_h^n} (\mathbf{z} \cdot \nabla_{\Gamma_h} \mathbf{P}_h \mathbf{u}) \cdot \mathbf{v} ds_h \right| &\lesssim \|\nabla_{\Gamma_h} \mathbf{P}_h \mathbf{u}\|_{\Gamma_h^n} \|\mathbf{P}_h \mathbf{z}\|_{L^4(\Gamma_h^n)} \|\mathbf{P}_h \mathbf{v}\|_{L^4(\Gamma_h^n)} \\ &\lesssim \|\mathbf{P}_h \mathbf{z}\|_{L^4(\Gamma_h^n)} \|\mathbf{P}_h \mathbf{u}\|_{H^1(\Gamma_h^n)} \|\mathbf{P}_h \mathbf{v}\|_{\Gamma_h^n}^{\frac{1}{2}} \|\mathbf{P}_h \mathbf{v}\|_{H^1(\Gamma_h^n)}^{\frac{1}{2}} \\ &\lesssim \|\mathbf{P}_h \mathbf{z}\|_{L^4(\Gamma_h^n)} \|\mathbf{u}\|_{U^n} \|\mathbf{v}\|_{U^n}. \end{aligned} \quad (4.21)$$

Combining this with the same estimate for the other trilinear term and with (4.20) we obtain the result (4.19). $\square$

4.3. Stability estimate. First we derive a stability result for the discrete velocity. Using this stability result and a suitable discrete inf-sup property, we then derive a stability estimate for the discrete pressure in Theorem 4.8.

THEOREM 4.7. *A solution of (3.4) satisfies the following estimate:*

$$\begin{aligned} \|\mathbf{u}_h^n\|_{\Gamma_h^n}^2 + \sum_{k=1}^n \|\mathbf{P}_h \mathbf{u}_h^k - \mathbf{u}_h^{k-1}\|_{\Gamma_h^k}^2 + \Delta t \sum_{k=1}^n (\|\mathbf{u}_h^k\|_{U^k}^2 + 2\rho_p \|\mathbf{n}_h \cdot \nabla p_h^k\|_{\omega_\Gamma^k}^2) \\ \lesssim \exp(ct_n) \left(\|\mathbf{u}_h^0\|_{\Gamma_h^0}^2 + \Delta t \|\mathbf{u}_h^0\|_{U^0}^2 + \Delta t \sum_{k=0}^n \|\mathbf{f}^{e,k}\|_{\Gamma_h^k}^2 \right), \end{aligned} \quad (4.22)$$

with c independent of h , Δt and n .

Proof. We test (4.2) with $\mathbf{v}_h = \mathbf{u}_h^n$, $q_h = -\Delta t p_h^n$. Adding the two identities leads to

$$\begin{aligned} \frac{1}{2}(\|\mathbf{P}_h \mathbf{u}_h^n\|_{\Gamma_h^n}^2 + \|\mathbf{P}_h \mathbf{u}_h^n - \mathbf{u}_h^{n-1}\|_{\Gamma_h^n}^2) + \Delta t a_n(\mathbf{u}_h^{n-1}; \mathbf{u}_h^n, \mathbf{u}_h^n) + \Delta t \rho_p \|\mathbf{n}_h \cdot \nabla p_h^n\|_{\omega_\Gamma^n}^2 \\ = \frac{1}{2}\|\mathbf{u}_h^{n-1}\|_{\Gamma_h^n}^2 + \Delta t(\mathbf{f}^{e,n}, \mathbf{P}_h \mathbf{u}_h^n)_{\Gamma_h^n}. \end{aligned} \quad (4.23)$$

We use the lower bound (4.18) and apply (4.10c):

$$\begin{aligned} \|\mathbf{P}_h \mathbf{u}_h^n\|_{\Gamma_h^n}^2 + \|\mathbf{P}_h \mathbf{u}_h^n - \mathbf{u}_h^{n-1}\|_{\Gamma_h^n}^2 + 2\Delta t(\|\mathbf{u}_h^n\|_{U^n}^2 + \rho_p \|\mathbf{n}_h \cdot \nabla p_h^n\|_{\omega_\Gamma^n}^2) \\ \leq \|\mathbf{u}_h^{n-1}\|_{\Gamma_h^n}^2 + 2\xi_h \Delta t \|\mathbf{P}_h \mathbf{u}_h^n\|_{\Gamma_h^n}^2 + \Delta t \|\mathbf{P}_h \mathbf{u}_h^n\|_{\Gamma_h^n}^2 + \Delta t \|\mathbf{f}^{e,n}\|_{\Gamma_h^n}^2 \\ \leq (1 + c \Delta t) \|\mathbf{u}_h^{n-1}\|_{\Gamma_h^{n-1}}^2 + \frac{1}{2} \Delta t \rho_u \|\mathbf{n}_h^{n-1} \cdot \nabla \mathbf{u}_h^{n-1}\|_{\mathcal{O}_\delta(\Gamma_h^{n-1})}^2 + C \Delta t \|\mathbf{u}_h^n\|_{\Gamma_h^n}^2 + \Delta t \|\mathbf{f}^{e,n}\|_{\Gamma_h^n}^2. \end{aligned} \quad (4.24)$$

Due to (3.19d) we have

$$\begin{aligned} \frac{1}{2} \Delta t \|\mathbf{u}_h^n\|_{U^n}^2 &\geq \frac{1}{2} \Delta t \tau \|\tilde{\mathbf{n}}_h \cdot \mathbf{u}_h^n\|_{\Gamma_h^n}^2 \geq \frac{1}{2} \Delta t \tau \|\mathbf{n}_h \cdot \mathbf{u}_h^n\|_{\Gamma_h^n}^2 - c \Delta t \tau h^{2q} \|\mathbf{u}_h^n\|_{\Gamma_h^n}^2 \\ &\geq \|\mathbf{n}_h \cdot \mathbf{u}_h^n\|_{\Gamma_h^n}^2 - c \Delta t h^{2q-2} \|\mathbf{u}_h^n\|_{\Gamma_h^n}^2. \end{aligned} \quad (4.25)$$

Using this and $\|\mathbf{u}_h^n\|_{\Gamma_h^n}^2 = \|\mathbf{P}_h \mathbf{u}_h^n\|_{\Gamma_h^n}^2 + \|\mathbf{n}_h \cdot \mathbf{u}_h^n\|_{\Gamma_h^n}^2$ in (4.24) we obtain

$$\begin{aligned} \|\mathbf{u}_h^n\|_{\Gamma_h^n}^2 + \|\mathbf{P}_h \mathbf{u}_h^n - \mathbf{u}_h^{n-1}\|_{\Gamma_h^n}^2 + \frac{3}{2} \Delta t \|\mathbf{u}_h^n\|_{U^n}^2 + 2\Delta t \rho_p \|\mathbf{n}_h \cdot \nabla p_h^n\|_{\omega_\Gamma^n}^2 \\ \leq (1 + c \Delta t) \|\mathbf{u}_h^{n-1}\|_{\Gamma_h^{n-1}}^2 + \frac{1}{2} \Delta t \|\mathbf{u}_h^{n-1}\|_{U^{n-1}}^2 + C \Delta t \|\mathbf{u}_h^n\|_{\Gamma_h^n}^2 + \Delta t \|\mathbf{f}^{e,n}\|_{\Gamma_h^n}^2. \end{aligned} \quad (4.26)$$

We sum up the inequalities for $n = 1, \dots, k$, and with $c^* = c + C$ we get

$$\begin{aligned} \|\mathbf{u}_h^k\|_{\Gamma_h^k}^2 + \sum_{n=1}^k \|\mathbf{P}_h \mathbf{u}_h^n - \mathbf{u}_h^{n-1}\|_{\Gamma_h^n}^2 + \Delta t \sum_{n=1}^k (\|\mathbf{u}_h^n\|_{U^n}^2 + 2\rho_p \|\mathbf{n}_h \cdot \nabla p_h^n\|_{\omega_\Gamma^n}^2) \\ \leq (1 + c \Delta t) \|\mathbf{u}_h^0\|_{\Gamma_h^0}^2 + \frac{1}{2} \Delta t \|\mathbf{u}_h^0\|_{U^0}^2 + \Delta t \sum_{n=0}^k c^* \|\mathbf{u}_h^n\|_{\Gamma_h^n}^2 + \Delta t \sum_{n=0}^k \|\mathbf{f}^{e,n}\|_{\Gamma_h^n}^2. \end{aligned}$$

Finally, we shift the term $\Delta t c^* \|\mathbf{u}_h^k\|_{\Gamma_h^k}^2$ from the right-hand side to the left hand side and apply, for $\Delta t \leq (2c^*)^{-1}$, a discrete Gronwall inequality. $\square$

We now derive a stability bound for the discrete pressure. The analysis is based on results derived in [33]. We recall some results from that paper. We use the scaled $H^1(\omega_\Gamma^n)$ -norm given by

$$\|q\|_{1, \omega_\Gamma^n, h} := h^{\frac{1}{2}} \|\nabla q\|_{\omega_\Gamma^n}. \quad (4.27)$$

Note that this norm is equivalent to the *scaled* $H^1(\Gamma(t_n))$ norm in the following sense. Assume $\Gamma(t_n) \subset \omega_\Gamma^n$ and q^e is the normal extension of $q \in H^1(\Gamma(t))$. Then $h \|\nabla_\Gamma q\|_{\Gamma(t)} \sim \|q^e\|_{1, \omega_\Gamma^n, h}$ holds.

For a given $q_h \in Q_h^n$ we define a corresponding function $\hat{\mathbf{v}}_h \in \mathbf{U}_h^n$ by

$$\hat{\mathbf{v}}_h(\mathbf{x}) = \sum_{E \in \mathcal{E}_{\text{reg}}} h_E^2 \varphi_E(\mathbf{x}) (\mathbf{t}_E \cdot \nabla q_h(\mathbf{x})) \mathbf{t}_E, \quad (4.28)$$

with $\mathcal{E}_{\text{reg}}$ being a subset of edges of tetrahedra in ω_Γ^n , $\mathbf{t}_E$ the unit tangent vector along the edge E and φ_E the quadratic nodal basis function corresponding to the edge E (extended by zero to Ω). Recall that the finite element function $\phi_h(t, \cdot)$ defines the surface approximation $\Gamma_h(t)$, cf. (3.10). The following result was shown in [18, Theorem 5.9]: For a fixed t_n and ϕ_h a piecewise *linear* finite element function satisfying (3.8)–(3.9b) there exist a subset $\mathcal{E}_{\text{reg}}$ and C independent of q_h , discretization parameters and the position of Γ_h in the mesh such that

$$\|q_h\|_{1, \omega_\Gamma^n, h}^2 \leq C(b(q_h, \hat{\mathbf{v}}_h) + \rho_p \|\mathbf{n}_h \cdot \nabla q_h\|_{\omega_\Gamma^n}^2) \quad (4.29)$$

$$\|\hat{\mathbf{v}}_h\|_{U^n} \leq C \|q_h\|_{1, \omega_\Gamma^n, h}. \quad (4.30)$$

ASSUMPTION 4.1. We assume that (4.29)–(4.30) hold for a P_k finite element function ϕ_h satisfying (3.8)–(3.9b), with C independent of q_h , discretization parameters, the position of Γ_h in the mesh and of t_n .

A result similar to (4.29)–(4.30) is also known if $\phi_h = \phi$, i.e., if the geometry approximation $\Gamma_h \approx \Gamma$ is exact (cf. [33]). This is a further support of the plausibility of Assumption 4.1. We claim that the analysis of [33] can be extended to the case of an arbitrary polynomial degree approximation ϕ_h of ϕ , with uniform constants as specified in Assumption 4.1. Such an extension, however, will not be straightforward and is expected to be rather technical and therefore is not addressed here.

