On the Clique Dynamics of Locally Cyclic Graphs

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Summary

In this thesis, it is proven that the clique graph operator k is divergent on a (not necessarily finite) locally cyclic graph G with minimum degree $\delta \geq 6$ if and only if the universal triangular cover of G contains arbitrarily large triangular-shaped subgraphs. For finite G, this is equivalent to G being 6-regular.

A graph is called locally cyclic if the open neighbourhood $N_G(v)$ of each vertex v induces a cycle. The clique graph kG of a graph G has the maximal complete subgraphs of G as its vertices and its edges are those pairs with non-empty intersection. The $(n+1)$-st iterated clique graph is recursively defined as the clique graph of the n-th iterated clique graph. If all iterated clique graphs of G are non-isomorphic, the graph G is called clique divergent; otherwise, it is clique convergent.

While it has been shown for finite locally cyclic graphs that those with minimum degree $\delta \geq 7$ are clique convergent while the 6-regular ones are clique divergent, this thesis gives a full characterisation of clique convergent locally cyclic graphs with minimum degree $\delta \geq 6$.

In the beginning, it is shown that a clique convergent connected graph has a clique convergent universal triangular cover. Conversely, a sufficient condition is given under which the clique convergence of the universal triangular cover of a graph implies the clique convergence of the graph itself.

Locally cyclic graphs with minimum degree $\delta \geq 6$ which are triangularly simply connected are their own universal covers and they are referred to as pikas throughout this thesis. On the class of pikas, clique convergence is characterised using an explicit construction of the iterated clique graphs and a finite yet divergent parameter for the clique divergent case. Furthermore, it is shown that locally cyclic graphs with minimum degree $\delta \geq 6$ are clique convergent if and only if their universal covers are clique convergent. This way, the characterisation is completed.

Zusammenfassung

In dieser Arbeit wird bewiesen, dass der Cliquenoperator k auf einem (nicht notwendigerweise endlichen) lokal zyklischen Graphen G mit Minimalgrad $\delta \geq 6$ genau dann divergent ist, wenn seine universelle Dreiecks-Überlagerung beliebig große dreieckförmige Teilgraphen enthält. Für einen endlichen Graphen G ist dies äquivalent dazu, dass G 6-regulär ist.

Ein Graph heißt lokal zyklisch, wenn die offene Nachbarschaft $N_G(v)$ jedes Knoten v einen Kreis induziert. Der Cliquengraph kG eines Graphen G hat die maximalen Cliquen von G als Knoten und seine Kanten sind durch Paare mit nichtleeren Schnitten gegeben. Der $(n+1)$-te iterierte Cliquengraph ist rekursiv als der Cliquengraph des n-ten iterierten Cliquengraphen definiert. Falls alle iterierten Cliquengraphen von G nicht-isomorph sind, wird der Graph als cliquen-divergent bezeichnet, anderenfalls als cliquen-konvergent.

Während es für endliche lokal zyklische Graphen bereits bekannt ist, dass solche mit Minimalgrad $\delta \geq 7$ cliquen-konvergent und 6-reguläre cliquen-divergent sind, gibt diese Arbeit eine vollständige Charakterisierung der Cliquen-Konvergenz lokal zyklischen Graphen mit Minimalgrad $\delta \geq 6$.

Zu Beginn wird gezeigt, dass ein cliquen-konvergenter, zusammenhängender Graph eine cliquen-konvergente universelle Dreiecks-Überlagerung hat. Umgekehrt wird eine hinreichende Bedingung angegeben, unter welcher die Konvergenz der universellen Dreiecks-Überlagerung die Konvergenz des Graphen selbst impliziert.

Lokal zyklische Graphen mit Minimalgrad $\delta \geq 6$, welche einfach dreiecks-zusammenhängend sind, sind ihre eigenen universellen Dreiecks-Überlagerungen. Für diese Graphenklasse wird Cliquen-Konvergenz mit Hilfe einer expliziten Konstruktion der iterierten Cliquengraphen sowie eines endlichen aber divergenten Graphenparameters für den cliquen-divergenten Fall charakterisiert. Darüber hinaus wird gezeigt, dass lokal zyklische Graphen mit Minimalgrad $\delta \geq 6$ genau dann cliquen-konvergent sind, wenn ihre universellen Dreiecks-Überlagerungen es auch sind. Auf diese Weise wird die Charakterisierung vervollständigt.

Contents

1 Introduction

Graphs are used in a wide variety of scientific areas to model how entities relate to each other. Their applications range from social networks to molecule structures and from planning the route of a delivery vehicle to improving cell coverage. Graphs consist of a set of points, called vertices, some pairs of which are linked by so-called edges. In the example of the social network, the people would be symbolised by vertices and their friendships by edges. Apart from being a common tool in applications, graphs are of great theoretical interest, because they are easy to define, yet a lot of questions remain open.

This thesis characterises the convergence of the clique graph operator on the class of locally cyclic graphs with minimum degree $\delta \geq 6$. A clique is a maximal subset of the vertices of a graph, in which every pair is linked by an edge, just like cliques in social networks are formed by people who are all friends with each other. The clique graph is built from the set of all cliques and their intersections. The vertices of the clique graphs correspond to the cliques of the original graph and they are linked by edges whenever the corresponding cliques intersect non-trivially. As the clique graph is a graph itself, this process can be repeated to generate the sequence of iterated clique graphs. This sequence can show two opposing dynamical behaviours. Either, it is convergent, i. e., it becomes periodic in the sense of eventually cycling on a finite succession of graphs, or it is divergent, i. e., the sequence of graphs never repeats.

As characterising clique dynamics is a hard problem in general, this thesis focusses on a restricted class of graphs, i. e., the locally cyclic graphs with minimum degree $\delta \geq 6$. The degree of a vertex is the number of its neighbours, i. e., the vertices connected with it by an edge. A graph is called locally cyclic if for each vertex its neighbours and the edges connecting them form a cycle.

The structure of this thesis is as follows: While Chapter 2 gives the standard definitions in group theory and graph theory as well as the relevant basics on the clique operator and locally cyclic graphs, Chapter 3 gives an overview of literature on clique dynamics with a focus on the tools used to derive information about the dynamic behaviour of one graph from the dynamic behaviour of another one. Triangular covering maps, as one of those tools, are central to this thesis. Thus, Chapter 4 gives a complete overview on them and their relationship with the clique graph operator. Furthermore, in Chapter 4 the corresponding universal object, i. e., the universal triangular cover, is defined and the following two theorems are proven:

Theorem A. *If a connected graph is clique convergent, so is its universal triangular cover.*

Theorem B. *Let* $p\colon \tilde{G} \to G$ *be a universal triangular covering map and let* $\Gamma := \{\gamma \in \mathrm{Aut}(\tilde{G}) \mid p \circ \gamma = p\}$*, which is called the corresponding deck transformation group. If there are* $n \in \mathbb{N}_0$ *and* $r \in \mathbb{N}$ *such that* $k^n\tilde{G}$ *and* $k^{n+r}\tilde{G}$ *are* Γ*-isomorphic, the quotient graphs* k^nG *and* $k^{n+r}G$ *are isomorphic and* G *is clique convergent.*

Chapter 5 gives an overview of literature on the clique dynamics of locally cyclic graphs. Chapter 6 introduces another central tool in form of the class of triangular-shaped graphs. Furthermore, it describes the local structure of triangularly simply connected locally cyclic graphs, which are called pikas for short. These prerequisites are a preparation for Chapter 7, in which the so-called n-th geometric clique graph is constructed and the following theorem is proven:

Theorem C. *For each* $n \in \mathbb{N}$ *and each pika* G*, its* n*-th geometric clique graph* G_n *is isomorphic to its* n*-th iterated clique graph* k^nG*.*

Chapter 8 characterises clique convergence on pikas using Theorem C as well as a graph parameter. Chapter 9 completes the characterisation with the following theorem as well as a simple but powerful corollary:

Theorem D. *A connected locally cyclic graph of minimum degree* $\delta \geq 6$ *is clique divergent if and only if its universal triangular cover contains arbitrarily large triangular-shaped subgraphs.*

Corollary E. *A finite and connected locally cyclic graph with minimum degree* $\delta \geq 6$ *is clique divergent if and only if it is* 6*-regular.*

2 Basics and Assumptions

In this chapter, the fundamental definitions used in this thesis are introduced. The chapter starts with notation around sets of numbers in Section 2.1 and moves on to basic group theory in Section 2.2, basic graph theory in Section 2.3, planarity of graphs in Section 2.4, locally cyclic graphs in Section 2.5, and the clique graph operator in Section 2.6. The last section contains a short list of assumptions, which apply for the whole thesis and which will not be repeated in each theorem.

We follow [Asc00] for the section on group theory, [Die12] for basic graph theory and graph planarity, [BP94] for basic analysis, and [Pri95] for the clique graph operator. The definition of a locally cyclic graph can be found in [LN00], but the generalisation to locally cyclic graphs with boundary is inspired by [LPV13].