THEOREM 4.8. Let Assumption 4.1 be satisfied. Denote the upper bound in (4.22) by $\mathbf{F}_n = \mathbf{F}_n(\mathbf{u}_h^0, \mathbf{f}^e)$. The discrete problem (4.2) has a unique solution and for the discrete pressure solution the following holds:

$$\Delta t \sum_{k=1}^n \|p_h^k\|_{1, \omega_{\Gamma}^k, h} \lesssim \mathbf{F}_n + \left(\frac{h}{\sqrt{\Delta t}} + 1\right) \mathbf{F}_n^{\frac{1}{2}} \lesssim \mathbf{F}_n + \mathbf{F}_n^{\frac{1}{2}}. \quad (4.31)$$

Proof. From the discrete inf-sup property (4.29) and standard results for saddle point problems it follows that the discrete problem (4.2) has a unique solution. Let p_h^n , $n = 1, \dots, N$, be the pressure solution and, for given n , $\widehat{\mathbf{v}}_h \in \mathbf{U}_h^n$ as in (4.28) a corresponding discrete velocity. We then have

$$\|p_h^n\|_{1, \omega_{\Gamma}^n, h}^2 \lesssim b(p_h^n, \widehat{\mathbf{v}}_h) + \rho_p \|\mathbf{n}_h \cdot \nabla p_h^n\|_{\omega_{\Gamma}^n}^2 \lesssim b(p_h^n, \widehat{\mathbf{v}}_h) + \rho_p^{\frac{1}{2}} \|\mathbf{n}_h \cdot \nabla p_h^n\|_{\omega_{\Gamma}^n} \|p_h^n\|_{1, \omega_{\Gamma}^n, h}. \quad (4.32)$$

We use the test function $\widehat{\mathbf{v}}_h$ in (4.2) and thus get

$$b(p_h^n, \widehat{\mathbf{v}}_h) = -\frac{1}{\Delta t} ((\mathbf{u}_h^n, \mathbf{P}_h \widehat{\mathbf{v}}_h)_{\Gamma_h^n} - (\mathbf{u}_h^{n-1}, \mathbf{P}_h \widehat{\mathbf{v}}_h)_{\Gamma_h^n}) - a_n(\mathbf{u}_h^{n-1}; \mathbf{u}_h^n, \widehat{\mathbf{v}}_h) + (\mathbf{f}^{e, n}, \mathbf{P}_h \widehat{\mathbf{v}}_h)_{\Gamma_h^n}.$$

Now note the following:

$$\|\widehat{\mathbf{v}}_h\|_{\Gamma_h^n}^2 \lesssim h^{-1} \|\widehat{\mathbf{v}}_h\|_{\omega_{\Gamma}^n}^2 \lesssim h^3 \|\nabla p_h^n\|_{\omega_{\Gamma}^n}^2 \lesssim h^2 \|p_h^n\|_{1, \omega_{\Gamma}^n, h}^2. \quad (4.33)$$

Using this and the continuity estimate in (4.19), (4.16) and (4.30) we obtain

$$\begin{aligned} |a_n(\mathbf{u}_h^{n-1}; \mathbf{u}_h^n, \widehat{\mathbf{v}}_h)| &\lesssim (\|\mathbf{u}_h^{n-1}\|_{U^{n-1}} + 1) \|\mathbf{u}_h^n\|_{U^n} \|\widehat{\mathbf{v}}_h\|_{U^n} \\ &\lesssim (\|\mathbf{u}_h^{n-1}\|_{U^{n-1}}^2 + \|\mathbf{u}_h^n\|_{U^n}^2 + \|\mathbf{u}_h^n\|_{U^n}) \|p_h^n\|_{1, \omega_{\Gamma}^n, h}. \end{aligned}$$

Using this and (4.33) we get

$$\begin{aligned} |b(p_h^n, \widehat{\mathbf{v}}_h)| &\lesssim \frac{h}{\Delta t} \|\mathbf{P}_h \mathbf{u}_h^n - \mathbf{u}_h^{n-1}\|_{\Gamma_h^n} \|p_h^n\|_{1, \omega_{\Gamma}^n, h} + h \|\mathbf{f}^{e, n}\|_{\Gamma_h^n} \|p_h^n\|_{1, \omega_{\Gamma}^n, h} \\ &\quad + (\|\mathbf{u}_h^{n-1}\|_{U^{n-1}}^2 + \|\mathbf{u}_h^n\|_{U^n}^2 + \|\mathbf{u}_h^n\|_{U^n}) \|p_h^n\|_{1, \omega_{\Gamma}^n, h}. \end{aligned}$$

Hence, using (4.32) we obtain

$$\begin{aligned} \|p_h^n\|_{1, \omega_{\Gamma}^n, h} &\lesssim \frac{h}{\Delta t} \|\mathbf{P}_h \mathbf{u}_h^n - \mathbf{u}_h^{n-1}\|_{\Gamma_h^n} + h \|\mathbf{f}^{e, n}\|_{\Gamma_h^n} + \rho_p^{\frac{1}{2}} \|\mathbf{n}_h \cdot \nabla p_h^n\|_{\omega_{\Gamma}^n} \\ &\quad + \|\mathbf{u}_h^{n-1}\|_{U^{n-1}}^2 + \|\mathbf{u}_h^n\|_{U^n}^2 + \|\mathbf{u}_h^n\|_{U^n}. \end{aligned} \quad (4.34)$$

Summing over $n = 1, \dots, k$, yields

$$\begin{aligned} \Delta t \sum_{n=1}^k \|p_h^k\|_{1, \omega_{\Gamma}^k, h} &\lesssim h \sum_{n=1}^k \|\mathbf{P}_h \mathbf{u}_h^n - \mathbf{u}_h^{n-1}\|_{\Gamma_h^n} + h \Delta t \sum_{n=1}^k \|\mathbf{f}^{e, n}\|_{\Gamma_h^n} \\ &\quad + \Delta t \sum_{n=1}^k \rho_p^{\frac{1}{2}} \|\mathbf{n}_h \cdot \nabla p_h^n\|_{\omega_{\Gamma}^n} + \Delta t \sum_{n=0}^k \|\mathbf{u}_h^n\|_{U^n}^2 + \Delta t \sum_{n=0}^k \|\mathbf{u}_h^n\|_{U^n}. \end{aligned} \quad (4.35)$$

We use the estimate (4.22), with upper bound denoted by $\mathbf{F}_n$, and $k \lesssim 1/\Delta t$. The different terms on the right-hand side can be bounded as follows, using Cauchy-Schwarz,

$$\begin{aligned}
h \sum_{n=1}^k \|\mathbf{P}_h \mathbf{u}_h^n - \mathbf{u}_h^{n-1}\|_{\Gamma_h^n} &\lesssim \frac{h}{\sqrt{\Delta t}} \left(\sum_{n=1}^k \|\mathbf{P}_h \mathbf{u}_h^n - \mathbf{u}_h^{n-1}\|_{\Gamma_h^n}^2 \right)^{\frac{1}{2}} \lesssim \frac{h}{\sqrt{\Delta t}} \mathbf{F}_k^{\frac{1}{2}}, \\
h \Delta t \sum_{n=1}^k \|\mathbf{f}^{e,n}\|_{\Gamma_h^n} + \Delta t \sum_{n=1}^k \rho_p^{\frac{1}{2}} \|\mathbf{n}_h \cdot \nabla p_h^n\|_{\omega_p^n} \\
&\lesssim h \left(\Delta t \sum_{n=1}^k \|\mathbf{f}^{e,n}\|_{\Gamma_h^n}^2 \right)^{\frac{1}{2}} + \left(\Delta t \sum_{n=1}^k \rho_p \|\mathbf{n}_h \cdot \nabla p_h^n\|_{\omega_p^n}^2 \right)^{\frac{1}{2}} \lesssim \mathbf{F}_k^{\frac{1}{2}}, \\
\Delta t \sum_{n=0}^k \|\mathbf{u}_h^n\|_{U^n}^2 + \Delta t \sum_{n=0}^k \|\mathbf{u}_h^n\|_{U^n} &\lesssim \mathbf{F}_k + \mathbf{F}_k^{\frac{1}{2}}.
\end{aligned} \tag{4.36}$$

Combining these estimates completes the proof. $\square$

Let us discuss the stability bound in (4.31). Note that the bound yields stability in a discrete $L^1(H^1)$ type norm with a uniform constant for $h^2 \lesssim \Delta t$, which covers parameter choices typically used in practice for $m = 1$ (the lowest order Taylor–Hood pair), namely $\Delta t \sim h^2$ (for BDF1) and $\Delta t \sim h$ (BDF2). The stability bound can be compared with bounds derived in the recent papers [4, 49] in which a similar discretization method for the Stokes problem on moving domains is treated. The analyses in these papers yield a uniform (in Δt and h) pressure bound only for $\Delta t^2 \sum_{k=1}^n \|p_h^k\|_{\Omega_k}$. Note the square in the scaling factor Δt^2 in this quantity. In particular these analyses do not yield uniform bounds for $\Delta t \sum_{k=1}^n \|p_h^k\|_{\Omega_k}$ if $\Delta t \sim h$ or $\Delta t \sim h^2$.

REMARK 4.2. We briefly explain why our analysis yields a uniform bound if $h^2 \lesssim \Delta t$. First note that (4.29)–(4.30) implies the discrete inf-sup property. In the stability analysis of the discrete pressure we do not use this discrete inf-sup property. We rather use the specific choice of the function $\hat{\mathbf{v}}_h$ in (4.29)–(4.30). For this function we have, cf. (4.33), $\|\hat{\mathbf{v}}_h\|_{\Gamma_h^n} \lesssim h \|p_h^n\|_{1, \omega_{\Gamma,h}^n}$. Here we gain a power of h compared to the more naive estimate $\|\hat{\mathbf{v}}_h\|_{\Gamma_h^n} \lesssim \|\hat{\mathbf{v}}_h\|_{U^n} \lesssim \|p_h^n\|_{1, \omega_{\Gamma,h}^n}$. Due to this we get the factor h in front of the time difference term $\|\mathbf{P}_h \mathbf{u}_h^n - \mathbf{u}_h^{n-1}\|_{\Gamma_h^n}$ in (4.34)–(4.35), which leads to $\frac{h}{\sqrt{\Delta t}}$, instead of $\frac{1}{\sqrt{\Delta t}}$ in (4.36). Hence we have a uniform bound for $h^2 \lesssim \Delta t$. It looks plausible that this approach can be used to improve the suboptimal pressure stability estimate in [49] as well.

REMARK 4.3. The treatment of non-homogeneous divergence condition in (2.4) involves handling the $(f^{e,n}, p_h)_{\Gamma_h^n}$ term on the right-hand side of eq. (4.23). Standard arguments would result in an additional term on the right hand side of the stability bound, which involves the quantity $\|f\|_* = \max_n \sup_{q_h \in Q_h^n} \frac{(f^{e,n}, q_h)_{\Gamma_h^n}}{\|q_h\|_{1, \omega_{\Gamma,h}^n}}$, which is dual to our discrete pressure norm. We cannot show that $\|f\|_*$ remains uniformly bounded for a smooth f if $h \rightarrow 0$. For this reason, our stability and convergence analysis is limited to the case of a homogeneous divergence condition. A formal way to reduce the original problem to the one with an homogeneous divergence condition would be to solve for the potential ψ satisfying $\Delta_{\Gamma} \psi = f$ on $\Gamma(t)$ and substitute $\mathbf{u}_T \rightarrow \mathbf{u}_T + \psi$. In practice, we do not use this approach and instead apply the FEM (3.16)–(3.17). We see optimal order convergence results, cf. Section 5.

We now proceed with an error estimate. Its proof combines the arguments we used for the stability analysis with geometric and interpolation error estimates. The geometric and interpolation error estimates are treated at each time instance t_n for ‘stationary’ surfaces Γ_h^n and so results already available in the literature (cf. [18, 35, 41]) can be used. We start with a consistency estimate for (3.4).

4.4. Consistency analysis. In this section we derive estimates for the consistency error. The analysis is based on a standard technique and uses estimates already available in the literature for surface vector-Laplace and surface Stokes equations. In the consistency and error bounds we need estimates on derivatives of the extended solution $\mathbf{u}^e(t, \mathbf{x}) = \mathbf{u}(t, \mathbf{p}(\mathbf{x}))$ in the strip $\mathcal{O}(\mathcal{G})$. To simplify the notation, the extension of (scalar or vector-valued) functions v defined on $\mathcal{G}$ is also denoted by v . By differentiating the identity $v(t, \mathbf{x}) = v(t, \mathbf{p}(\mathbf{x}))$, $(t, \mathbf{x}) \in \mathcal{O}(\mathcal{G})$, $k \geq 0$ times one finds that for

C^{k+1} -smooth manifold $\mathcal{G}$ and $v \in C^k(\mathcal{G})$ the following bound holds:

$$\|v\|_{W^{k,\infty}(\mathcal{O}(\mathcal{G}))} \lesssim \|v\|_{W^{k,\infty}(\mathcal{G})}. \quad (4.37)$$

Further calculations, see, for example, [41, Lemma 3.1], yield

$$\|v\|_{H^k(U_\varepsilon(\Gamma(t)))} \lesssim \varepsilon^{\frac{1}{2}} \|v\|_{H^k(\Gamma(t))}, \quad \text{for } t \in [0, T] \quad (4.38)$$

and any $\varepsilon > 0$ such that $U_\varepsilon(\Gamma(t)) \subset \mathcal{O}(\Gamma(t))$, where $U_\varepsilon(\Gamma(t))$ is an ε -neighborhood in $\mathbb{R}^3$.

We next observe that the smooth solution $\mathbf{u}^n = \mathbf{u}(t_n)$, $\mathbf{u}^{n-1} = \mathbf{u}(t_{n-1}) = \mathbf{u}(t_{n-1})^e$, $p^n = p(t_n)$ of (2.4) satisfies the identity

$$\int_{\Gamma_h^n} \left(\frac{\mathbf{u}^n - \mathbf{u}^{n-1}}{\Delta t} \right) \cdot \mathbf{P}_h \mathbf{v}_h \, ds + a_n(\mathbf{u}^{n-1}; \mathbf{u}^n, \mathbf{v}_h) + b_n(p^n, \mathbf{v}_h) = \mathcal{E}_C^n(\mathbf{v}_h) + \int_{\Gamma^n} \mathbf{f}^n \cdot \mathbf{v}_h^\ell \, ds, \quad (4.39)$$

for any $\mathbf{v}_h \in \mathbf{U}_h^n$ and with $a_n(\cdot; \cdot, \cdot)$, $b_n(\cdot, \cdot)$ as in (4.1) and $\mathcal{E}_C^n(\mathbf{v}_h)$ collecting consistency terms due to geometric errors, time derivative approximation and linearization, i.e.

$$\begin{aligned} \mathcal{E}_C^n(\mathbf{v}_h) := & \underbrace{\int_{\Gamma_h^n} \left(\frac{\mathbf{u}^n - \mathbf{u}^{n-1}}{\Delta t} \right) \cdot \mathbf{P}_h \mathbf{v}_h \, ds_h - \int_{\Gamma^n} \mathbf{u}_t(t_n) \cdot \mathbf{v}_h^\ell \, ds + \rho_u \int_{\mathcal{O}_\delta(\Gamma_h^n)} ((\mathbf{n}_h - \mathbf{n}) \cdot \nabla \mathbf{u}^n)(\mathbf{n}_h \cdot \nabla \mathbf{v}_h) \, d\mathbf{x}}_{I_1} \\ & + \underbrace{\frac{1}{2} \int_{\Gamma_h^n} (\mathbf{u}^{n-1} \cdot \nabla_{\Gamma_h} \mathbf{P}_h \mathbf{u}^n) \cdot \mathbf{v}_h - (\mathbf{u}^{n-1} \cdot \nabla_{\Gamma_h} \mathbf{P}_h \mathbf{v}_h) \cdot \mathbf{u}^n \, ds_h - \frac{1}{2} \int_{\Gamma^n} (\mathbf{u}^n \cdot \nabla_{\Gamma} \mathbf{u}^n) \cdot \mathbf{v}_h^\ell - (\mathbf{u}^n \cdot \nabla_{\Gamma} \mathbf{P} \mathbf{v}_h^\ell) \cdot \mathbf{u}^n \, ds}_{I_3} \\ & + \underbrace{2\mu \int_{\Gamma_h^n} E_{s,h}(\mathbf{P}_h \mathbf{u}^n) : E_{s,h}(\mathbf{P}_h \mathbf{v}_h) \, ds_h - 2\mu \int_{\Gamma^n} E_s(\mathbf{u}^n) : E_s(\mathbf{P} \mathbf{v}_h^\ell) \, ds}_{I_4} \\ & + \underbrace{\int_{\Gamma_h^n} w_N^{e,n} \mathbf{v}_h \cdot \mathbf{H}_h \mathbf{u}^n \, ds_h - \int_{\Gamma^n} w_N^n \mathbf{v}_h^\ell \cdot \mathbf{H} \mathbf{u}^n \, ds + \tau \int_{\Gamma_h^n} (\tilde{\mathbf{n}}_h \cdot \mathbf{u}^n)(\tilde{\mathbf{n}}_h \cdot \mathbf{v}_h) \, ds_h - \tau \int_{\Gamma^n} (\mathbf{n} \cdot \mathbf{u}^n)(\mathbf{n} \cdot \mathbf{v}_h^\ell) \, ds}_{I_5} \\ & + \underbrace{\int_{\Gamma_h^n} \nabla_{\Gamma_h} p^n \cdot \mathbf{v}_h \, ds_h - \int_{\Gamma^n} \nabla_{\Gamma} p^n \cdot \mathbf{v}_h^\ell \, ds}_{I_7}. \end{aligned}$$

All terms above except I_3 have been considered in consistency analyses of TraceFEM in the literature, [18, 20, 26]. In the next lemma we collect results which are essentially known and then treat the term I_3 in Lemma 4.10.

LEMMA 4.9. *The following uniform estimates hold*

$$|I_1| \lesssim (\Delta t + h^q) \|\mathbf{v}_h\|_{\Gamma_h^n} \quad (4.40)$$

$$|I_2| \lesssim h^q \rho_u^{\frac{1}{2}} \|\mathbf{n}_h \cdot \nabla \mathbf{v}_h\|_{\mathcal{O}_\delta(\Gamma_h^n)} \quad (4.41)$$

$$|I_4| \lesssim h^q \left(\|\mathbf{v}_h\|_{H^1(\Gamma_h^n)} + h^{-1} \|\tilde{\mathbf{n}}_h \cdot \mathbf{v}_h\|_{\Gamma_h^n} \right) \quad (4.42)$$

$$|I_5| \lesssim h^q \|\mathbf{v}_h\|_{H^1(\Gamma_h^n)} \quad (4.43)$$

$$|I_6| \lesssim h^{q+1} \tau \|\tilde{\mathbf{n}}_h \cdot \mathbf{v}_h\|_{\Gamma_h^n} \lesssim h^q \tau^{\frac{1}{2}} \|\tilde{\mathbf{n}}_h \cdot \mathbf{v}_h\|_{\Gamma_h^n} \quad (4.44)$$

$$|I_7| \lesssim h^q \|\mathbf{v}_h\|_{\Gamma_h^n} \quad (4.45)$$

where $q+1$ is the order of geometry recovery defined in (3.8) and (3.13).

Proof. Componentwise application of the arguments in the proof of Lemma 11 in [26] and $\|\mathbf{P}_h - \mathbf{P}\|_{L^\infty(\Gamma_h^n)} \lesssim h^q$ yields the bound in (4.40). In the same proof the result (4.41) is derived, using the assumption (3.19b). Recall that $E_{s,h}(\mathbf{P}_h \mathbf{v}_h)$ represents $E_{s,h}(\mathbf{v}_h) - (\mathbf{n}_h \cdot \mathbf{v}_h) \mathbf{H}_h$, cf. Remark 3.4. The result (4.42) is shown in Lemma 5.15 and (5.42) in [20]. For the estimate (4.43) we use assumption (3.13) and an appropriate partial integration to shift the derivatives in $\mathbf{H}_h$ and $\mathbf{H}$ to the function $\mathbf{v}_h$.

Details of a proof are given in Appendix C. The result (4.44) also follows with similar arguments, using (3.18), cf. proof of Lemma 5.18 in [20]. The result (4.45) is obtained with the same techniques. In the literature cited, bounds (4.42)–(4.45) were proved for the case of a stationary surface. However, the arguments need only norm equivalences as in (4.6) and geometry approximation inequalities formulated in section 3.2.1. Since in our setting these results hold uniformly in time, this implies uniform boundedness of the constants in (4.42)–(4.45) also with respect to time. $\square$

The nonlinear term requires a more careful handling, which is carried out below.

LEMMA 4.10. *It holds $|I_3| \lesssim (\Delta t + h^q) \|\mathbf{v}_h\|_{H^1(\Gamma_h^n)}$.*

Proof. Recall that the spatial-normal extension of $\mathbf{u}$ to $\mathcal{O}(\mathcal{G})$ is denoted by $\mathbf{u}$, too. We start with estimating the differences between the quadratic nonlinear terms, i.e. the third and fourth terms in I_3 , and the corresponding linearized ones on the *exact* surface at time t_n . For the fourth term in I_3 we get, using the smoothness of $\mathbf{u}$

$$\begin{aligned} \left| \int_{\Gamma^n} (\mathbf{u}^n \cdot \nabla_{\Gamma} \mathbf{P} \mathbf{v}_h^\ell) \cdot \mathbf{u}^n ds - \int_{\Gamma^n} (\mathbf{u}^{n-1} \cdot \nabla_{\Gamma} \mathbf{P} \mathbf{v}_h^\ell) \cdot \mathbf{u}^n ds \right| &= \left| \int_{\Gamma^n} \left(\int_{t_{n-1}}^{t_n} \mathbf{u}_t dt \cdot \nabla_{\Gamma} \mathbf{P} \mathbf{v}_h^\ell \right) \cdot \mathbf{u}^n ds \right| \\ &\leq \Delta t \|\nabla_{\Gamma} \mathbf{P} \mathbf{v}_h^\ell\|_{L^2(\Gamma^n)} \lesssim \Delta t \|\mathbf{v}_h^\ell\|_{H^1(\Gamma^n)} \lesssim \Delta t \|\mathbf{v}_h\|_{H^1(\Gamma_h^n)}. \end{aligned}$$

With very similar arguments we obtain for the third term in I_3 :

$$\left| \int_{\Gamma^n} (\mathbf{u}^n \cdot \nabla_{\Gamma} \mathbf{u}^n) \cdot \mathbf{v}_h^\ell ds - \int_{\Gamma^n} (\mathbf{u}^{n-1} \cdot \nabla_{\Gamma} \mathbf{u}^n) \cdot \mathbf{v}_h^\ell ds \right| \lesssim \Delta t \|\mathbf{v}_h\|_{\Gamma_h^n}.$$

For the approximate surface $\nabla_{\Gamma_h} \mathbf{P}_h \mathbf{w}$ represents $\nabla_{\Gamma_h} \mathbf{w} - (\mathbf{n}_h \cdot \mathbf{w}) \mathbf{H}_h$. We now compare the linearized terms on the *approximate* surface Γ_h^n (first and second one in I_3) with the corresponding ones on the exact surface. For the second term in I_3 we get

$$\begin{aligned} &\left| \int_{\Gamma_h^n} (\mathbf{u}^{n-1} \cdot \nabla_{\Gamma_h} \mathbf{P}_h \mathbf{v}_h) \cdot \mathbf{u}^n ds_h - \int_{\Gamma^n} (\mathbf{u}^{n-1} \cdot \nabla_{\Gamma} \mathbf{P} \mathbf{v}_h^\ell) \cdot \mathbf{u}^n ds \right| \\ &\leq \underbrace{\left| \int_{\Gamma_h^n} (\mathbf{u}^{n-1} \cdot \nabla_{\Gamma_h} \mathbf{v}_h) \cdot \mathbf{u}^n ds_h - \int_{\Gamma^n} (\mathbf{u}^{n-1} \cdot \nabla_{\Gamma} \mathbf{v}_h^\ell) \cdot \mathbf{u}^n ds \right|}_{J_1} \\ &\quad + \underbrace{\left| \int_{\Gamma_h^n} (\mathbf{n}_h \cdot \mathbf{v}_h) \mathbf{u}^{n-1} \cdot \mathbf{H}_h \mathbf{u}^n ds_h - \int_{\Gamma^n} (\mathbf{n} \cdot \mathbf{v}_h^\ell) \mathbf{u}^{n-1} \cdot \mathbf{H} \mathbf{u}^n ds \right|}_{J_2} \end{aligned}$$

For the term J_1 we use $\nabla_{\Gamma_h} \mathbf{v}_h = \mathbf{P}_h \nabla_{\Gamma_h^\ell} \mathbf{P}_h$, the transformation relation (4.5) applied to $\mathbf{v}_h$, $\|\mathbf{P} - \mathbf{P}_h\|_{L^\infty(\Gamma_h^n)} \lesssim h^q$ and the bound in (A.1) for the change in surface measure. Thus we get

$$J_1 \lesssim h^q \|\mathbf{v}_h^\ell\|_{H^1(\Gamma^n)} \lesssim h^q \|\mathbf{v}_h\|_{H^1(\Gamma_h^n)}.$$

For the term J_2 we proceed as in the derivation of the bound for I_5 in (4.43), cf. Appendix C. As in (C.1) we obtain

$$\int_{\Gamma^n} (\mathbf{n} \cdot \mathbf{v}_h^\ell) \mathbf{u}^{n-1} \cdot \mathbf{H} \mathbf{u}^n ds = - \int_{\Gamma^n} (\mathbf{n} \cdot \mathbf{v}_h^\ell) \mathbf{n} \cdot \nabla (\mathbf{P} \mathbf{u}^{n-1}) \mathbf{u}^n ds. \quad (4.46)$$

In the other term in J_2 we replace $\mathbf{n}_h$ by $\mathbf{n}$, with error bounded by $Ch^q \|\mathbf{v}_h\|_{\Gamma_h^n}$. With the same arguments as in (C.2)–(C.4) the resulting term $\int_{\Gamma_h^n} (\mathbf{n} \cdot \mathbf{v}_h) \mathbf{u}^{n-1} \cdot \mathbf{H}_h \mathbf{u}^n ds_h$ can be replaced by $-\int_{\Gamma^n} (\mathbf{n} \cdot \mathbf{v}_h^\ell) \bar{\mathbf{n}}_h^\ell \cdot \nabla (\mathbf{P} \mathbf{u}^{n-1}) \mathbf{u}^n ds$ with error bounded by $Ch^q \|\mathbf{v}_h\|_{\Gamma_h^n}$. Comparing the latter term with the one on the right-hand side in (4.46) we get, using the assumption (3.13), a bound $Ch^q \|\mathbf{v}_h\|_{\Gamma_h^n}$. Summarizing we get $J_1 + J_2 \lesssim h^q \|\mathbf{v}_h\|_{H^1(\Gamma_h^n)}$.

For the remaining first term in I_3 we use similar arguments. First note

$$\begin{aligned} & \left| \int_{\Gamma_h^n} (\mathbf{u}^{n-1} \cdot \nabla_{\Gamma_h} \mathbf{P}_h \mathbf{u}^n) \cdot \mathbf{v}_h ds_h - \int_{\Gamma^n} (\mathbf{u}^{n-1} \cdot \nabla_{\Gamma} \mathbf{u}^n) \cdot \mathbf{v}_h^\ell ds \right| \\ & \leq \left| \int_{\Gamma_h^n} (\mathbf{u}^{n-1} \cdot \nabla_{\Gamma_h} \mathbf{u}^n) \cdot \mathbf{v}_h ds_h - \int_{\Gamma^n} (\mathbf{u}^{n-1} \cdot \nabla_{\Gamma} \mathbf{u}^n) \cdot \mathbf{v}_h^\ell ds \right| + \left| \int_{\Gamma_h^n} (\mathbf{n}_h \cdot \mathbf{u}^n) \mathbf{u}^{n-1} \cdot \mathbf{H}_h \mathbf{v}_h ds_h \right|. \end{aligned}$$

The first term $|\cdot|$ can be bounded by $Ch^q \|\mathbf{v}_h\|_{\Gamma_h^n}$ using the same arguments as for J_1 above. For the second $|\cdot|$ term we also obtain such a bound using $\mathbf{n}_h \cdot \mathbf{u}^n = (\mathbf{n}_h - \mathbf{n}) \cdot \mathbf{u}^n$ and $\|\mathbf{n}_h - \mathbf{n}\|_{L^\infty(\Gamma_h^n)} \lesssim h^q$. Combining these estimates we obtain the result of the lemma. $\square$

From the estimates in Lemmas 4.9, 4.10 and eq. (4.9) we obtain the following corollary.