2.1 Numbers and Maps

We choose the notation $\mathbb{N} := \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ for the sets of natural numbers without and with zero. We write $t\mathbb{N}$ and $t\mathbb{N}_0$ to denote multiples of t. The set of integers between s and t is usually denoted as $\{s, \ldots, t\}$. For real numbers x, y, the interval between them is defined by $[x, y] := \{z \in \mathbb{R} \mid x \leq z \leq y\}$. For $n \in \mathbb{N}$, the ring of integers modulo n is denoted by $\mathbb{Z}/n\mathbb{Z}$.

If A, B, and C are sets with $A \subseteq B$ and if $f : A \to C$ and $g : B \to C$ are maps such that for all $x \in A$ we have $f(x) = g(x)$, we call f a **restriction** of g, we call g an **extension** of f and we write $g|_A = f$.

2.2 Basics of Group Theory

This section contains the basic definitions for groups and group actions, which will be used to give structure to automorphisms of graphs (see Section 2.3) and to formalise the concept of factoring out symmetries of a graph (see Section 4.5).

A **group** $(\Gamma, \cdot)$ is a pair of a set Γ and a binary operator $\cdot\colon \Gamma \times \Gamma \to \Gamma$ that is associative, i. e., for all $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$, we have $(\gamma_1 \cdot \gamma_2) \cdot \gamma_3 = \gamma_1 \cdot (\gamma_2 \cdot \gamma_3)$; it contains a neutral element $1 \in \Gamma$, i. e., for each $\gamma \in \Gamma$, we have $1 \cdot \gamma = \gamma \cdot 1 = \gamma$; and it contains inverses, i. e., for each $\gamma \in \Gamma$, there is a $\gamma^{-1} \in \Gamma$ fulfilling $\gamma^{-1} \cdot \gamma = \gamma \cdot \gamma^{-1} = 1$. Usually, we do not write out the operator explicitly. This way, for $\gamma_1, \gamma_2 \in \Gamma$, their product $\gamma_1 \cdot \gamma_2$ is denoted by $\gamma_1 \gamma_2$ and the group $(\Gamma, \cdot)$ is referred to as Γ.

A **subgroup** of a group Γ is a subset $\Lambda \subseteq \Gamma$ which is also a group with regard to the restricted binary operator $\cdot_{\Lambda \times \Lambda}$.

For two groups Γ and Λ, a map $\phi\colon \Gamma \to \Lambda$ is called a **homomorphism of groups** if it is compatible with the operations of the groups, i. e., for each pair of elements $\gamma_1, \gamma_2 \in \Gamma$, we have $\phi(\gamma_1 \gamma_2) = \phi(\gamma_1)\phi(\gamma_2)$. A bijective homomorphism is called an **isomorphism** and the inverse of an isomorphism is again an isomorphism. Two groups Γ and Λ are called **isomorphic** and denoted by $\Gamma \cong \Lambda$ if there is an isomorphism $\phi\colon \Gamma \to \Lambda$.

A **group action** is a triple $(\Gamma, M, \cdot)$ of a group Γ, a set M and a binary operator $\cdot\colon G \times M \to M$ which is associative, i. e., for all $\gamma_1, \gamma_2 \in \Gamma$ and all $m \in M$, we have $(\gamma_1 \gamma_2)m = \gamma_1(\gamma_2 m)$, and that fixes everything paired with the neutral element, i. e., for each $m \in M$, $1 \cdot m = m$. The **orbit** of an element $m \in M$ is defined as the subset $\Gamma m := \{\gamma m \mid \gamma \in \Gamma\} \subseteq M$. The orbits of an action form a partition of M. If an action has only one orbit, it is called **transitive**. If for each pair of elements $m_1, m_2 \in M$, there is at most one element $\gamma \in \Gamma$ such that $\gamma m_1 = m_2$, the action is called **free**. If a group Γ acts on two sets M and N, a bijection $\varphi\colon M \to N$ is called **Γ-equivariant** if for each $\gamma \in \Gamma$ and each $m \in M$ we have $\varphi(\gamma m) = \gamma \varphi(m)$.

2.3 Basics of Graph Theory

This section gives definitions for all the standard graph-theoretical concepts which are repeatedly used throughout the whole thesis.

For a set X, its power set is denoted by $\mathbf{2}^{\boldsymbol{X}}$ and its set of 2-element subsets is denoted by $\binom{\boldsymbol{X}}{\mathbf{2}}$. A **(simple) graph** is a pair of sets $G = (V, E)$ in which $E \subseteq \binom{V}{2}$. If V and E are finite sets, G is called **finite**.

The elements of V are called the **vertices** and the elements of E are called the **edges** of G. For clarification, V and E can be called $V(G)$ and $E(G)$. For an edge $e = \{v, w\}$ we use the short notation $e = vw = wv$. Two vertices $v, w \in V$ are called **adjacent** if there is an edge $e = vw \in E$. A vertex $v \in V$ is called **incident** to an edge $e \in E$ if there is a vertex $w \in V$ such that $e = vw$.

A graph $G' = (V', E')$ is called a **subgraph** of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. If $E' = E \cap \binom{V'}{2}$, G' is called the **subgraph induced by V'** and is denoted by $\boldsymbol{G[V']}$.

For a graph $G = (V, E)$, the **open neighbourhood $N_G(v)$** of a vertex v is defined as the set of vertices adjacent to v. The **closed neighbourhood** of v is defined as $\boldsymbol{N_G[v]} := N_G(v) \cup \{v\}$. For a set $M \subseteq V$ in G, its **closed neighbourhood $N_G[M]$** is defined as the union of the closed neighbourhoods of its vertices and its **common neighbourhood $N_G^\cap[M]$** is defined as their intersection. For a subgraph $G' = (V', E')$ of G, $\boldsymbol{N_G[G']} := N_G[V']$ and $\boldsymbol{N_G^\cap[G']} := N_G^\cap[V']$. We call the induced subgraphs of the open/closed/common neighbourhoods by the same names and symbols, but the meaning is always clear from context.

For two graphs G and H their difference $\boldsymbol{G \setminus H}$ is defined by omitting all vertices of H from G as well as all edges that have an end vertex in H.

For a vertex $v \in V$, its **(vertex) degree** is defined as $\boldsymbol{\deg_G(v)} := |N_G(v)|$. If all vertex degrees of G are finite, G is called **locally finite** and the **minimum degree $\delta(G)$** is the minimum of all vertex degrees of G. If the set of vertex degrees has a maximum, it is called the **maximum degree** and it is denoted by $\boldsymbol{\Delta(G)}$. If all vertices $v \in V$ fulfil $\deg_G(v) = r$ for some $r \in \mathbb{N}_0$, G is called **r-regular**.

For each $s \in \mathbb{N}_0$ the graph

$$\boldsymbol{K_s} := \left(\{v_1, \ldots, v_s\}, \binom{\{v_1, \ldots, v_s\}}{2}\right)$$

is called the **complete graph on s vertices** and for each $s \in \mathbb{N}_{\geq 3}$ the graph

$$\boldsymbol{C_s} := (\{v_1, \ldots, v_s\}, \{v_i v_{i+1} \mid 1 \leq i \leq s-1\} \cup \{v_1 v_s\})$$

is called the **s-cycle**. The graph $C_3 = K_3$ is called **three-cycle** or a **triangle**.

Additionally, for each graph G, a subgraph which is isomorphic to some K_s is called a **complete subgraph** and each subgraph which is isomorphic to some

C_s is called a **cycle** in G. The maximal complete subgraphs in a graph G (with respect to inclusion) are called the **cliques**. For simplification, the vertex sets of the cliques are also called cliques.

We extend the incidence terminology of a graph to three-cycles. Thus, the three vertices and the three edges of a three-cycle are each **incident** to the three-cycle itself.

Furthermore, a **walk of length $\boldsymbol{\ell}$** in a graph G is a finite sequence of vertices $\alpha = v_0 \dots v_\ell$ such that each pair of consecutive vertices $v_{i-1}v_i$ with $i \in \{1, \dots, \ell\}$ is adjacent. The vertex v_0 is called the **start vertex** and is denoted by **start($\boldsymbol{\alpha}$)** while the vertex v_ℓ is called the **end vertex** and is denoted by **end($\boldsymbol{\alpha}$)**. A walk is called **closed** if start and end vertex coincide and it is called **trivial** if it has length 0. A walk consisting of pairwise distinct vertices is called a **path**, and a closed walk is called a **cycle** if deleting its end vertex yields a path.

Two vertices $v, w \in V$ are called **connected**, if there is a walk $\alpha = v_0 \dots v_\ell$ with $v_0 = v$ and $v_\ell = w$. A graph is called **connected** if each pair of its vertices is connected. The length of a shortest path between two connected vertices v, w is called their **distance** and is denoted by $\mathbf{dist}_{\boldsymbol{G}}(\boldsymbol{v}, \boldsymbol{w})$. Similarly, for two sets $U, W \subseteq V(G)$ their distance $\mathbf{dist}_{\boldsymbol{G}}(\boldsymbol{U}, \boldsymbol{W})$ is defined as the minimum distance of two vertices from U and W, respectively. The maximal distance between a pair of vertices of a connected graph is called the graphs **diameter**.