COROLLARY 4.11. *For the consistency error the following uniform estimate holds*

$$|\mathcal{E}_C^n(\mathbf{v}_h)| \lesssim (\Delta t + h^q) \|\mathbf{v}_h\|_{U^n} \quad \text{for all } \mathbf{v}_h \in \mathbf{U}_h^n. \quad (4.47)$$

4.5. Error estimates. In this section we derive a bound for the velocity error $\mathbb{E}^n := \mathbf{u}^n - \mathbf{u}_h^n \in H^1(\mathcal{O}_\delta(\Gamma_h^n))$ and for the pressure error $r^n := p^n - p_h^n$, $r^n \in H^1(\omega_\Gamma^n)$. From (4.2) and (4.39) we get the error equation, for arbitrary $\mathbf{v}_h \in \mathbf{U}_h^n$,

$$\frac{1}{\Delta t} (\mathbb{E}^n - \mathbb{E}^{n-1}, \mathbf{P}_h \mathbf{v}_h)_{\Gamma_h^n} + c_n(\mathbb{E}^{n-1}; \mathbf{u}^n, \mathbf{v}_h) + a_n(\mathbf{u}_h^{n-1}; \mathbb{E}^n, \mathbf{v}_h) + b_n(r^n, \mathbf{v}_h) = \mathcal{E}_C^n(\mathbf{v}_h) + \delta_f^n(\mathbf{v}_h), \quad (4.48)$$

with the data error term $\delta_f^n(\mathbf{v}_h) := \int_{\Gamma^n} \mathbf{f}^n \cdot \mathbf{v}_h^\ell ds - \int_{\Gamma_h^n} \mathbf{f}^{e,n} \mathbf{P}_h \cdot \mathbf{v}_h^\ell ds_h$. Let $\mathbf{u}_I^n \in \mathbf{U}_h^n$ and $p_I^n \in Q_h^n$ be the nodal interpolants for $\mathbf{u}^n$ in $\mathcal{O}_\delta(\Gamma_h^n)$ and p^n in ω_Γ^n , respectively.

We proceed using standard techniques, based on splitting the error $\mathbb{E}^n$ into approximation and finite element parts,

$$\mathbb{E}^n = \underbrace{(\mathbf{u}^n - \mathbf{u}_I^n)}_{\mathbf{e}_I^n} + \underbrace{(\mathbf{u}_I^n - \mathbf{u}_h^n)}_{\mathbf{e}_h^n}.$$

Using $a_n(\mathbf{u}_h^{n-1}; \mathbb{E}^n, \mathbf{v}_h) = \widehat{a}_n(\mathbf{e}_I^n, \mathbf{v}_h) + \widehat{a}_n(\mathbf{e}_h^n, \mathbf{v}_h) + c_n(\mathbf{u}_h^{n-1}; \mathbb{E}^n, \mathbf{v}_h)$ equation (4.48) can be reformulated as

$$\frac{1}{\Delta t} (\mathbf{e}_h^n - \mathbf{e}_h^{n-1}, \mathbf{P}_h \mathbf{v}_h)_{\Gamma_h^n} + \widehat{a}_n(\mathbf{e}_h^n, \mathbf{v}_h) + b_n(p_I^n - p_h^n, \mathbf{v}_h) = \mathcal{E}_C^n(\mathbf{v}_h) + \mathcal{E}_I^n(\mathbf{v}_h) + \mathcal{C}^n(\mathbf{v}_h) + \delta_f^n(\mathbf{v}_h), \quad (4.49)$$

with the interpolation and nonlinear terms

$$\begin{aligned} \mathcal{E}_I^n(\mathbf{v}_h) &:= -\frac{1}{\Delta t} (\mathbf{e}_I^n - \mathbf{e}_I^{n-1}, \mathbf{P}_h \mathbf{v}_h)_{\Gamma_h^n} - \widehat{a}_n(\mathbf{e}_I^n, \mathbf{v}_h) - b_n(p^n - p_I^n, \mathbf{v}_h), \\ \mathcal{C}^n(\mathbf{v}_h) &:= -c_n(\mathbb{E}^{n-1}; \mathbf{u}^n, \mathbf{v}_h) - c_n(\mathbf{u}_h^{n-1}; \mathbb{E}^n, \mathbf{v}_h). \end{aligned}$$

An estimate for the interpolation terms is given in the following lemma.

LEMMA 4.12. *It holds*

$$|\mathcal{E}_I^n(\mathbf{v}_h)| \lesssim (h^{m+1} + h^q) \|\mathbf{v}_h\|_{U^n}, \quad \mathbf{v}_h \in \mathbf{U}_h^n. \quad (4.50)$$

Proof. For this to hold it is important to use the nodal interpolation. This operator has the key property that it interpolates velocity in the fixed (i.e., time independent) finite element space $\mathbf{U}_h$, which allows to shift time differentiation from outside the interpolation operator to inside. Using this approach, which is standard in the analysis of parabolic problems on Euclidean domains, one avoids a factor $\frac{1}{\Delta t}$ while handling the first term in $\mathcal{E}_I^n(\mathbf{v}_h)$, cf. [26, Lemma 12]. Estimates of the remaining two terms in $\mathcal{E}_I^n(\mathbf{v}_h)$ follow by standard arguments of applying the Cauchy-Schwarz and uniform trace FE interpolation inequalities [35]; see e.g. [20, § 5.3] $\square$

For the nonlinear terms a bound is given in the following lemma.

LEMMA 4.13. *The following holds, with $\mathbf{v}_h \in \mathbf{U}_h^n$,*

$$|\mathcal{C}^n(\mathbf{v}_h)| \lesssim (h^{m+2} + \|\mathbf{e}_h^{n-1}\|_{\Gamma_h^{n-1}} + h \|\mathbf{e}_h^{n-1}\|_{U^{n-1}} + \|\mathbf{u}_h^{n-1}\|_{U^{n-1}}^{\frac{1}{2}} (h^{m+1} + \|\mathbf{e}_h^n\|_{U^n})) \|\mathbf{v}_h\|_{U^n} \quad (4.51)$$

$$|\mathcal{C}^n(\mathbf{e}_h^n)| \lesssim (h^{m+2} + \|\mathbf{e}_h^{n-1}\|_{\Gamma_h^{n-1}} + h \|\mathbf{e}_h^{n-1}\|_{U^{n-1}} + \|\mathbf{u}_h^{n-1}\|_{U^{n-1}}^{\frac{1}{2}} h^{m+1}) \|\mathbf{e}_h^n\|_{U^n}. \quad (4.52)$$

Proof. We estimate, using (4.11),

$$\begin{aligned}
|c_n(\mathbb{E}^{n-1}; \mathbf{u}^n, \mathbf{v}_h)| &\lesssim (\|\mathbf{e}_I^{n-1}\|_{\Gamma_h^n} + \|\mathbf{e}_h^{n-1}\|_{\Gamma_h^n}) \|\mathbf{u}^n\|_{W^{1,\infty}(\Gamma_h^n)} \|\mathbf{v}_h\|_{U^n} \\
&\lesssim (h^{m+2} \|\mathbf{u}^n\|_{W^{m+2,\infty}} + \|\mathbf{e}_h^{n-1}\|_{\Gamma_h^n}) \|\mathbf{u}^n\|_{W^{1,\infty}} \|\mathbf{v}_h\|_{U^n} \\
&\lesssim (h^{m+2} + \|\mathbf{e}_h^{n-1}\|_{\Gamma_h^{n-1}} + \delta_{n-1}^{\frac{1}{2}} \|\mathbf{n}_h^{n-1} \cdot \nabla \mathbf{e}_h^{n-1}\|_{\mathcal{O}_\delta(\Gamma_h^{n-1})}) \|\mathbf{v}_h\|_{U^n} \\
&\lesssim (h^{m+2} + \|\mathbf{e}_h^{n-1}\|_{\Gamma_h^{n-1}} + h \|\mathbf{e}_h^{n-1}\|_{U^{n-1}}) \|\mathbf{v}_h\|_{U^n}.
\end{aligned}$$

For the other term in $\mathcal{C}^n(\mathbf{v}_h)$ we use the estimates in (4.21), (4.12), (4.8) and (4.9)

$$\begin{aligned}
|c_n(\mathbf{u}_h^{n-1}; \mathbb{E}^n, \mathbf{v}_h)| &\lesssim \|\mathbf{u}_h^{n-1}\|_{L^4(\Gamma_h^n)} (h^{m+1} + \|\mathbf{e}_h^n\|_{U^n}) \|\mathbf{v}_h\|_{U^n} \\
&\lesssim (\|\mathbf{u}_h^{n-1}\|_{L^4(\Gamma_h^{n-1})} + \|\mathbf{n}_h \cdot \nabla \mathbf{u}_h^{n-1}\|_{\mathcal{O}_\delta(\Gamma_h^{n-1})}) (h^{m+1} + \|\mathbf{e}_h^n\|_{U^n}) \|\mathbf{v}_h\|_{U^n} \\
&\lesssim (\|\mathbf{u}_h^{n-1}\|_{\Gamma_h^{n-1}}^{\frac{1}{2}} \|\mathbf{u}_h^{n-1}\|_{U^{n-1}}^{\frac{1}{2}} + h^{\frac{1}{2}} \|\mathbf{u}_h^{n-1}\|_{U^{n-1}}) (h^{m+1} + \|\mathbf{e}_h^n\|_{U^n}) \|\mathbf{v}_h\|_{U^n} \\
&\lesssim (\|\mathbf{u}_h^{n-1}\|_{\Gamma_h^{n-1}}^{\frac{1}{2}} + h^{\frac{1}{2}} \|\mathbf{u}_h^{n-1}\|_{U^{n-1}}^{\frac{1}{2}}) \|\mathbf{u}_h^{n-1}\|_{U^{n-1}}^{\frac{1}{2}} (h^{m+1} + \|\mathbf{e}_h^n\|_{U^n}) \|\mathbf{v}_h\|_{U^n}.
\end{aligned}$$

From the stability result in Theorem 4.7 and $h^2 \lesssim \Delta t$ we get $\|\mathbf{u}_h^{n-1}\|_{\Gamma_h^{n-1}} \leq C$ and $h^{\frac{1}{2}} \|\mathbf{u}_h^{n-1}\|_{U^{n-1}}^{\frac{1}{2}} \lesssim \Delta t^{\frac{1}{4}} \|\mathbf{u}_h^{n-1}\|_{U^{n-1}}^{\frac{1}{2}} \leq C$, with C specified in (4.22), in particular depending only on the data. Combining these results yields (4.51). If $\mathbf{v}_h = \mathbf{e}_h^n$ we use skew-symmetry and obtain

$$|c_n(\mathbf{u}_h^{n-1}; \mathbb{E}^n, \mathbf{e}_h^n)| = |c_n(\mathbf{u}_h^{n-1}; \mathbf{e}_I^n, \mathbf{e}_h^n)| \lesssim \|\mathbf{u}_h^{n-1}\|_{U^{n-1}}^{\frac{1}{2}} h^{m+1} \|\mathbf{e}_h^n\|_{U^n}.$$

□

Now we are prepared to prove the main result of the paper. Let $\mathbf{u}_h^0 = \mathbf{u}_I^0 \in \mathbf{U}_h^0$ be a suitable interpolant to $\mathbf{u}^0 \in \mathcal{O}(\Gamma_h^0)$.

THEOREM 4.14. *Let Assumption 4.1 be satisfied. Let $(\mathbf{u}, p)$ be the solution of (2.4). Let $\mathbf{u}_h^n, p_h^n, n = 1, \dots, N$, be the finite element solution of (4.2). For the errors $\mathbb{E}^n = \mathbf{u}_h^n - \mathbf{u}^n$, $r^n = p_h^n - p^n$ the following estimate holds:*

$$\|\mathbb{E}^n\|_{\Gamma_h^n}^2 + \frac{1}{8} \Delta t \sum_{k=1}^n (\|\mathbb{E}^k\|_{U^k}^2 + \rho_p \|\mathbf{n}_h \cdot \nabla r^k\|_{\omega_\Gamma^n}^2) \lesssim \exp(c t_n) (\Delta t^2 + h^{2(m+1)} + h^{2q}), \quad (4.53)$$

$$\Delta t \sum_{k=1}^n \|r^k\|_{1, \omega_\Gamma^n, h} \lesssim \exp(c t_n) (\Delta t + h^{m+1} + h^q), \quad (4.54)$$

with c depending on the problem data and independent of h , Δt , n and of the positions of the surface in the background mesh.

Proof. The arguments used to prove (4.53) largely repeat those used to show the stability result in Theorem 4.7 and involve estimates from Corollary 4.11, Lemmas 4.12 and 4.13 to bound the arising right-hand side terms. We set $\mathbf{v}_h = \Delta t \mathbf{e}_h^n$ in (4.49). This yields

$$\begin{aligned}
&\|\mathbf{P}_h \mathbf{e}_h^n\|_{\Gamma_h^n}^2 + \|\mathbf{P}_h \mathbf{e}_h^n - \mathbf{e}_h^{n-1}\|_{\Gamma_h^n}^2 + 2\Delta t \|\mathbf{e}_h^n\|_{U^n}^2 + 2\Delta t b_n(p_I^n - p_h^n, \mathbf{e}_h^n) \\
&\leq \|\mathbf{e}_h^{n-1}\|_{\Gamma_h^n}^2 + c\Delta t \|\mathbf{e}_h^n\|_{\Gamma_h^n}^2 + 2\Delta t (|\mathcal{E}_I^n(\mathbf{e}_h^n)| + |\mathcal{E}_C^n(\mathbf{e}_h^n)| + |\mathcal{C}^n(\mathbf{e}_h^n)| + |\delta_f^n(\mathbf{e}_h^n)|).
\end{aligned} \quad (4.55)$$

As in (4.25) we get

$$\|\mathbf{e}_h^n\|_{\Gamma_h^n}^2 \leq \|\mathbf{P}_h \mathbf{e}_h^n\|_{\Gamma_h^n}^2 + \frac{1}{2} \Delta t \|\mathbf{e}_h^n\|_{U^n}^2 + c\Delta t h^{2q} \|\mathbf{e}_h^n\|_{\Gamma_h^n}^2.$$

Using this, (4.10c) and the relation

$$b_n(p_I^n - p_h^n, \mathbf{e}_h^n) = b_n(p_I^n - p_h^n, \mathbf{u}_I^n) + \rho_p \|\mathbf{n}_h \cdot \nabla (p_h^n - p_I^n)\|_{\omega_\Gamma^n}^2 + \underbrace{\rho_p (\mathbf{n}_h \cdot \nabla p_I^n, \mathbf{n}_h \cdot \nabla (p_h^n - p_I^n))}_{\mathcal{E}_1(p_h^n - p_I^n)} \Big|_{\omega_\Gamma^n}$$

one gets

$$\begin{aligned}
& \|\mathbf{e}_h^n\|_{\Gamma_h^n}^2 + \|\mathbf{P}_h \mathbf{e}_h^n - \mathbf{e}_h^{n-1}\|_{\Gamma_h^n}^2 + \frac{3}{2} \Delta t \|\mathbf{e}_h^n\|_{U^n}^2 + 2 \Delta t \rho_p \|\mathbf{n}_h \cdot \nabla(p_h^n - p_I^n)\|_{\omega_\Gamma^n}^2 \\
& \leq \|\mathbf{e}_h^{n-1}\|_{\Gamma_h^{n-1}}^2 + c \Delta t (\|\mathbf{e}_h^{n-1}\|_{\Gamma_h^{n-1}}^2 + \|\mathbf{e}_h^n\|_{\Gamma_h^n}^2) + \frac{1}{2} \Delta t \|\mathbf{e}_h^{n-1}\|_{U^{n-1}}^2 \\
& \quad + 2 \Delta t (|\mathcal{E}_I^n(\mathbf{e}_h^n)| + |\mathcal{E}_C^n(\mathbf{e}_h^n)| + |\mathcal{C}^n(\mathbf{e}_h^n)| + |\delta_f^n(\mathbf{e}_h^n)| + |b_n(p_I^n - p_h^n, \mathbf{u}_I^n)| + |\mathcal{E}_1(p_h^n - p_I^n)|).
\end{aligned} \tag{4.56}$$

We estimate the different terms in the last line of (4.56). From (4.50) we get

$$|\mathcal{E}_I^n(\mathbf{e}_h^n)| \leq C(h^{m+1} + h^q) \|\mathbf{e}_h^n\|_{U^n} \leq \tilde{C}(h^{2m+2} + h^{2q}) + \frac{1}{16} \|\mathbf{e}_h^n\|_{U^n}^2. \tag{4.57}$$

For the second term we obtain, using (4.47):

$$|\mathcal{E}_C^n(\mathbf{e}_h^n)| \leq C(\Delta t + h^q) \|\mathbf{e}_h^n\|_{U^n} \leq \tilde{C}(\Delta t^2 + h^{2q}) + \frac{1}{16} \|\mathbf{e}_h^n\|_{U^n}^2. \tag{4.58}$$

For the third term we have the result (4.52):

$$\begin{aligned}
|\mathcal{C}^n(\mathbf{e}_h^n)| & \leq C(h^{m+2} + \|\mathbf{e}_h^{n-1}\|_{\Gamma_h^{n-1}} + h \|\mathbf{e}_h^{n-1}\|_{U^{n-1}} + h^{m+1} \|\mathbf{u}_h^{n-1}\|_{U^{n-1}}^{\frac{1}{2}}) \|\mathbf{e}_h^n\|_{U^n} \\
& \leq \tilde{C}(h^{2m+2} (1 + \|\mathbf{u}_h^{n-1}\|_{U^{n-1}}) + \|\mathbf{e}_h^{n-1}\|_{\Gamma_h^{n-1}}^2 + h^2 \|\mathbf{e}_h^{n-1}\|_{U^{n-1}}^2) + \frac{1}{16} \|\mathbf{e}_h^n\|_{U^n}^2.
\end{aligned} \tag{4.59}$$

For the data error term we have $|\delta_f^n(\mathbf{e}_h^n)| \leq Ch^q \|\mathbf{e}_h^n\|_{\Gamma_h^n} \leq \tilde{C}h^{2q} + \frac{1}{2} \|\mathbf{e}_h^n\|_{\Gamma_h^n}^2$. Furthermore we have, using $\mathbf{n} \cdot \nabla p^n = 0$ and $\|\mathbf{n}_h \cdot \nabla(p_I^n - p^n)\|_{\omega_\Gamma^n} \lesssim h^m \|p^n\|_{H^{m+1}(\omega_\Gamma^n)} \lesssim h^{m+\frac{1}{2}} \|p^n\|_{H^{m+1}(\Gamma^n)}$:

$$\begin{aligned}
|\mathcal{E}_1(p_h^n - p_I^n)| & = \rho_p \left| (\mathbf{n}_h \cdot \nabla(p_I^n - p^n), \mathbf{n}_h \cdot \nabla(p_h^n - p_I^n))_{\omega_\Gamma^n} + ((\mathbf{n}_h - \mathbf{n}) \cdot \nabla p^n, \mathbf{n}_h \cdot \nabla(p_h^n - p_I^n))_{\omega_\Gamma^n} \right| \\
& \leq C \rho_p (h^{2m+1} + h^{2q}) + \frac{1}{2} \rho_p \|\mathbf{n}_h \cdot \nabla(p_h^n - p_I^n)\|_{\omega_\Gamma^n}^2 \\
& \leq C(h^{2m+2} + h^{2q+1}) + \frac{1}{2} \rho_p \|\mathbf{n}_h \cdot \nabla(p_h^n - p_I^n)\|_{\omega_\Gamma^n}^2.
\end{aligned} \tag{4.60}$$

It remains to estimate the term $|b_n(p_I^n - p_h^n, \mathbf{u}_I^n)|$ in the last line of (4.56). We use the splitting

$$|b_n(p_I^n - p_h^n, \mathbf{u}_I^n)| \leq |b_n(p_I^n - p_h^n, \mathbf{u}_I^n - \mathbf{u})| + |b_n(p_I^n - p_h^n, \mathbf{u})|$$

For the first term on the right-hand side we get

$$\begin{aligned}
|b_n(p_I^n - p_h^n, \mathbf{u}_I^n - \mathbf{u})| & \lesssim h^{m+2} \|\nabla_{\Gamma_h}(p_I^n - p_h^n)\|_{\Gamma_h^n} \lesssim h^{m+\frac{3}{2}} \|\nabla(p_I^n - p_h^n)\|_{\omega_\Gamma^n} \\
& \lesssim h^{m+1} \|p_I^n - p_h^n\|_{1, \omega_\Gamma^n, h}.
\end{aligned} \tag{4.61}$$

For the second term we define $q_h := p_I^n - p_h^n \in H^1(\Gamma_h)$ and note the following. For the lifting of this function we have $\int_\Gamma \nabla_\Gamma q_h^\ell \cdot \mathbf{u} ds = 0$. Recall the transformation formula $\nabla_{\Gamma_h} q_h(\mathbf{y}) = \mathbf{P}_h(\mathbf{y})(\mathbf{P}(\mathbf{y}) - d(\mathbf{y})\mathbf{H}(\mathbf{y}))\nabla q^\ell(\mathbf{p}(\mathbf{y}))$. Using this and $\mathbf{P}\mathbf{u} = \mathbf{u}$ we get

$$b_n(q_h, \mathbf{u}) = \int_{\Gamma_h^n} \mathbf{P} \mathbf{P}_h (\mathbf{P} - d\mathbf{H}) \nabla q^\ell(\mathbf{p}(\cdot)) \cdot \mathbf{u} ds_h.$$

Using $\|d\|_{L^\infty(\Gamma_h^n)} \lesssim h^{q+1}$ and $\|\mathbf{P} \mathbf{P}_h \mathbf{P} - \mathbf{P}\|_{L^\infty(\Gamma_h^n)} = \|\mathbf{P} \mathbf{n}_h \mathbf{n}_h^T \mathbf{P}\|_{L^\infty(\Gamma_h^n)} \lesssim h^{2q}$ we get

$$\left| b_n(q_h, \mathbf{u}) - \int_{\Gamma_h^n} \mathbf{P} \nabla q_h^\ell(\mathbf{p}(\cdot)) \cdot \mathbf{u} ds_h \right| \lesssim h^{q+1} \|\nabla_{\Gamma_h} q_h\|_{\Gamma_h^n}.$$

Using $\int_\Gamma \nabla_\Gamma q_h^\ell \cdot \mathbf{u} ds = 0$ and the bound for the change in surface measure in (A.1) we obtain $\left| \int_{\Gamma_h^n} \mathbf{P} \nabla q_h^\ell(\mathbf{p}(\cdot)) \cdot \mathbf{u} ds_h \right| \lesssim h^{q+1} \|\nabla_{\Gamma_h} q_h\|_{\Gamma_h^n}$. Combining these results yields the estimate

$$|b_n(p_I^n - p_h^n, \mathbf{u})| \lesssim h^{q+1} \|\nabla_{\Gamma_h}(p_I^n - p_h^n)\|_{\Gamma_h^n} \lesssim h^{q+\frac{1}{2}} \|\nabla(p_I^n - p_h^n)\|_{\omega_\Gamma^n} \lesssim h^q \|p_I^n - p_h^n\|_{1, \omega_\Gamma^n, h}.$$

Collecting these results, we see that the term between brackets (...) in the last line in (4.56) can be bounded by

$$\begin{aligned} & \frac{3}{16} \|\mathbf{e}_h^n\|_{U^n}^2 + \frac{1}{2} \rho_p \|\mathbf{n}_h \cdot \nabla(p_h^n - p_I^n)\|_{\omega_\Gamma^n}^2 + C \|\mathbf{e}_h^{n-1}\|_{\Gamma_h^{n-1}}^2 + Ch^2 \|\mathbf{e}_h^{n-1}\|_{U^{n-1}}^2 \\ & + \tilde{C}(\Delta t^2 + h^{2q} + h^{2m+2}(1 + \|\mathbf{u}_h^{n-1}\|_{U^{n-1}}) + (h^{m+1} + h^q) \|p_I^n - p_h^n\|_{1, \omega_\Gamma^n, h}). \end{aligned} \quad (4.62)$$

The first two terms in (4.62) can be shifted to the left in (4.56). We assume that h is sufficiently small such that $Ch^2 \leq \frac{1}{16}$. Then we get

$$\begin{aligned} & \|\mathbf{e}_h^n\|_{\Gamma_h^n}^2 + \|\mathbf{P}_h \mathbf{e}_h^n - \mathbf{e}_h^{n-1}\|_{\Gamma_h^n}^2 + \Delta t \|\mathbf{e}_h^n\|_{U^n}^2 + \Delta t \rho_p \|\mathbf{n}_h \cdot \nabla(p_h^n - p_I^n)\|_{\omega_\Gamma^n}^2 \\ & \leq \|\mathbf{e}_h^{n-1}\|_{\Gamma_h^{n-1}}^2 + c \Delta t (\|\mathbf{e}_h^{n-1}\|_{\Gamma_h^{n-1}}^2 + \|\mathbf{e}_h^n\|_{\Gamma_h^n}^2) + \frac{5}{8} \Delta t \|\mathbf{e}_h^{n-1}\|_{U^{n-1}}^2 \\ & + C \Delta t (\Delta t^2 + h^{2q} + h^{2m+2}(1 + \|\mathbf{u}_h^{n-1}\|_{U^{n-1}}) + (h^{m+1} + h^q) \|p_I^n - p_h^n\|_{1, \omega_\Gamma^n, h}). \end{aligned} \quad (4.63)$$

For deriving a bound for $\|p_I^n - p_h^n\|_{1, \omega_\Gamma^n, h}$ we use the same approach as in the proof of Theorem 4.8. For the given $q_h := p_I^n - p_h^n \in Q_h^n$ we take the corresponding $\hat{\mathbf{v}}_h \in \mathbf{U}_h^n$ as in (4.28), for which the estimates (4.29) and (4.30) hold. We take $\mathbf{v}_h = \hat{\mathbf{v}}_h$ in (4.49). For the right-hand side in (4.49) we introduce the notation $G_h(\mathbf{v}_h) := \mathcal{E}_C^n(\mathbf{v}_h) + \mathcal{E}_I^n(\mathbf{v}_h) + \mathcal{C}^n(\mathbf{v}_h) + \delta_f^n(\mathbf{v}_h)$. As in (4.34) we then get

$$\begin{aligned} \|p_I^n - p_h^n\|_{1, \omega_\Gamma^n, h} & \lesssim \frac{h}{\Delta t} \|\mathbf{P}_h \mathbf{e}_h^n - \mathbf{e}_h^{n-1}\|_{\Gamma_h^n} + \rho_p^{\frac{1}{2}} \|\mathbf{n}_h \cdot \nabla(p_I^n - p_h^n)\|_{\omega_\Gamma^n} \\ & + \|\mathbf{e}_h^{n-1}\|_{U^{n-1}}^2 + \|\mathbf{e}_h^n\|_{U^n}^2 + \|\mathbf{e}_h^n\|_{U^n} + |G_h(\hat{\mathbf{v}}_h)| \|p_I^n - p_h^n\|_{1, \omega_\Gamma^n, h}^{-1}. \end{aligned} \quad (4.64)$$