To emphasize the graph with respect to which all these quantities are computed, we use the name of the graph as a prefix, i. e., we write G-degree, G-neighbourhood, or G-distance.

A **graph homomorphism** $\phi\colon G \to H$ is any adjacency-preserving vertex map, i. e., it maps adjacent vertices to adjacent vertices. The **image** of a subgraph $G' \subseteq G$ under a graph homomorphism $\phi\colon G \to H$ is the subgraph $\boldsymbol{\phi}(\boldsymbol{G'}) \subseteq H$ whose vertex set consists of the images of the vertices of G' under ϕ and for which a pair of vertices is adjacent if and only if it has a pair of adjacent preimages in G'.

Injective homomorphisms are called **monomorphisms** or **embeddings**. An **isomorphism** is a bijective homomorphism whose inverse is also a homomorphism. An isomorphism $\phi\colon G \to G$ is called an **automorphism**. The set of all automorphisms of a given graph G forms a group, in which the group operation is the concatenation of maps. This group is called the **automorphism group** of G, it is denoted by **Aut(G)**, and its subgroups are called **groups of automorphisms** of G.

We say that a group Γ **acts** on a graph G if it acts on the vertices of G in a way that each $\gamma \in \Gamma$ induces an automorphism of G. The pair of a graph G and a group Γ acting on it is called a **Γ-graph**. For each subgroup $\Gamma \leq \mathrm{Aut}(G)$, G is a Γ-graph in a natural way. For two Γ-graphs G and H, we call a graph isomorphism $\phi\colon G \to H$ a **Γ-isomorphism**, if it is a Γ-equivariant isomorphism i. e., if $\phi(\gamma v) = \gamma\phi(v)$ for each $v \in V(G)$ and each $\gamma \in \Gamma$.

2.4 Euclidean Geometry and Graph Planarity

In this section, graph planarity is defined, which is later used to describe local areas of graphs from a specific class. For explicit calculations around planarity, we need some basic definitions from Euclidean geometry.

The Euclidean space $(\mathbb{R}^2, |\cdot|)$ is called the **plane** and the point $\boldsymbol{\mathcal{O}} = (0,0) \in \mathbb{R}^2$ is called the **origin**. For three points $x, y, z \in \mathbb{R}^2$, the line segment connecting x and y is denoted by $\overline{\boldsymbol{xy}} := \{x + \lambda(y-x) \mid \lambda \in [0,1]\}$, the line through x and y is denoted by $\boldsymbol{xy} := \{x + \lambda(y-x) \mid \lambda \in \mathbb{R}\}$, and the non-negative, non-reflex angle between $\overline{xy}$ and $\overline{yz}$ is denoted by $\boldsymbol{\angle(xyz)} \in [0, \pi)$. The euclidean distance between x and y is denoted by $\mathbf{dist}_{\mathbb{R}^2}\boldsymbol{(x,y)} = |x-y|$ and the euclidean distance between two sets $X, Y \subseteq \mathbb{R}^2$ is $\mathbf{dist}_{\mathbb{R}^2}\boldsymbol{(X,Y)} = \inf_{x \in X, y \in Y} \mathrm{dist}_{\mathbb{R}^2}(x,y)$. In both cases, the subscript $\mathbb{R}^2$ is omitted if this does not lead to ambiguity.

A set $X \subseteq \mathbb{R}^2$ is called **open** if for each $x \in X$ there is an $\varepsilon > 0$ such that $\{z \in \mathbb{R}^2 \mid |x-z| < \varepsilon\} \subseteq X$ and it is called **connected** if there is no pair of disjoint open sets $Y, Z \subseteq \mathbb{R}^2$ such that $X \cap Y \neq \emptyset$, $X \cap Z \neq \emptyset$, and $X \subset Y \cup Z$. The maximal connected subsets of a set $X \subseteq \mathbb{R}^2$ are called its **connected components**. The **boundary** of a set $X \subseteq \mathbb{R}^2$ is the set of all points $y \in \mathbb{R}^2$ such that for all $\varepsilon > 0$ the set $\{z \in \mathbb{R}^2 \setminus \{y\} \mid |y-z| < \varepsilon\} \cap X$ is non-empty. A set $X \subseteq \mathbb{R}^2$ is called **bounded** if there is an $R > 0$ such that each $x \in X$ fulfils $|x| < R$; otherwise, it is called **unbounded**.

A **polygonal arc** is a subset $L \subseteq \mathbb{R}^2$ of the form $L = \bigcup_{i \in \{1,\dots,s\}} L_i$ with $L_i = \overline{v_{i-1}v_i}$ for $s \in \mathbb{N}$ and for some points $v_0, v_1, \dots, v_s \in \mathbb{R}^2$ which fulfils that for $0 \leq i < j \leq s$, we have $L_i \cap L_j = \{v_i\}$, if $j = i+1$, and $L_i \cap L_j = \emptyset$, otherwise. The points v_0

and v_s are called the **endpoints** of L and all the other elements of L are called the **interior points**.

A **(finite, simple) plane graph** is a pair (V, E) of finite sets, which are called the sets of vertices and edges, respectively, such that the vertices are points in the plane, the edges are polygonal arcs with both endpoints lying in V such that different edges have different sets of endpoints and such that the interior points of an edge do neither belong to V nor to any other edge.

The **regions** of a plane graph G are the connected components of $\mathbb{R}^2 \setminus (V \cup \bigcup E)$. A plane graph is called a **triangulated disc** if each of its bounded regions is a triangle, i. e., the region's boundary consists of exactly three of the graph's edges.

A graph is called **planar** if it is isomorphic to a plane graph. An isomorphism from a planar graph to a plane graph is called a **planar embedding**.

2.5 Locally Cyclic Graphs With or Without Boundary

This section gives the definitions for the main class of graphs which occurs in this thesis, i. e., the class of locally cyclic graphs with boundary, and for its important subclass of locally cyclic graphs.

A **locally cyclic graph with boundary** is a simple graph $G = (V, E)$ such that for each vertex $v \in V$ the open neighbourhood $N_G(v)$ is either a cycle graph or a path graph, see Figure 2.1.

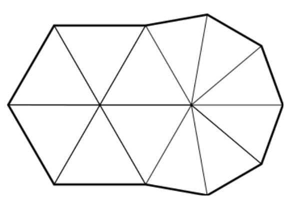

Figure 2.1: A locally cyclic graph with boundary.

If $N_G(v)$ is a cycle, v is called an **inner vertex** of G; otherwise, v is called a **boundary vertex.**

An edge $xy \in E$ is called an **inner edge** if its incident vertices x and y have two common neighbours, and a **boundary edge** if not. Note that the two end vertices of a boundary edge always have a common neighbour, as their open neighbourhoods would otherwise induce disconnected graphs and not paths.

The **boundary graph ∂G** is the subgraph of G consisting of the boundary vertices and the boundary edges. The boundary graph ∂G is well-defined, as the edges incident to inner vertices are inner edges. This can be seen in the following way: for each inner vertex x and each edge xy, the vertex y lies in the cyclic neighbourhood $N_G(x)$ and has, therefore, two neighbours in $N_G(x)$.

G is called **locally cyclic** if $\partial G = \emptyset$, i. e., the set of vertices adjacent to a given vertex v always induces a cycle. The three-cycles of locally cyclic graphs with boundary are sometimes called **faces**. Frequently studied locally cyclic graphs are the octahedron, the icosahedron, and the hexagonal grid, which are displayed in Figure 2.2.

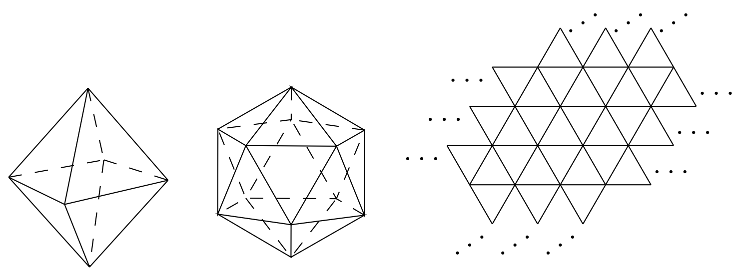

Figure 2.2: Octahedron, icosahedron, and the hexagonal grid.

2.6 The Clique Graph Operator

The main object studied in this thesis is the clique graph operator, which is defined next.

The **clique graph operator k** constructs from a given graph G its **clique graph kG**. The vertex set of kG corresponds bijectively to the set of distinct cliques in G and two vertices are adjacent if and only if their corresponding cliques intersect non-trivially in G. For $n \in \mathbb{N}_0$, the **n-th iterated clique graph k^nG** is recursively defined by $k^0G = G$ and $k^nG = k(k^{n-1}G)$ for $n \geq 1$.

The **clique dynamics** of a graph G is the long-term behaviour of the sequence of iterated clique graphs $(k^n G)_{n \in \mathbb{N}_0}$. If the members of $(k^n G)_{n \in \mathbb{N}_0}$ are pairwise non-isomorphic graphs, it is called **clique divergent**, otherwise **clique convergent**.