We estimate the terms in $|G_h(\hat{\mathbf{v}}_h)|$:

$$\begin{aligned} |\mathcal{E}_C^n(\hat{\mathbf{v}}_h)| & \lesssim (\Delta t + h^q) \|\hat{\mathbf{v}}_h\|_{U^n} \lesssim (\Delta t + h^q) \|p_I^n - p_h^n\|_{1, \omega_\Gamma^n, h} \\ |\mathcal{E}_I^n(\hat{\mathbf{v}}_h)| & \lesssim (h^{m+1} + h^q) \|\hat{\mathbf{v}}_h\|_{U^n} \lesssim (h^{m+1} + h^q) \|p_I^n - p_h^n\|_{1, \omega_\Gamma^n, h} \\ |\mathcal{C}^n(\hat{\mathbf{v}}_h)| & \lesssim (h^{m+2} + \|\mathbf{e}_h^{n-1}\|_{\Gamma_h^{n-1}} + h \|\mathbf{e}_h^{n-1}\|_{U^{n-1}} + \|\mathbf{u}_h^{n-1}\|_{U^{n-1}}^{\frac{1}{2}} (h^{m+1} + \|\mathbf{e}_h^n\|_{U^n})) \|\hat{\mathbf{v}}_h\|_{U^n} \\ & \lesssim (h^{m+2} + \|\mathbf{e}_h^{n-1}\|_{U^{n-1}} + \|\mathbf{u}_h^{n-1}\|_{U^{n-1}}^{\frac{1}{2}} (h^{m+1} + \|\mathbf{e}_h^n\|_{U^n})) \|p_I^n - p_h^n\|_{1, \omega_\Gamma^n, h} \\ |\delta_f^n(\hat{\mathbf{v}}_h)| & \lesssim h^q \|\hat{\mathbf{v}}_h\|_{\Gamma_h^n} \lesssim h^{q+1} \|p_I^n - p_h^n\|_{1, \omega_\Gamma^n, h}. \end{aligned}$$

Thus we obtain

$$\begin{aligned} \|p_I^n - p_h^n\|_{1, \omega_\Gamma^n, h} & \lesssim \frac{h}{\Delta t} \|\mathbf{P}_h \mathbf{e}_h^n - \mathbf{e}_h^{n-1}\|_{\Gamma_h^n} + \rho_p^{\frac{1}{2}} \|\mathbf{n}_h \cdot \nabla(p_I^n - p_h^n)\|_{\omega_\Gamma^n} \\ & + \|\mathbf{e}_h^{n-1}\|_{U^{n-1}}^2 + \|\mathbf{e}_h^n\|_{U^n}^2 + \|\mathbf{e}_h^{n-1}\|_{U^{n-1}} + \|\mathbf{e}_h^n\|_{U^n} \\ & + \Delta t + h^{m+1} + h^q + \|\mathbf{u}_h^{n-1}\|_{U^{n-1}}^{\frac{1}{2}} (h^{m+1} + \|\mathbf{e}_h^n\|_{U^n}). \end{aligned} \quad (4.65)$$

Now note that

$$C \Delta t (h^{m+1} + h^q) \|\mathbf{u}_h^{n-1}\|_{U^{n-1}}^{\frac{1}{2}} (h^{m+1} + \|\mathbf{e}_h^n\|_{U^n}) \leq \frac{\Delta t}{16} \|\mathbf{e}_h^n\|_{U^n}^2 + \tilde{C} \Delta t (h^{2q} + h^{2m+2}) (1 + \|\mathbf{u}_h^{n-1}\|_{U^{n-1}}).$$

Using this, (4.65) and $h^2 \lesssim \Delta t$ we obtain (for h sufficiently small) for the last term in (4.63):

$$\begin{aligned} C \Delta t (h^{m+1} + h^q) \|p_I^n - p_h^n\|_{1, \omega_\Gamma^n, h} & \leq \frac{1}{2} \|\mathbf{P}_h \mathbf{e}_h^n - \mathbf{e}_h^{n-1}\|_{\Gamma_h^n}^2 + \frac{1}{2} \Delta t \rho_p \|\mathbf{n}_h \cdot \nabla(p_I^n - p_h^n)\|_{\omega_\Gamma^n}^2 \\ & + \frac{1}{8} \Delta t (\|\mathbf{e}_h^{n-1}\|_{U^{n-1}}^2 + \|\mathbf{e}_h^n\|_{U^n}^2) + \tilde{C} \Delta t (\Delta t^2 + (h^{2q} + h^{2m+2})(1 + \|\mathbf{u}_h^{n-1}\|_{U^{n-1}})). \end{aligned}$$

Substituting this in (4.63) and shifting terms from the right to the left-hand side we get

$$\begin{aligned} & \|\mathbf{e}_h^n\|_{\Gamma_h^n}^2 + \frac{1}{2} \|\mathbf{P}_h \mathbf{e}_h^n - \mathbf{e}_h^{n-1}\|_{\Gamma_h^n}^2 + \frac{7}{8} \Delta t \|\mathbf{e}_h^n\|_{U^n}^2 + \frac{1}{2} \Delta t \rho_p \|\mathbf{n}_h \cdot \nabla(p_h^n - p_I^n)\|_{\omega_\Gamma^n}^2 \\ & \leq \|\mathbf{e}_h^{n-1}\|_{\Gamma_h^{n-1}}^2 + c_1 \Delta t (\|\mathbf{e}_h^{n-1}\|_{\Gamma_h^{n-1}}^2 + \|\mathbf{e}_h^n\|_{\Gamma_h^n}^2) + \frac{6}{8} \Delta t \|\mathbf{e}_h^{n-1}\|_{U^{n-1}}^2 \\ & + c_2 \Delta t (\Delta t^2 + (h^{2q} + h^{2m+2})(1 + \|\mathbf{u}_h^{n-1}\|_{U^{n-1}})). \end{aligned} \quad (4.66)$$

We drop the term $\frac{1}{2}\|\mathbf{P}_h \mathbf{e}_h^n - \mathbf{e}_h^{n-1}\|_{\Gamma_h^n}^2$, sum over $n = 1, \dots, k$, apply the discrete Gronwall inequality and use the discrete stability estimate $\Delta t \sum_{n=1}^k \|\mathbf{u}_h^{n-1}\|_{U^{n-1}} \lesssim (\Delta t \sum_{n=1}^k \|\mathbf{u}_h^{n-1}\|_{U^{n-1}}^2)^{\frac{1}{2}} \leq C$ to get

$$Q_{e,k} := \|\mathbf{e}_h^k\|_{\Gamma_h^k}^2 + \frac{1}{8}\Delta t \sum_{n=1}^k (\|\mathbf{e}_h^n\|_{U^n}^2 + \rho_p \|\mathbf{n}_h \cdot \nabla(p_h^n - p_I^n)\|_{\omega_\Gamma^n}^2) \lesssim \exp(ct_k)(\Delta t^2 + h^{2m+2} + h^{2q}). \quad (4.67)$$

The triangle inequality and standard FE interpolation properties give

$$\begin{aligned} & \|\mathbb{E}^k\|_{\Gamma_h^k}^2 + \frac{1}{8}\Delta t \sum_{n=1}^k (\|\mathbb{E}^n\|_{U^n}^2 + \rho_p \|\mathbf{n}_h \cdot \nabla r^n\|_{\omega_\Gamma^n}^2) \\ & \leq 2Q_{e,k} + 2\|\mathbf{e}_h^k\|_{\Gamma_h^k}^2 + \frac{1}{4}\Delta t \sum_{n=1}^k (\|\mathbf{e}_h^n\|_{U^n}^2 + \rho_p \|\mathbf{n}_h \cdot \nabla(p^n - p_I^n)\|_{\omega_\Gamma^n}^2) \lesssim Q_{e,k} + Ch^{2m+2}. \end{aligned}$$

This completes the proof of (4.53).

We now derive a pressure error bound. The approach is similar to the one used in the proof of Theorem 4.8. First note that if we do not drop the term $\frac{1}{2}\|\mathbf{P}_h \mathbf{e}_h^n - \mathbf{e}_h^{n-1}\|_{\Gamma_h^n}^2$ when going from (4.66) to (4.67) we get

$$\tilde{Q}_{e,k} := Q_{e,k} + \frac{1}{2} \sum_{n=1}^k \|\mathbf{P}_h \mathbf{e}_h^n - \mathbf{e}_h^{n-1}\|_{\Gamma_h^n}^2 \lesssim \exp(ct_k)(\Delta t^2 + h^{2m+2} + h^{2q}) =: F_k. \quad (4.68)$$

From (4.65) we obtain, with the same arguments as in the proof of Theorem 4.8:

$$\Delta t \sum_{n=1}^k \|p_I^n - p_h^n\|_{1, \omega_\Gamma^n, h} \lesssim \left(\frac{h}{\sqrt{\Delta t}} + 1\right) \tilde{Q}_{e,k}^{\frac{1}{2}} + Q_{e,k} + \Delta t + h^{m+1} + h^q + \Delta t \sum_{n=1}^k \|\mathbf{u}_h^{n-1}\|_{U^{n-1}}^{\frac{1}{2}} (h^{m+1} + \|\mathbf{e}_h^n\|_{U^n})$$

Similar to above, using discrete stability for the last term we get

$$\Delta t \sum_{n=1}^k \|\mathbf{u}_h^{n-1}\|_{U^{n-1}}^{\frac{1}{2}} (h^{m+1} + \|\mathbf{e}_h^n\|_{U^n}) \lesssim h^{m+1} + Q_{e,k}^{\frac{1}{2}}.$$

Recall that $h^2 \lesssim \Delta t$. Hence, we obtain

$$\Delta t \sum_{n=1}^k \|p_I^n - p_h^n\|_{1, \omega_\Gamma^n, h} \lesssim \tilde{Q}_{e,k}^{\frac{1}{2}} + \tilde{Q}_{e,k} + \Delta t + h^{m+1} + h^q \lesssim F_k^{\frac{1}{2}} + F_k \lesssim F_k^{\frac{1}{2}}.$$

Finally we combine this result with the interpolation error estimate

$$\|p_I^n - p^n\|_{1, \omega_\Gamma^n, h} = h^{\frac{1}{2}} \|\nabla(p_I^n - p^n)\|_{\omega_\Gamma^n} \lesssim h^{m+\frac{1}{2}} \|p\|_{H^m(\omega_\Gamma^n)} \lesssim h^{m+1} \|p\|_{H^m(\Gamma)}$$

which then proves the result (4.54). $\square$

Both bounds in (4.53) and in (4.54) are optimal in terms of convergence order in Δt and h .

5. Numerical experiments. In this section, we present the results of numerical experiments with the proposed method. We consider a simple geometry evolution, namely a slowly moving sphere which does not change its shape. The exact level set function is given by $\phi(t, \mathbf{x}) = \|\mathbf{x} - \mathbf{g}(t)\|^2 - 1$, with $\mathbf{g}(t) = (0.2t, 0, 0)^T$. The normal velocity can be determined using $w_N = -\frac{\partial \phi}{\partial t} / \|\nabla \phi\|$. The exact tangential velocity $\mathbf{u}_T$ is chosen as $\mathbf{u}_T = \mathbf{P}(\mathbf{n} \times \nabla_\Gamma \psi)$ with the stream function $\psi = xy - 2t$. Hence, $\text{div}_\Gamma \mathbf{u}_T = 0$ holds. The pressure is taken as $p = (x - 0.2t)y + z$, which satisfies $\int_{\Gamma(t)} p \, ds = 0$. The time interval is $I = [0, 2]$ and the evolving sphere is embedded in the domain $\Omega = [-\frac{4}{3}, \frac{10}{3}] \times [-\frac{4}{3}, \frac{4}{3}]^2$. Using Maple [27], we calculate the corresponding exact force terms $\mathbf{f}$ and f .

The method introduced in this paper is implemented in Netgen/NGSolve [1] with the add-on ngsxfem [25]. The geometry approximation is based on an Oswald type quasi-interpolation in the finite element space $V_{h,q}$ (cf. (3.6)), denoted by I_h^q :

$$\phi_h^n = I_h^q(\phi(t_n, \cdot)), \quad \tilde{\phi}_h^n = I_h^{q+1}(\phi(t_n, \cdot)), \quad \mathbf{n}_h^n := \frac{\nabla \phi_h^n}{\|\nabla \phi_h^n\|}, \quad \tilde{\mathbf{n}}_h^n := \frac{\nabla \tilde{\phi}_h^n}{\|\nabla \tilde{\phi}_h^n\|}.$$

For the approximation of the Weingarten mapping we use $\mathbf{H}_h^n = \nabla_{\Gamma_h} \bar{\mathbf{n}}_h^n$, with $\bar{\mathbf{n}}_h^n = I_h^q \mathbf{n}_h^n$ the (Oswald type) componentwise quasi-interpolation of $\mathbf{n}_h^n$ in the finite element space $(V_{h,q})^3$. In the experiments below we use the Taylor-Hood pair (cf. (3.7)) with $m = 1$, $q = 1$ or $q = 2$, and $m = 2$, $q = 3$. For the cases $q = 2$ and $q = 3$ the zero level of ϕ_h^n is not easy to determine, cf. Remark 3.1, and we use the parametric finite element technique developed in [12, 24]. For the time discretization, we use the BDF1, BDF2, or BDF3 scheme. We start with a uniform tetrahedral triangulation of Ω with maximum mesh size $h_0 = 0.5$. In each refinement step, the mesh size is halved. As the initial time step size, we chose $\Delta t_0 = 0.2$. In each spatial refinement step, we halve the time step size in the BDF2- and BDF3-scheme and we divide it by four when the BDF1 scheme is used.

The parameter choices we use are consistent with (3.19a)–(3.19f). For the stabilization and the penalization we take $\rho_u = h^{-1}$, $\rho_p = h$ and $\tau = h^{-2}$. To define the extension area $\mathcal{O}_\delta(\Gamma_h)$ the approximative level set function ϕ_h (instead of the distance function) is used in (3.14). We choose the thickness of the narrow band as $\delta_n = \tilde{c}_{\delta_n} R \Delta t \|w_N\|_{\infty, I_n}$, with $R = 1, 2, 3$, for BDF1, BDF2, and BDF3, respectively. The default value is $\tilde{c}_{\delta_n} = 2.5$ and we check in each step whether δ_n is sufficiently large such that $\omega_\Gamma^n \subset \mathcal{O}_\delta(\Gamma_h^{n-i})$, $i = 1, \dots, R$, holds, cf. (3.20).

In case of the BDF1 scheme we use the first order accurate linearization $(\nabla_{\Gamma_h} \mathbf{u}_h^n) \mathbf{u}_h^{n-1}$, cf. Section 3. For the BDF2- and BDF3-scheme we consider the second and third order accurate linearization $(\nabla_{\Gamma_h} \mathbf{u}_h^n) \mathbf{u}_h^n \approx (\nabla_{\Gamma_h} \mathbf{u}_h^n) (2\mathbf{u}_h^{n-1} - \mathbf{u}_h^{n-2})$ and $(\nabla_{\Gamma_h} \mathbf{u}_h^n) \mathbf{u}_h^n \approx (\nabla_{\Gamma_h} \mathbf{u}_h^n) (3\mathbf{u}_h^{n-1} - 3\mathbf{u}_h^{n-2} + \mathbf{u}_h^{n-3})$ respectively.

The error quantities we use are defined as follows. The velocity and pressure errors in each time step are denoted by $\mathbb{E}^n = \mathbf{u}_h^n - \mathbf{u}(t_n)$ and $r^n := p_h^n - p(t_n)$ for $n = 1, \dots, N$. We consider the following error quantities:

$$\begin{aligned} \left(E_{\mathbf{u}}^{L^2 L^2}\right)^2 &:= \Delta t \sum_{n=1}^N \|\mathbb{E}^n\|_{L^2(\Gamma_h^n)}^2, & \left(E_{\mathbf{u}}^{L^2 U}\right)^2 &:= \Delta t \sum_{n=1}^N \|\mathbb{E}^n\|_{U^n}^2, \\ \left(E_p^{L^2 L^2}\right)^2 &:= \Delta t \sum_{n=1}^N \|r^n\|_{L^2(\Gamma_h^n)}^2, & E_p^{L^1 H^1 \omega_\Gamma^n} &:= \Delta t \sum_{n=1}^N \|r^n\|_{1, \omega_\Gamma^n, h}. \end{aligned}$$

Recall that the norm $\|\cdot\|_{1, \omega_\Gamma^n, h}$ is closely related (cf. discussion below (4.27)) to the *scaled* $H^1(\Gamma_h^n)$ norm $h \|\nabla \cdot\|_{\Gamma_h^n}$, consistent with the error term h^{m+1} in (4.54).

We start with results for the BDF1 method, presented in the Figures 5.1 and 5.2. The first order convergence for the error $E_{\mathbf{u}}^{L^2 U}$ in Figure 5.1(a) shows that the geometric error bound h^q for this term in (4.53) is sharp. The result for the pressure error $E_p^{L^1 H^1 \omega_\Gamma^n}$ in Figure 5.1(b) indicates that the geometric error term h^q in (4.54) might be not sharp. The results for $E_{\mathbf{u}}^{L^2 U}$ and $E_p^{L^1 H^1 \omega_\Gamma^n}$ in Figure 5.2 confirm the second order convergence predicted by Theorem 4.14. Although we did not derive a bound for the error $E_{\mathbf{u}}^{L^2 L^2}$, this error has the expected (approximately) second order convergence in Figure 5.2(a), due to the term $\Delta t \sim h^2$ in the time discretization error.

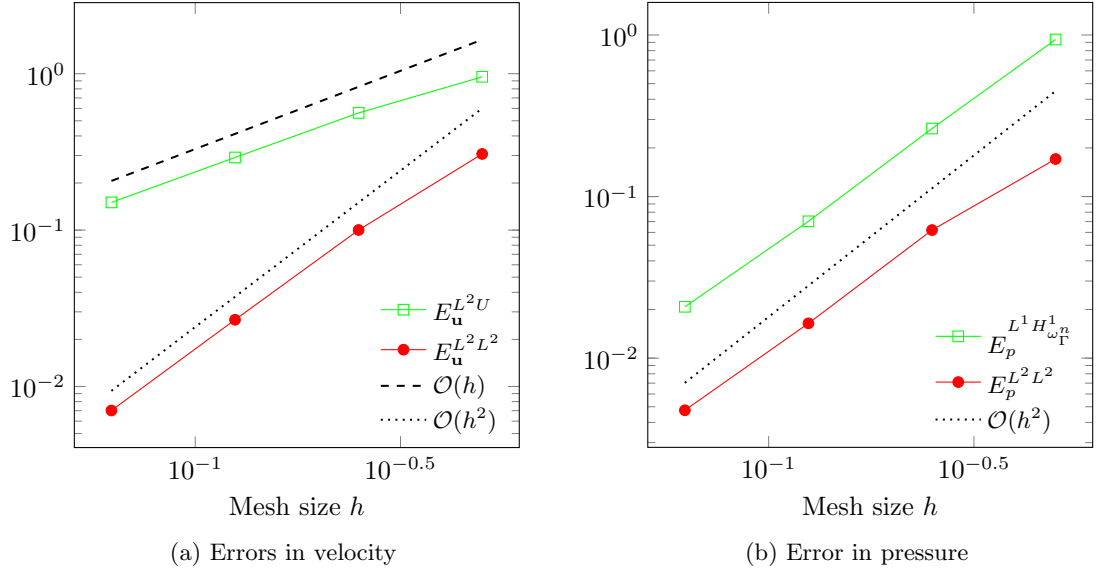


Fig. 5.1: $m = 1$, $q = 1$, BDF = 1, $\Delta t \sim h^2$

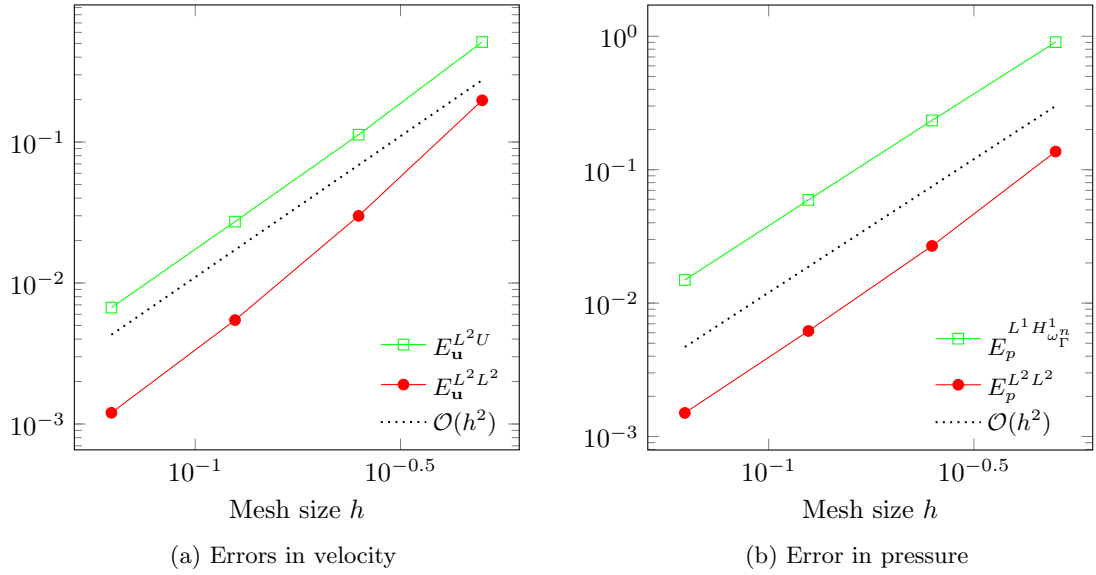


Fig. 5.2: $m = 1$, $q = 2$, BDF = 1, $\Delta t \sim h^2$

Results for the BDF2 method are shown in Figures 5.3, and 5.4. We claim that our theoretical analysis can be extended to this case, in which then the time discretization error terms Δt in (4.53)-(4.54) would be replaced by Δt^2 . In Figure 5.3(a) we observe again that for $E_u^{L^2 U}$ we only have first order convergence due to the h^1 geometric error term. The results in Figure 5.4 show close to second order convergence for all error quantities (as in Figure 5.2).

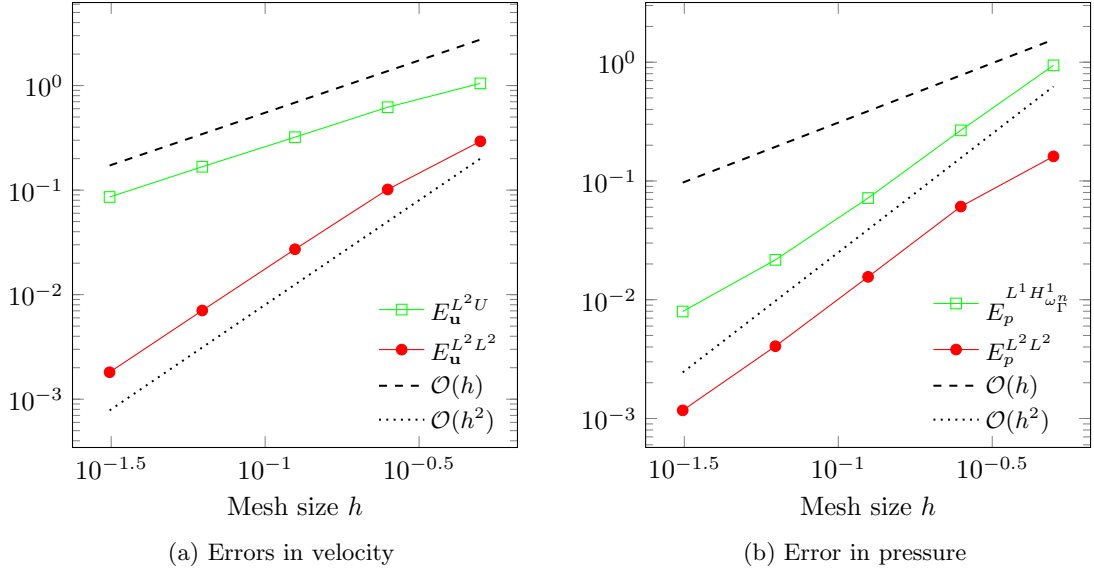


Fig. 5.3: $m = 1$, $q = 1$, BDF = 2, $\Delta t \sim h$

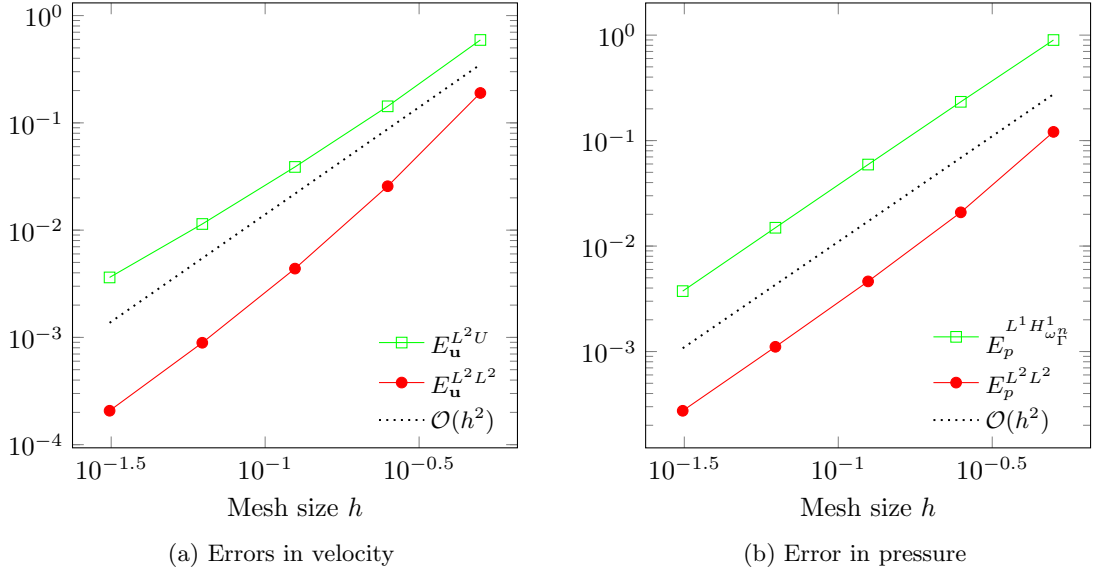


Fig. 5.4: $m = 1$, $q = 2$, BDF = 2, $\Delta t \sim h$

We also consider the higher-order Taylor-Hood pair with $m = 2$. We combine it with a higher geometric order ($q = 3$) and the higher order time integration scheme BDF3. All parameters are taken the same as in the experiments above. The results are shown in figure 5.5. We observe optimal third order convergence in the energy norm for the velocity and in the scaled $H^1(\Gamma_h^n)$ -norm for the pressure.