2.7 General Assumptions

All graphs in this thesis are non-empty and locally finite but not necessarily finite. If not stated otherwise, they are connected.

3 Literature on the Clique Graph Operator

The study of graph operators has a long history in discrete mathematics. While a graph operator is any rule that constructs from a given graph a new one, the dynamics of an operator looks at the possible behaviours when applying the operator iteratively to a given graph. The possible behaviours are convergence, i. e., isomorphic copies of finitely many graphs appear repeatedly, and divergence i. e., the operator generates a sequence of pairwise non-isomorphic graphs. The line graph as the first graph operator was defined by Krausz [Kra43]. A good resource on graph operators and their dynamics is the book [Pri95] by Prisner.

The focus of this thesis is on the clique graph operator, which was introduced by Hedetniemi and Slater [HS72], who used it for studying line graphs and triangle-free graphs. In general, the classification of graphs with respect to their clique dynamics, i. e., the dynamics of the clique graph operator, is hard. In fact, the decision problem, whether a given graph is clique convergent or clique divergent, is algorithmically undecidable on the class of automatic graphs, i. e., graphs whose vertices and edges can be recognised by finite automata, and the decidability for finite graphs is unknown, see [CP21].

3.1 Proof Techniques for Clique Convergence and Divergence

While there is no general criterion for clique convergence, it is possible to find characterisations for restricted classes, such as graphs of low degree in [Vil22], circular arc graphs in [LSS10] or locally H graphs (e. g. locally cyclic graphs in [LN00] or shoal graphs in [LPV16]).

A particularly well-understood class consists of the clique-Helly graphs. A graph is called clique-Helly if its set of cliques fulfils the Helly property, i. e., for every set of pairwise intersecting cliques, the total intersection of these cliques is non-empty. It was shown by Escalante [Esc73] that all finite clique-Helly graphs are clique convergent and that they have period length 1 or 2, i. e., there is some $n_0 \in \mathbb{N}_0$ such that $k^n G \cong k^{n+2} G$ for every $n \geq n_0$.

Proving clique convergence of a graph can be done in a straightforward way by giving an explicit isomorphism between $k^n G$ and $k^{n+r} G$ for some $n \in \mathbb{N}_0$ and some $r \in \mathbb{N}$. In contrast, a more complex argument is needed for proving that such an isomorphism is absent for all possible n and r. To show clique divergence, usually a graph parameter is identified which is unbounded on the sequence of iterated clique graphs or on some convenient subsequence. Other techniques, which use knowledge about the dynamics of other graphs, are described in the next section.

3.2 Transfer Arguments for Clique Divergence

There are several tools to deduce the clique divergence of one graph from the clique divergence of another one. In the following, we describe retracts, two applications of admissible graph relations, and triangular covering maps, which are three of the common tools for finite graphs.

For the first two of them, we discuss whether there is hope to generalise them to infinite graphs. As triangular covering maps are extensively used in this thesis, their applicability to infinite graphs is only touched on, as it is discussed in Chapter 4 in detail.

Before we can explain the tools, we need a few definitions which stem from [LNP06] and [Neu78]. Given two sets A and B, a subset $f \subseteq A \times B$ is called a **relation** and it is denoted by $f\colon A \to B$. For $a \in A$, the set $\boldsymbol{f(a)} := \{b \in B \mid (a,b) \in f\}$ is called the **image** of a and for a subset $A' \subseteq A$, its **image** is given by $\boldsymbol{f(A')} := \bigcup_{a \in A'} f(a)$. For two relations $f\colon A \to B$ and $g\colon B \to C$, their **composition** $\boldsymbol{g \circ f}\colon A \to C$ is given by $(g \circ f)(a) := g(f(a))$ for every $a \in A$. For relations that are also maps we write $f(a) = b$ instead of $f(a) = \{b\}$.

For two vertices v, w from a graph G, we write $\boldsymbol{v \simeq w}$ if they are either equal or adjacent. For two graphs G and H a **graph relation** $\phi\colon G \to H$ is a relation of the vertex sets such that for each $v \in V(G)$, we have $\phi(v) \neq \emptyset$ and such that the image of each complete subgraph of G is a complete subgraph of H. Especially, for $v_1, v_2 \in V(G)$ and $w_1, w_2 \in V(H)$ with $v_1 \simeq v_2$, $w_1 \in \phi(v_1)$ and $w_2 \in \phi(v_2)$, we have $w_1 \simeq w_2$. A graph relation which is also a map is called a **weak graph morphism**. We note that every homomorphism of graphs is a weak graph morphism and, thus, a graph relation.

3.2.1 Retractions

A **retraction** is a weak graph morphism $\phi\colon G \to H$ for which H is a subgraph of G and $\phi(v) = v$ for every $v \in V(H)$. In this situation, the subgraph H is called a **retract** of G. It was shown in [Neu78] that a finite graph is clique divergent if its retract is. Unfortunately, as the proof of divergence relies on a pigeon hole principle, this idea does not carry over to infinite graphs.

An infinite counterexample works the following way: Let G be the graph in Figure 3.1 given by

$$V(G) = \{(x, y) \in \mathbb{N}_0 \times \mathbb{N}_0 \mid y \leq x\} \text{ and}$$
$$E(G) = \{(x, y_1)(x, y_2) \mid |y_1 - y_2| = 1\} \cup \{(x_1, 0)(x_2, 0) \mid |x_1 - x_2| = 1\}.$$

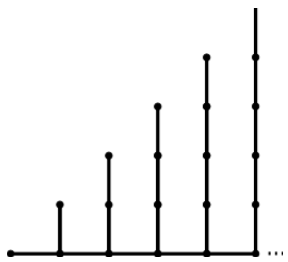

Figure 3.1: The graph G.

As G has no triangles and as $\{v \in V(G) \mid \deg(G) = 1\} = \{(x, y) \in V(G) \mid x = y\}$, by [HS72, Theorem 3] the second clique graph k^2G is isomorphic to

$$G[(x, y) \in \mathbb{N}_0 \times \mathbb{N}_0 \mid y \leq x - 1].$$

Thus, the map $\psi\colon G \to k^2G, (x, y) \to (x + 1, y)$ is an isomorphism and G is clique convergent. Now, let H be the induced subgraph of G with vertex set

$$V(H) = \left\{(2^\ell, 0) \mid \ell \in \mathbb{N}_0\right\} \cup \left\{(x, y) \in \mathbb{N}_0 \times \mathbb{N}_0 \mid x \notin \{2^\ell \mid \ell \in \mathbb{N}_0\} \text{ and } y \leq x\right\},$$

which is depicted in Figure 3.2.

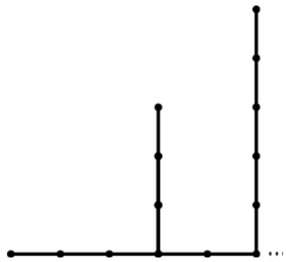

Figure 3.2: The graph H.

As

$$\phi\colon G \to H, (x,y) \mapsto \begin{cases} (x,0), & \text{if } x = 2^\ell \text{ for some } \ell \in \mathbb{N}_0, \\ (x,y), & \text{otherwise,} \end{cases}$$

is a weak graph morphism, H is a retract of G. Again by [HS72, Theorem 3], the second clique graph k^2H is isomorphic to

$$H\Big[\{(2^\ell,0) \mid \ell \in \mathbb{N}_0 \text{ and } 2^\ell \geq 1\} \\ \cup \Big\{(x,y) \in \mathbb{N}_0 \times \mathbb{N}_0 \mid x \notin \{2^\ell \mid \ell \in \mathbb{N}_0\} \text{ and } y \leq x-1\Big\}\Big]$$

and analogously for $r \in \mathbb{N}_0$, k^{2r} is isomorphic to

$$H\Big[\{(2^\ell,0) \mid \ell \in \mathbb{N}_0 \text{ and } 2^\ell \geq r\} \\ \cup \Big\{(x,y) \in \mathbb{N}_0 \times \mathbb{N}_0 \mid x \notin \{2^\ell \mid \ell \in \mathbb{N}_0\} \text{ and } y \leq x-r\Big\}\Big].$$

As the graph parameter $p(G) := \{\text{dist}(\{v\}, \text{DEG}_3(G)) \mid v \in \text{DEG}_1(G)\}$ with $\text{DEG}_i(G) = \{w \in V(G) \mid \deg(w) = i\}$ for $i \in \{1,3\}$ fulfils

$$p(k^{2r}H) = \mathbb{N}_0 \setminus \{2^\ell - r \mid \ell \in \mathbb{N}_0\},$$

it has pairwise distinct values on $(k^{2r}H)_{r \in N_0}$ and H is clique divergent.

3.2.2 Coaffine Automorphic Graphs and Admissible Relations

A pair $\boldsymbol{(G,\gamma)}$ of a graph G and an automorphism γ of that graph is called an **automorphic graph**. For some integer $r \geq 2$, an automorphic graph (G,γ) is called $\boldsymbol{r}$**-coaffine** if $\text{dist}(v,\gamma(v)) \geq r$ for every $v \in V(G)$.

For an automorphism γ of some graph G we get a corresponding automorphism γ_k of kG via $\gamma_k\colon kG \to kG, Q \mapsto \gamma(Q) = \{\gamma(v) \in V(G) \mid v \in Q\}$. If (G, γ) is an r-coaffine automorphic graph, then the automorphic graph (kG, γ_k) is r-coaffine as shown in [LNP06].

A graph relation $\phi\colon G \to H$ between two automorphic graphs (G, γ) and (H, σ) is called **admissible** if $\phi \circ \gamma = \sigma \circ \phi$.

Rank Divergence

The **rank** of an r-coaffine automorphic graph (G, γ) is the maximum $s \in \mathbb{N}$ such that there are non-empty, γ-invariant, and pairwise disjoint sets $V_1, \ldots, V_s \subseteq V(G)$, such that $\operatorname{dist}(v_i, v_j) < r$ for all $i \neq j$ and for all $v_i \in V_i$ and $v_j \in V_j$. Hence, the definition of the rank depends on γ as well as r.

We say that a r-coaffine automorphic graph (G, γ) is **rank divergent** if the sequence of the ranks of its iterated clique graphs $(k^n G, \gamma_{k^n})_{n \in \mathbb{N}_0}$ is not bounded.

As the rank can be at most the number of vertices, for finite graphs rank divergence implies clique divergence. For a single graph, however, this technique is not very useful as proving rank divergence is not necessarily easier than proving clique divergence. But in [LNP06] it was shown that rank divergence is preserved under admissible graph relations, which paves the ground for transfer arguments.

Because of the dependence on γ, the rank is not purely a graph parameter and thus two iterated clique graphs with different rank might be isomorphic graphs. Thus, there is no obvious way to apply it to prove clique divergence of infinite graphs. However, there does not seem to be an obvious example of a clique convergent infinite graph such that the rank is finite but unbounded on its iterated clique graphs. Thus, the question whether admissible relations and rank divergence can be used to show clique divergence in infinite graphs remains open.

Absolute Saturation and Absolute Freedom

For a Γ-graph and an automorphism $\gamma \in \Gamma$, the automorphic graph (G, γ) is called **r-saturated** if $\operatorname{dist}(v, w) + \operatorname{dist}(w, \gamma v) = r$ for all $v, w \in V(G)$ and it is called **absolutely r-saturated** if all its iterated clique graphs $(k^n G, \gamma_{k^n})$ are saturated. A Γ-graph is called **free** if Γ acts freely on the vertices of G. It is called **absolutely free** if all its iterated clique graphs $(k^n G, \gamma_{k^n})$ are free.

In [Piz03] it was shown that for a weak graph morphism $\phi\colon G \to H$ between two automorphic Γ-graphs (G, γ) and (H, σ), the clique divergence of H follows from G being clique divergent, absolutely r-saturated and absolutely free and H being is r-coaffine. As in the proof divergence is achieved by showing that H has at least as many vertices as G, it does not transfer easily to infinite graphs. Moreover, as saturation implies a finite diameter, this method does require G to be finite or locally infinite and locally infinite graphs are a lot more complicated to investigate. Thus, a generalisation to infinite but locally finite graphs is unlikely.

3.2.3 Triangular Covering Maps

A graph homomorphism $p\colon \tilde{G} \to G$ between two connected graphs is called a **triangular covering map** if it is locally isomorphic, i. e., if the restriction $p|_{N[\tilde{v}]}\colon N[\tilde{v}] \to N[p(\tilde{v})]$ to the closed neighbourhood of any vertex $\tilde{v}$ of $\tilde{G}$ is an isomorphism. In this case, $\tilde{G}$ is called a **triangular cover** of G. The term “triangular” refers to the unique triangle lifting property which can be used as an alternative definition and is defined in the next chapter.

We relate the n-th iterated clique graph of a graph to the n-th iterated clique graph of its cover as follows: For a triangular covering map $p\colon \tilde{G} \to G$, we define the map $\boldsymbol{p_{k^n}}\colon k^n\tilde{G} \to k^nG$ which is constructed from p recursively by $p_{k^0} = p$ and $p_{k^n}(Q) = \{p_{k^{n-1}}(v) \mid v \in Q\}$ for $n \geq 1$. By [LN00, Proposition 2.2], p_{k^n} is a triangular covering map as well.

It has been shown in [LN00] that for a finite triangular cover of a finite graph, the graph is clique convergent if and only if its cover is. As this proof also relies on a pigeon hole principle, it does not generalise to the infinite setting. In the following chapter, we prove that the transfer of convergence holds in one direction and in a specific setting and we give a condition under which it also holds in the other direction.

4 Triangular Covering Maps and the Clique Operator

This chapter describes the interaction between the clique graph operator, triangular covering maps and a topological property called triangular simple connectivity. Section 4.1 defines triangular simple connectivity, Section 4.2 shows that this property is preserved under the clique graph operator, Section 4.3 shows that every graph has a triangularly simply connected triangular cover, which is unique up to isomorphism, Section 4.4 shows that clique convergence is transferred from a graph to its universal triangular cover, and Section 4.5 gives a sufficient condition under which the clique convergence of the universal triangular cover implies the clique convergence of the graph itself.

In the following, we need results from [LN00] and [Rot73], whose ways of notation look incompatible at first glance. We combine them by using the setting and definitions from [LN00] and translating the results from [Rot73], as well as their proofs, to that language and restrict them to the conditions under which we need them. This chapter is based on joint work with Martin Winter in [LW23].

4.1 Triangular Simple Connectivity

This section introduces homotopy of walks as well as triangular simple connectivity. In order to define the homotopy relation, we need four types of **elementary moves** (see Figure 4.1).

Given a walk α containing three consecutive vertices that form a triangle in G, the **triangle removal** shortens α by removing the middle one of them. Inversely, if α contains two consecutive vertices that are contained in a common triangle of G, the **triangle insertion** lengthens α by inserting the third vertex of the triangle between the other two. The **dead end removal** shortens a walk α that contains a

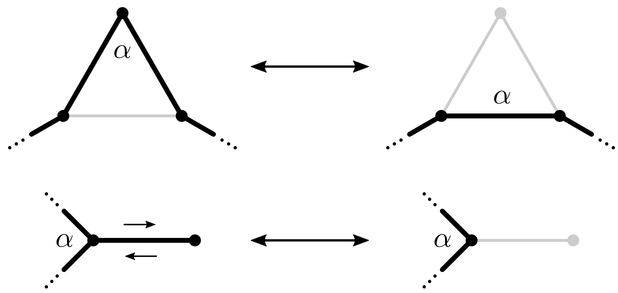

Figure 4.1: Visualisations of the elementary moves performed on a walk α.

vertex twice with distance two in the walk by removing one of the two occurrences as well as the vertex between them. Inversely, the **dead end insertion** lengthens α by inserting behind one vertex an adjacent one and then the vertex itself again. Note that elementary moves do not change the start or end vertex of a walk, not even if the walk is closed.

Two walks are called **homotopic** if it is possible to transform one into the other by performing a finite number of elementary moves. The graph G is called **triangularly simply connected** if it is connected and if every closed walk is homotopic to a trivial one. Examples of triangularly simply connected graphs are trees, complete graphs, and all Whitney triangulations (see Chapter 5) of the sphere or the plane. Conversely, the cycle graphs C_s with $s \geq 4$ are not triangularly simply connected and neither is any Whitney triangulation of the torus.

4.2 The Clique Operator and Triangular Simple Connectivity

In this section, we show that triangular simple connectivity is preserved under the clique graph operator. A weaker version was obtained by Prisner [Pri92] in 1992, who proved that the clique graph operator preserves the first $\mathbb{Z}/2\mathbb{Z}$ Betti number, which is a measure for the number of two-dimensional holes of a simplicial complex. Larrión and Neumann-Lara [LN00] then extended this in 2000 to the isomorphism type of the triangular fundamental group. An extension to more general graph operators (including the clique graph operator and the line graph operator) was proven by Larrión, Pizaña, and Villarroel-Flores [LPV09] in 2009.

The proof for triangular simple connectivity being preserved under the clique graph operator, which is presented below, is new and completely elementary, as it explicitly constructs a sequence of elementary moves that transforms a given closed walk to the trivial one.

As triangular simple connectivity requires connectivity, we start with a lemma about connectivity before we show that closed walks are homotopic to trivial ones.

Lemma 4.1. *For a connected graph G, the clique graph kG is also connected.*

Proof. Let $Q, Q' \in V(kG)$ be two cliques of G. We choose two vertices $v \in Q$ and $v' \in Q'$. As G is connected, there is a shortest walk $v_0 \dots v_\ell$ in G connecting $v_0 = v$ to $v_\ell = v'$. For each $i \in \{1, \dots, \ell\}$ we choose a clique Q_i that contains the pair of consecutive vertices v_{i-1} and v_i of this walk. Thus, for each $i \in \{1, \dots, \ell - 1\}$, the cliques Q_i and Q_{i+1} intersect in v_i and they are distinct, as otherwise the vertices v_{i-1} and v_{i+1} would be adjacent, in contradiction to the minimality of the walk $v_0 \dots v_\ell$. Thus, $Q_1 \dots Q_\ell$ is a walk in kG. If $Q \neq Q_1$ we add Q to the start of the walk and if $Q_\ell \neq Q'$ we append Q'. The resulting walk connects Q and Q' in kG and, thus, kG is connected. □

We establish a concept of correspondence between a walk in G and a walk in kG in order to use the elementary moves that transform the former one into a trivial one as a guideline for doing the same with the latter one.