Appendix A. Proof of (4.6), (4.7) and (4.9). Define $\mathbf{B} = \mathbf{B}(t, \mathbf{x}) = \mathbf{P}(\mathbf{I} - d\mathbf{H})\mathbf{P}_h$ for $\mathbf{x} \in \Gamma_h(t)$. Geometry approximation assumptions from section 3.2.1 imply that $\mathbf{B}$ is invertible on the range of $\mathbf{P}$ (for h small enough, cf. (3.19a)) and the following uniform in time and discretization

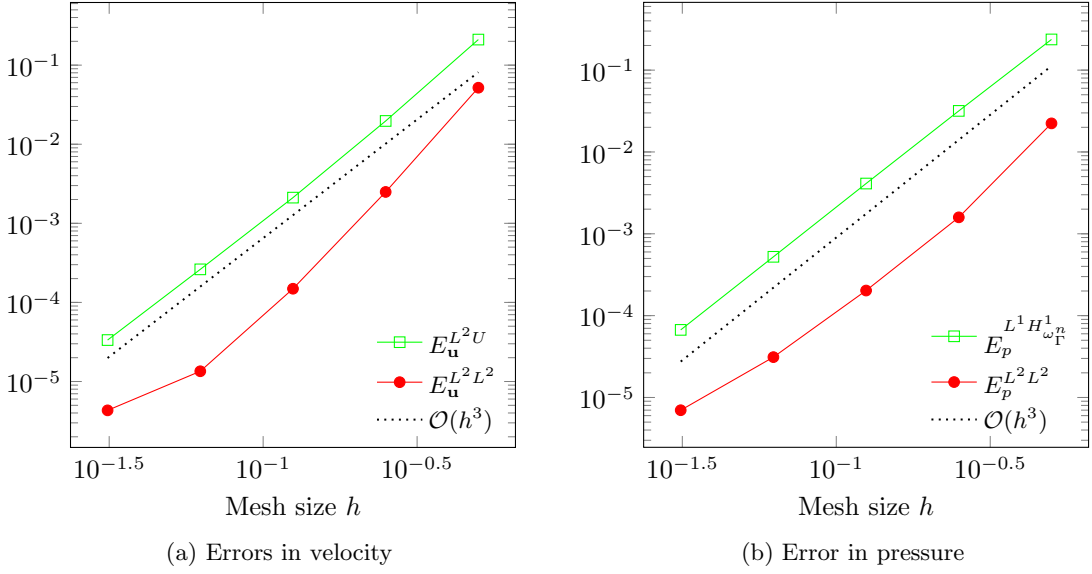


Fig. 5.5: $m = 2$, $q = 3$, BDF = 3, $\Delta t \sim h$

parameters estimates hold (see [16]):

$$\begin{aligned} \|\mathbf{B}\|_{L^\infty(\Gamma_h)} &\leq C, \quad \|\mathbf{P}_h \mathbf{B}^{-1} \mathbf{P}\|_{L^\infty(\Gamma_h)} \leq C, \\ \|\mathbf{P}_h \mathbf{B}^{-1} \mathbf{P} - \mathbf{P}_h \mathbf{P}\|_{L^\infty(\Gamma_h)} &\leq Ch^{q+1}, \quad \|1 - |\det(\mathbf{B})|\|_{L^\infty(\Gamma_h)} \leq Ch^{q+1}. \end{aligned} \quad (\text{A.1})$$

These bounds and the identities

$$d\Gamma = |\det(\mathbf{B})| d\Gamma_h \quad \text{and} \quad \nabla_\Gamma u^\ell(\mathbf{x}) = \mathbf{P} \mathbf{B}^{-T} \nabla_{\Gamma_h} u(\mathbf{x}), \quad \mathbf{x} \in \Gamma_h, \quad u \in H^1(\Gamma_h), \quad (\text{A.2})$$

and the vector analogues, see [20, section 5.4.1], imply the norm equivalences in (4.6). Same uniform estimates (A.1) together with (3.12) and (3.13) yield (cf. [20, Lemma 5.14]) for any $\mathbf{v} \in H^1(\Gamma_h)^3$,

$$\begin{aligned} \|(\nabla_\Gamma \mathbf{v}^\ell)^e - \nabla_{\Gamma_h} \mathbf{v}\|_{\Gamma_h^n} &\lesssim h^q \|\mathbf{v}\|_{H^1(\Gamma_h^n)}, \\ \|E_s(\mathbf{P} \mathbf{v}^\ell)^e - E_h(\mathbf{P}_h \mathbf{v})\|_{L^2(\Gamma_h)} &\lesssim h^q (\|\mathbf{v}\|_{H^1(\Gamma_h)} + h^{-1} \|\mathbf{v} \cdot \mathbf{n}_h\|_{L^2(\Gamma_h)}). \end{aligned} \quad (\text{A.3})$$

Now the uniform Korn-type inequality follows from the uniform in time Korn inequality on $\Gamma(t)$ from [37, Lemma 3.2], estimates (4.6), (A.3) and FE trace and inverse inequalities by the arguments in [20, Lemma 5.16]. Finally, (4.7) follows from the uniform in time interpolation inequality on $\Gamma(t)$ from [37, Lemma 3.4] and (4.6).

Appendix B. Proof of Lemma 4.2. To prove Lemma 4.2 we first note that due to condition (3.20) we have $\Gamma_h^n \subset \mathcal{U}_\delta(\Gamma_h^{n-1}) \subset \mathcal{O}_\delta(\Gamma_h^{n-1}) \subset \mathcal{O}(\Gamma_h^{n-1})$, where the last inclusion holds for $\Delta t, h$ small enough, see condition (3.19a). Hence, for $v \in L^2(\Gamma_h^{n-1})$ we define a lift $v^\ell \in L^2(\mathcal{U}_\delta(\Gamma_h^{n-1}))$ as in (4.4), with n replaced by $n - 1$. We use the splitting

$$\|v_h\|_{\Gamma_h^n}^2 = \int_{\Gamma_h^n} (|v_h|^2 - |v_h^\ell|^2) ds_h + \|v_h^\ell\|_{\Gamma_h^n}^2. \quad (\text{B.1})$$

For the second term on the right-hand side we apply the estimate $\|v_h^\ell\|_{\Gamma_h^n} \lesssim \|v_h\|_{\Gamma_h^{n-1}}$ (see, e.g. [26,

Lemma 6]). For the first term we obtain (we abbreviate $U_\delta = U_\delta(\Gamma_h^{n-1})$ here):

$$\begin{aligned}
& \int_{\Gamma_h^n} (|v_h|^2 - |v_h^\ell|^2) ds_h \lesssim \int_{U_\delta} |\mathbf{n}^{n-1} \cdot \nabla (|v_h|^2 - |v_h^\ell|^2)| dx & (|v_h|^2 = |v_h^\ell|^2 \text{ on } \Gamma_h^{n-1}) \\
& \leq \int_{U_\delta} |\mathbf{n}_h^{n-1} \cdot \nabla |v_h|^2| dx + \int_{U_\delta} |(\mathbf{n}^{n-1} - \mathbf{n}_h^{n-1}) \cdot \nabla |v_h|^2| dx & (\text{as } \mathbf{n}^{n-1} \cdot \nabla |v_h^\ell|^2 = 0) \\
& \lesssim \|v_h\|_{U_\delta} \|\mathbf{n}_h^{n-1} \cdot \nabla v_h\|_{U_\delta} + \|\mathbf{n}^{n-1} - \mathbf{n}_h^{n-1}\|_{L^\infty(U_\delta)} \|\nabla v_h\|_{U_\delta} \|v_h\|_{U_\delta} \\
& \lesssim \|v_h\|_{U_\delta} (\|\mathbf{n}_h^{n-1} \cdot \nabla v_h\|_{U_\delta} + h^{q-1} \|v_h\|_{\mathcal{O}_\delta(\Gamma_h^{n-1})}) & (\text{eq. (3.12) and FE inv. ineq.}) \\
& \lesssim \|v_h\|_{U_\delta} (\|\mathbf{n}_h^{n-1} \cdot \nabla v_h\|_{U_\delta} + \|v_h\|_{\Gamma_h^{n-1}} + \|\mathbf{n}_h^{n-1} \cdot \nabla v_h\|_{\mathcal{O}_\delta(\Gamma_h^{n-1})}) & (\text{eq. (4.10b) and } \delta_{n-1} + h \lesssim 1) \\
& \lesssim (\delta_{n-1}^{\frac{1}{2}} \|v_h\|_{\Gamma_h^{n-1}} + \delta_{n-1} \|\mathbf{n}_h^{n-1} \cdot \nabla v_h\|_{\mathcal{O}_\delta(\Gamma_h^{n-1})}) & (\text{eq. (4.10a)}) \\
& \quad \times (\|v_h\|_{\Gamma_h^{n-1}} + \|\mathbf{n}_h^{n-1} \cdot \nabla v_h\|_{\mathcal{O}_\delta(\Gamma_h^{n-1})}) \\
& \lesssim \|v_h\|_{\Gamma_h^{n-1}}^2 + \delta_{n-1} \|\mathbf{n}_h^{n-1} \cdot \nabla v_h\|_{\mathcal{O}_\delta(\Gamma_h^{n-1})}^2.
\end{aligned}$$

This completes the proof.

Appendix C. Proof of (4.43). The term that has to be bounded is given by

$$A := \left| \int_{\Gamma^n} w_N^n \mathbf{v}_h^\ell \cdot \mathbf{H} \mathbf{u}^n ds - \int_{\Gamma_h^n} w_N^{e,n} \mathbf{v}_h \cdot \mathbf{H}_h \mathbf{u}^n ds_h \right|.$$

Using $\mathbf{H} = \mathbf{P} \nabla \mathbf{n} \mathbf{P}$, $\mathbf{P} \mathbf{u}^n = \mathbf{u}^n$ and the product rule $\nabla(\mathbf{w} \cdot \mathbf{n})^T = \mathbf{w} \cdot \nabla \mathbf{n} + \mathbf{n} \cdot \nabla \mathbf{w}$ we get

$$\begin{aligned}
& \int_{\Gamma^n} w_N^n \mathbf{v}_h^\ell \cdot \mathbf{H} \mathbf{u}^n ds = \int_{\Gamma^n} w_N^n \nabla(\mathbf{P} \mathbf{v}_h^\ell \cdot \mathbf{n}) \cdot \mathbf{u}^n ds - \int_{\Gamma^n} w_N^n \mathbf{n} \cdot \nabla(\mathbf{P} \mathbf{v}_h^\ell) \mathbf{u}^n ds \\
& = \int_{\Gamma^n} w_N^n \nabla_\Gamma(\mathbf{v}_h^\ell \cdot \mathbf{P} \mathbf{n}) \cdot \mathbf{u}^n ds - \int_{\Gamma^n} w_N^n \mathbf{n} \cdot \nabla(\mathbf{P} \mathbf{v}_h^\ell) \mathbf{u}^n ds = - \int_{\Gamma^n} w_N^n \mathbf{n} \cdot \nabla(\mathbf{P} \mathbf{v}_h^\ell) \mathbf{u}^n ds.
\end{aligned} \tag{C.1}$$

Using the definition $\mathbf{H}_h = \nabla_{\Gamma_h} \bar{\mathbf{n}}_h = \mathbf{P}_h \nabla \bar{\mathbf{n}}_h^\ell \mathbf{P}_h$, the transformation formula (4.5) applied to $\bar{\mathbf{n}}_h$, $\|d\|_{L^\infty(\Gamma_h^n)} \lesssim h^{q+1}$, $\|\mathbf{P} - \mathbf{P}_h\|_{L^\infty(\Gamma_h^n)} \lesssim h^q$, the bound for the surface measure change in (A.1) and the smoothness of $\mathbf{u}$ we obtain

$$\left| \int_{\Gamma_h^n} w_N^{e,n} \mathbf{v}_h \cdot \mathbf{H}_h \mathbf{u}^n ds_h - \int_{\Gamma^n} w_N^n \mathbf{v}_h^\ell \cdot \mathbf{P} \nabla \bar{\mathbf{n}}_h^\ell \mathbf{P} \mathbf{u}^n ds \right| \lesssim h^q \|\mathbf{v}_h\|_{\Gamma_h^n}. \tag{C.2}$$

We apply partial integration as in (C.1), which yields

$$\int_{\Gamma^n} w_N^n \mathbf{v}_h^\ell \cdot \mathbf{P} \nabla \bar{\mathbf{n}}_h^\ell \mathbf{P} \mathbf{u}^n ds = \int_{\Gamma^n} w_N^n \nabla_\Gamma(\mathbf{v}_h^\ell \cdot \mathbf{P} \bar{\mathbf{n}}_h^\ell) \cdot \mathbf{u}^n ds - \int_{\Gamma^n} w_N^n \bar{\mathbf{n}}_h^\ell \cdot \nabla(\mathbf{P} \mathbf{v}_h^\ell) \mathbf{u}^n ds. \tag{C.3}$$

For the first term on the right-hand side we apply partial integration and using $\|\mathbf{P} \bar{\mathbf{n}}_h^\ell\|_{L^\infty(\Gamma^n)} = \|\mathbf{P}(\bar{\mathbf{n}}_h^\ell - \mathbf{n})\|_{L^\infty(\Gamma^n)} \lesssim h^q$, cf. (3.13), we get

$$\left| \int_{\Gamma^n} w_N^n \nabla_\Gamma(\mathbf{v}_h^\ell \cdot \mathbf{P} \bar{\mathbf{n}}_h^\ell) \cdot \mathbf{u}^n ds \right| = \left| \int_{\Gamma^n} (\mathbf{v}_h^\ell \cdot \mathbf{P} \bar{\mathbf{n}}_h^\ell) \operatorname{div}_\Gamma(w_N^n \mathbf{u}^n) ds \right| \lesssim h^q \|\mathbf{v}_h\|_{\Gamma_h^n}. \tag{C.4}$$

With the results (C.1)-(C.4) we get, using again (3.13),

$$\begin{aligned}
A & \leq \left| \int_{\Gamma^n} w_N^n \mathbf{n} \cdot \nabla(\mathbf{P} \mathbf{v}_h^\ell) \mathbf{u}^n ds - \int_{\Gamma^n} w_N^n \bar{\mathbf{n}}_h^\ell \cdot \nabla(\mathbf{P} \mathbf{v}_h^\ell) \mathbf{u}^n ds \right| + h^q \|\mathbf{v}_h\|_{\Gamma_h^n} \\
& = \left| \int_{\Gamma^n} w_N^n (\mathbf{n} - \bar{\mathbf{n}}_h^\ell) \cdot \nabla(\mathbf{P} \mathbf{v}_h^\ell) \mathbf{u}^n ds \right| + h^q \|\mathbf{v}_h\|_{\Gamma_h^n} \lesssim h^q \|\mathbf{v}_h\|_{H^1(\Gamma_h^n)},
\end{aligned}$$

which completes the proof.

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