We say that a closed walk α in G and a closed walk $\alpha' = Q_0 \dots Q_\ell$ in kG with $Q_0 = Q_\ell$ **correspond** if for each $i \in \{0, \dots, \ell - 1\}$ there is a walk $v_{i,0} \dots v_{i,t_i}$ of length $t_i \in \mathbb{N}_0$ that lies completely in Q_i and α is the concatenation of those walks, i. e., $v_{i,t_i} = v_{i+1,0}$ for each $i \in \{0, \dots, \ell - 2\}$, see Figure 4.2.

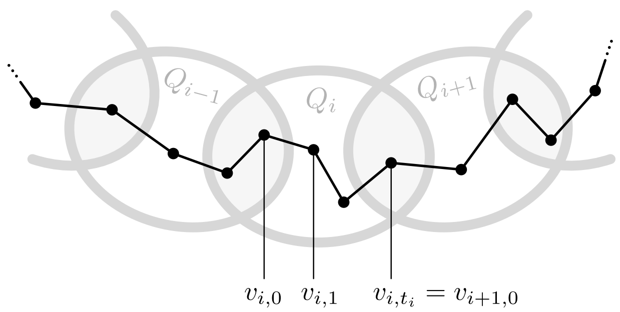

Figure 4.2: The correspondence relation between a walk in G and one in kG.

As α is closed, we have $v_{0,0} = v_{\ell-1,t_{\ell-1}} =: v_{\ell,0}$. Note that for each closed walk in kG there is a corresponding one in G, which is obtained as follows. Let $\alpha' =$

$Q_0 \dots Q_\ell$ with $Q_0 = Q_\ell$ be a closed walk in kG. For each $i \in \{1, \dots, \ell\}$, we choose $w_i \in Q_{i-1} \cap Q_i$, we define $w_0 := w_\ell$, and we drop repeated consecutive vertices. This way, we obtain a walk $\alpha = w_0 \dots w_\ell$ which clearly corresponds to α'.

Lemma 4.2. *If G is a triangularly simply connected graph, so is kG.*

Proof. Let G be a triangularly simply connected graph. Thus, G is connected and, by Lemma 4.1, so is kG. Next, we show that every closed walk in kG can be transformed to a single vertex by a sequence of elementary moves. Let $\alpha' = Q_0 \dots Q_\ell$ with $Q_0 = Q_\ell$ be a closed walk in kG. Let α be any corresponding walk in G, thus α consists of the $\ell + 1$ subwalks $v_{i,0} \dots v_{i,t_i}$ with $i \in \{0, \dots, \ell\}$ as described above.

Since G is triangularly simply connected, there is a sequence of elementary moves from α to a trivial walk. We now describe how we use the first of these moves as a guideline for elementary moves on α'; for the other moves in the sequence, it works by induction on the number of moves.

Let β be the walk in G that is obtained from α by the first move. We now perform two steps in order to construct a walk β' in kG, which is homotopic to α' and which corresponds to β. While the first step yields a walk in kG which still corresponds to α but is now in some kind of standard form that reduces the number of special cases needed afterwards, the second step performs moves that lead to the walk β' corresponding to β.

The first step consists of repeated triangle removals and dead end removals on α' that preserve the correspondence to α until α' cannot be shortened any further in that way. As no elementary move can change the start and end vertex of a walk, we do not remove $Q_0 = Q_\ell$ this way. For every $i \in \{1, \dots, \ell - 1\}$ with $t_i = 0$, the clique Q_i can be removed in a triangle or dead end removal. Consequently, the only t_i which can be zero after the first step is t_0.

For the second step, we distinguish whether the move applied to α is an insertion or a removal.

<u>Case 1:</u> insertion move. If the elementary move from α to β is a triangle insertion or dead end insertion, let the indices $i \in \{0, \dots, \ell-1\}$ and $j \in \{0, \dots, t_i-1\}$ be chosen such that the additional one or two vertices are inserted between $v_{i,j}$ and $v_{i,j+1}$. For the triangle insertion, the subwalk $v_{i,0} \dots v_{i,t_i}$ becomes $v_{i,0} \dots v_{i,j} v^* v_{i,j+1} \dots v_{i,t_i}$ and for the dead end insertion, it becomes $v_{i,0} \dots v_{i,j} v^* v_{i,j}\ v_{i,j+1} \dots v_{i,t_i}$. If $v^* \in Q_i$,

$\beta' := \alpha'$ corresponds to β and we are finished. If $v^* \notin Q_i$, let Q^* be a clique that contains $v^*, v_{i,j}$ and, in the case of a triangle insertion, also $v_{i,j+1}$. Then, the dead end insertion of Q^* and Q_i behind Q_i yields a walk β'. In the case of a dead end insertion, it corresponds to β because $v_{i,0} \dots v_{i,j}$ and $v_{i,j} \dots v_{i,t_i}$ lie in Q_i and $v_{i,j}v^*v_{i,j}$ lies in Q^*. In the case of a triangle insertion, it corresponds to β because $v_{i,0} \dots v_{i,j}$ and $v_{i,j+1} \dots v_{i,t_i}$ lie in Q_i and $v_{i,j}v^*v_{i,j+1}$ lies in Q^*. Both cases are depicted in Figure 4.3.

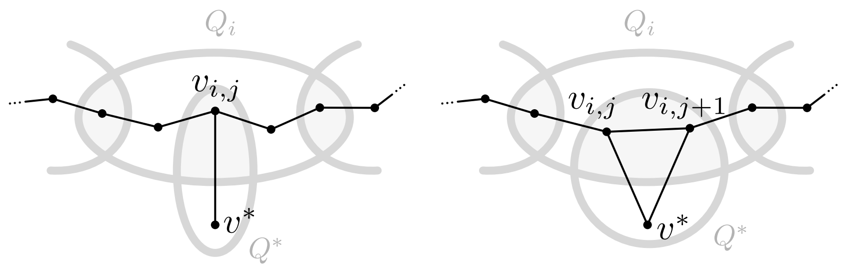

Figure 4.3: The elementary move in kG that corresponds to a dead end insertion (left) or triangle insertion (right) of a vertex which is not in Q_i.

<u>Case 2:</u> removal move. If the elementary move from α to β is a triangle removal or dead end removal, let the indices $i \in \{0, \dots, \ell - 1\}$ and $j \in \{0, \dots, t_i - 1\}$ be chosen such that $v_{i,j}$ (triangle removal) or $v_{i,j}$ and $v_{i,j+1}$ (dead end removal) are removed from Q_i. This choice is possible, as the (first) removed vertex and its successor lie in a common Q_i. If $j \geq 1$, the walk $\beta' = \alpha'$ corresponds to β as $v_{i,0} \dots v_{i,j-1}v_{i,j+1} \dots v_{i,t_i}$ or $v_{i,0} \dots v_{i,j-1}v_{i,j+2} \dots v_{i,t_i}$ respectively, still lie in Q_i. In case of a dead end removal, this works even if $t_i = j + 1$, as then $v_{i,j-1} = v_{i,j+1} = v_{i+1,0}$.

If $j = 0$, we know that $i \neq 0$, as otherwise $v_{i,j} = v_{0,0}$ would be removed. Furthermore, we know that if $i = 1$, $t_0 \neq 0$ as this also would imply that $v_{0,0} = v_{1,0}$ is removed. In any case, $v_{i,j}$ lies between $v_{i-1,t_{i-1}-1}$ and $v_{i,1}$. We now distinguish between two subcases.

<u>Case 2.1:</u> $v_{i-1,t_{i-1}-1} \notin Q_i$ and $v_{i,1} \notin Q_{i-1}$. As $v_{i,1} \in Q_i$, it is immediately clear that $v_{i-1,t_{i-1}-1} \neq v_{i,1}$, thus it is a triangle removal step and $v_{i-1,t_{i-1}-1}v_{i,0}v_{i,1}$ is a triangle. Let Q^* be a clique that contains $v_{i-1,t_{i-1}-1}$ and $v_{i,1}$. As Q^* is neither Q_{i-1} nor Q_i, the insertion of Q^* between Q_{i-1} and Q_i is a triangle insertion and thus the resulting walk β' is homotopic to α'. Furthermore, β and β' correspond, because $v_{i-1,0} \dots v_{i-1,t_{i-1}-1}$ lies in Q_{i-1}, $v_{i-1,t_{i-1}-1}v_{i,1}$ lies in Q^* and $v_{i,1} \dots v_{i,t_i}$ lies in Q_i, see Figure 4.4.

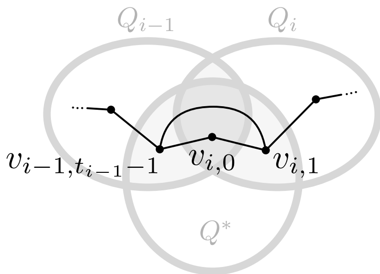

Figure 4.4: The elementary move in kG that corresponds to triangle removal in G.

Case 2.2: $v_{i-1,t_{i-1}-1} \in Q_i$ or $v_{i,1} \in Q_{i-1}$. We start by assuming that $v_{i,1} \in Q_{i-1}$. We subdivide α differently in subwalks that each lie in one clique Q_i. Let $t'_{i-1} := t_{i-1}+1$, $t'_i := t_i - 1$ and $t'_s := t_s$ for every $s \in \{0, \ldots, \ell-1\} \setminus \{i-1, i\}$. Furthermore, let $v'_{i-1,t'_{i-1}} := v_{i,1}$, let $v'_{i,u} := v_{i,u+1}$ for every $u \in \{0, \ldots, t'_i\}$, and let $v'_{s,u} := v_{s,u}$ for every $s \in \{0, \ldots, \ell-1\} \setminus \{i-1, i\}$ and every $u \in \{0, \ldots, t'_s\}$. Now, the removed vertex is $v'_{i-1,t'_{i-1}}$ and as $t'_{i-1} \geq 1$ we are in the $(j \geq 1)$-part of Case 2, which we have already dealt with. The step for $v_{i-1,t_{i-1}-1} \in Q_i$ is analogous.

After proceeding inductively for the other moves of the sequence, we reach a closed walk in kG which corresponds to a trivial walk in G. Thus, all vertices of that walk in kG are pairwise connected, as they all contain the single vertex of that trivial walk, and the walk in kG can easily be transformed into a trivial one. □

4.3 The Universal Triangular Covering Map

As triangular simple connectivity is preserved under the clique graph operator, the characterisation of clique dynamics for a given graph class can be split into two steps. The first step defines for every graph of the class a triangularly simply connected one that behaves similarly under the clique graph operator and the second step gives a characterisation of clique convergence for the triangularly simply connected graphs in the class. A candidate for a suitable triangularly simply connected version of an arbitrary graph is the universal triangular cover. We spend this section to define this special cover and show its existence and uniqueness.

We start by naming some properties of triangular covering maps and proving a lemma about lifting walks. Triangular covering maps fulfil the **unique edge lifting property**, i. e., for each pair of adjacent vertices $v, w \in V(G)$ and each

$\tilde{v} \in V(\tilde{G})$ such that $p(\tilde{v}) = v$, there is a unique $\tilde{w} \in V(\tilde{G})$ such that $\tilde{v}$ and $\tilde{w}$ are adjacent and $p(\tilde{w}) = w$. This property is equivalent to the **unique walk lifting property**, which says that for each walk α in G and each preimage of its start vertex there is a unique walk $\tilde{\alpha}$ in $\tilde{G}$ which is mapped to α. Furthermore, triangular covering maps fulfil the **triangle lifting property**, i. e., for each triangle (i. e. three-cycle) $\{u, v, w\}$ in G and each preimage $\tilde{u}$ of u, there exists a unique triangle $\{\tilde{u}, \tilde{v}, \tilde{w}\}$ in $\tilde{G}$ that is bijectively mapped to $\{u, v, w\}$. It follows from the unique walk lifting property that each triangular covering map between two connected graphs is surjective. Lastly, we remark that triangular covering maps preserve vertex degrees.

Throughout this section, we repeatedly make use of the following lemma connecting triangular covering maps and homotopy of walks. The lemma is equivalent to [Rot73, Lemma 2.2] but instead of proving this equivalence, we reprove the lemma in the language of [LN00].

Lemma 4.3. *Given a triangular covering map $p\colon \tilde{G} \to G$ and two homotopic walks $\alpha = v_0 \dots v_\ell$ and $\beta = v'_0 \dots v'_{\ell'}$ in G, for a fixed vertex $\tilde{v}_0$ from the preimage of their common start vertex $v_0 = v'_0$ the unique walks $\tilde{\alpha} = \tilde{v}_0 \dots \tilde{v}_\ell$ with $p(\tilde{v}_i) = v_i$ and $\tilde{\beta} = \tilde{v}'_0 \dots \tilde{v}'_{\ell'}$ with $p(\tilde{v}'_i) = v'_i$ are homotopic as well. Especially, they have the same end vertex $\tilde{v}_\ell = \tilde{v}'_{\ell'}$.*

Proof. As homotopy is defined by a finite sequence of elementary moves, it suffices to show that an elementary move in the image implies an elementary move in the preimage. Thus, let $\alpha = v_0 \dots v_\ell$ be a walk in G and let $\tilde{\alpha} = \tilde{v}_0 \dots \tilde{v}_\ell$ be from its preimage with $p(\tilde{v}_i) = v_i$. Let β be reached from α by inserting a vertex v^* and possibly v_i again between v_i and v_{i+1} for some $i \in \{0, \dots, \ell - 1\}$. As lifting a walk is done vertex by vertex from start to end, the lift of β begins with the vertices $\tilde{v}_0$ to $\tilde{v}_i$. As the restriction of p to the neighbourhood of v_{i-1} is an isomorphism, the lift of β starting in $\tilde{v}_0$ still has $\tilde{v}_{i+1}$ as the preimage of v_{i+1}, and consequently, the lift of β agrees with that of α in all following vertices. Thus, the lift of β arises from the lift of α by inserting a vertex $\tilde{v}^*$, possibly followed by $\tilde{v}_i$, between $\tilde{v}_i$ and $\tilde{v}_{i+1}$, which is an elementary move. For the elementary moves that remove vertices, exchange α and β. □

Next, we show that every connected graph has a triangularly simply connected triangular cover. The proof of the following lemma follows the ideas from [Rot73, Theorems 2.5, 2.8, and 3.6].

Lemma 4.4. *Every connected graph G has a triangular covering map $p\colon \tilde{G} \to G$ such that $\tilde{G}$ is triangularly simply connected.*

Proof. We give a construction for a graph $\tilde{G}$ and a map p. Then, we show that p is in fact a triangular covering map, that $\tilde{G}$ is connected and that $\tilde{G}$ is triangularly simply connected.

Construction of $\tilde{G}$ and p: We fix a vertex v of G. For each walk α, we denote by $[\alpha]$ its homotopy class, i. e., the set of walks that can be reached from α by a finite sequence of elementary moves. A walk β is called a continuation of a walk α if β arises from α by appending exactly one vertex to its end. Now we can define the graph $\tilde{G}$ by

$$V(\tilde{G}) = \{[\alpha] \mid \alpha \text{ is a walk in } G \text{ starting at vertex } v\} \text{ and}$$
$$E(\tilde{G}) = \{[\alpha][\beta] \mid \beta \text{ is a continuation of } \alpha\}.$$

Note that $[\alpha][\beta] \in E(\tilde{G})$ does not imply by definition that β is a continuation of α, but only that there are $\alpha' \in [\alpha]$ and $\beta' \in [\beta]$ such that β' is a continuation of α' or α' is a continuation of β'. However, it is quite easy to see, that if $[\alpha][\beta] \in E(\tilde{G})$, there is a $\beta' \in [\beta]$ such that β' is a continuation of α. We define

$$p\colon \tilde{G} \to G, [\alpha] \mapsto \operatorname{end}(\alpha),$$

in which $\operatorname{end}(\alpha)$ is the end vertex of α. The map p is well defined as homotopic walks have the same start and end vertex.

Triangular covering map: For an edge $[\alpha][\beta] \in E(\tilde{G})$, without loss of generality, we assume that β is a continuation of α. Thus, the end vertices of the two walks are adjacent and p is a graph homomorphism. Next we show that the restriction of p to neighbourhoods is bijective. Thus, let $[\alpha_w]$ be a class of walks from v to some vertex w. As noted above, the neighbourhood of $[\alpha_w]$ consists of the classes of continuations of α_w to the neighbours of w. Especially, the restriction of p to the neighbourhoods of $[\alpha_w]$ and w, respectively, is bijective. Let now α_x and α_y be the continuations of α_w by two distinct neighbours x and y of w. As we have already shown that the adjacency of $[\alpha_x]$ and $[\alpha_y]$ implies the adjacency of x and y, it remains to show the reverse. Thus, let x and y be adjacent. Hence, we can construct the walk α'_y as the continuation of α_x by the vertex y, thus, $[\alpha_x]$ and $[\alpha'_y]$ are adjacent. Since α'_y is reached from α_y by the elementary move of inserting x between w and y, they are homotopic and thus $[\alpha_y] = [\alpha'_y]$ is adjacent to $[\alpha_x]$.

Connectivity: We show that each vertex $[\alpha]$ of $\tilde{G}$ is connected to the class of the trivial walk $[\alpha_v]$ with vertex v. Thus, let α be any walk in G. The vertices $[\alpha_v]$ and $[\alpha]$ are connected by the walk $[\beta_0]\dots[\beta_\ell]$ in $\tilde{G}$, where ℓ is the length of α, and β_i is the initial subwalk of length i of α.

Triangular simple connectivity: For a closed walk $[\alpha_0]\dots[\alpha_\ell]$ with $[\alpha_0] = [\alpha_\ell]$ in $\tilde{G}$, as we noted above, we can assume without loss of generality that α_i is a continuation of α_{i-1} for each $i \in \{1,\dots,\ell\}$. Furthermore, we can assume that α_0 is the trivial walk as all the walks $\alpha_0,\dots,\alpha_\ell$ coincide with α_0 on their initial subwalks, anyway. We prove that the closed walk $[\alpha_0]\dots[\alpha_\ell]$ and the trivial walk $[\alpha_0]$ are homotopic. As α_0 and α_ℓ are homotopic, there is a finite sequence of elementary moves that transforms α_ℓ into α_0. To each walk α' in G that occurs in this homotopy between α_0 and α_ℓ, we associate the walk $[\alpha'_0]\dots[\alpha'_{\ell'}]$ where ℓ' is the length of α', and α'_i is the initial subwalk of length i of α'. Then, $[\alpha'_0]\dots[\alpha'_{\ell'}]$ is a walk by construction and it fulfils $\alpha'_0 = \alpha_0$ and $\alpha'_{\ell'} = \alpha'$. This way, we associate the final (trivial) walk α_0 to the trivial walk $[\alpha_0]$. If the walks α' and α'' are connected by an elementary move in G, their associated walks in $\tilde{G}$ are connected by the corresponding elementary move in the following way: A triangle insertion move that inserts v^* after v_i corresponds to the insertion of the class of the continuation of α_i by v^* and changing the representative of the following classes to the one, in which v^* is inserted after v_i. The other elementary moves work analogously. □

A triangular covering map $p\colon \tilde{G} \to G$ is called **universal** if it fulfils the following universal property: for each triangular covering map $q\colon \bar{G} \to G$ and for each pair of vertices $\tilde{v} \in V(\tilde{G})$ and $\bar{v} \in V(\bar{G})$ such that $p(\tilde{v}) = q(\bar{v})$, there exists a unique triangular covering map $\tilde{q}_{\tilde{v},\bar{v}}\colon \tilde{G} \to \bar{G}$ such that $\tilde{q}_{\tilde{v},\bar{v}}(\tilde{v}) = \bar{v}$ and $p = q \circ \tilde{q}_{\tilde{v},\bar{v}}$ (see the commuting diagram in Figure 4.5). In the next lemma, we show that

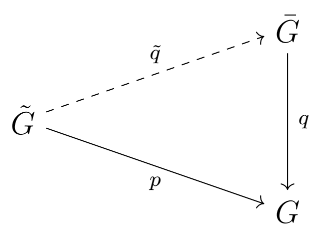

Figure 4.5: The commuting diagram depicting the universal property.

triangularly simply connected triangular covers are in fact universal. The proof is a combination of special cases from the proofs of [Rot73, Theorem 3.2 and Theorem 3.3].

Lemma 4.5. *A triangular covering map $p\colon \tilde{G} \to G$ with a triangularly simply connected graph $\tilde{G}$ is universal.*

Proof. Let $p\colon \tilde{G} \to G$ be a triangular covering map such that $\tilde{G}$ is triangularly simply connected and let $q\colon \bar{G} \to G$ be any triangular covering map. We fix a vertex $v \in V(G)$ as well as vertices $\tilde{v} \in V(\tilde{G})$ and $\bar{v} \in V(\bar{G})$ that are in the preimage of v under p and q, respectively. We construct $\tilde{q}_{\tilde{v},\bar{v}}$ from p and q and show that it is in fact a well-defined triangular covering map.

Construction of $\tilde{q}_{\tilde{v},\bar{v}}$: For each $\tilde{u} \in V(\tilde{G})$, we choose a walk $\alpha_{\tilde{v},\tilde{u}}$ from $\tilde{v}$ to $\tilde{u}$. The image of $\alpha_{\tilde{v},\tilde{u}}$ under p is a walk, which we call $\beta_{\tilde{u}}$, from $p(\tilde{v})$ to $p(\tilde{u})$. As $p(\tilde{v}) = v = q(\bar{v})$, by the unique walk lifting property, there is exactly one walk $\alpha_{\bar{v},\bar{u}}$ starting at $\bar{v}$ that is mapped to $\beta_{\tilde{u}}$ by q. We define $\tilde{q}_{\tilde{v},\bar{v}}(\tilde{u})$ to be the end vertex $\bar{u}$ of $\alpha_{\bar{v},\bar{u}}$.

Well-Definedness: We need to show that $\tilde{q}_{\tilde{v},\bar{v}}(\tilde{u})$ is independent of the choice of the walk $\alpha_{\tilde{v},\tilde{u}}$. Thus, let $\alpha'_{\tilde{v},\tilde{u}}$ be a different walk from $\tilde{v}$ to $\tilde{u}$. Its image under p is called $\beta'_{\tilde{u}}$ which has the same start and end vertices as $\beta_{\tilde{u}}$. As $\tilde{G}$ is triangularly simply connected, the walks $\alpha_{\tilde{v},\tilde{u}}$ and $\alpha'_{\tilde{v},\tilde{u}}$ are homotopic and, consequently, so are $\beta_{\tilde{u}}$ and $\beta'_{\tilde{u}}$. By Lemma 4.3 also the preimages under q, which are called $\alpha_{\bar{v},\bar{u}}$ and $\alpha'_{\bar{v},\bar{u}}$, are homotopic and, thus, have the same end vertex, implying $\tilde{q}_{\tilde{v},\bar{v}}$ being well defined. Additionally, $p = q \circ \tilde{q}_{\tilde{v},\bar{v}}$ holds by construction.

Homomorphy: Let $\tilde{x}, \tilde{y}$ be adjacent vertices in $\tilde{G}$. Let $\alpha_{\tilde{v},\tilde{y}}$ be a walk from $\tilde{v}$ to $\tilde{y}$ such that $\tilde{x}$ is its penultimate vertex. Via the same construction as above, we obtain a walk $\alpha_{\bar{v},\bar{y}}$ such that its penultimate vertex $\bar{x}$ fulfils $p(\bar{x}) = q(\tilde{x})$. Consequently, $\tilde{q}_{\tilde{v},\bar{v}}(\tilde{x}) = \bar{x}$ and $\tilde{q}_{\tilde{v},\bar{v}}(\tilde{y}) = \bar{y}$ are adjacent, and thus $\tilde{q}_{\tilde{v},\bar{v}}$ is a graph homomorphism.

Triangular covering map: Let $\tilde{u}$ be a vertex of $\tilde{G}$ and let $u = p(\tilde{u})$ and $\bar{u} = \tilde{q}_{\tilde{v},\bar{v}}(\tilde{u})$ be its images. As $p|_{N[\tilde{u}]}\colon N[\tilde{u}] \to N[u]$ and $q|_{N[\bar{u}]}\colon N[\bar{u}] \to N[u]$ are isomorphisms, so is $\tilde{q}_{\tilde{v},\bar{v}}|_{N[\tilde{u}]} = q|_{N[\bar{u}]}^{-1} \circ p|_{N[\tilde{u}]}$.

Uniqueness of $\tilde{q}_{\tilde{v},\bar{v}}$: Let $\tilde{q}\colon \tilde{G} \to \bar{G}$ be any triangular covering map such that $p = q \circ \tilde{q}$ and $\tilde{q}(\tilde{v}) = \bar{v}$. With the definitions from above, both the image of $\alpha_{\tilde{v},\tilde{u}}$ under $\tilde{q}$ and $\alpha_{\bar{v},\bar{u}}$ are lifts of the walk $\beta_{\tilde{u}}$ and they share the start vertex $\bar{v}$. By the unique walk lifting property, they are equal and so is their end vertex, implying $\tilde{q}(\tilde{u}) = \bar{u} = \tilde{q}_{\tilde{v},\bar{v}}(\tilde{u})$. □

Lemma 4.6. *If for a graph G there are two graphs $\tilde{G}$ and $\bar{G}$ and two universal triangular covering maps $p\colon \tilde{G} \to G$ and $q : \bar{G} \to G$, the graphs $\tilde{G}$ and $\bar{G}$ are isomorphic.*

Proof. Let $p\colon \tilde{G} \to G$ and $q\colon \bar{G} \to G$ be two triangular covering maps which both fulfil the universal property. Furthermore, let $\tilde{v} \in V(\tilde{G})$ and $\bar{v} \in V(\bar{G})$ be chosen such that $p(\tilde{v}) = q(\bar{v})$. By the universal properties, there are (unique) triangular covering maps $\tilde{p}\colon \bar{G} \to \tilde{G}$ and $\tilde{q}\colon \tilde{G} \to \bar{G}$ such that $p = q \circ \tilde{q}$, $\tilde{q}(\tilde{v}) = \bar{v}$, $q = p \circ \tilde{p}$, and $\tilde{p}(\bar{v}) = \tilde{v}$. Consequently, $p = p \circ \tilde{p} \circ \tilde{q}$ and $(\tilde{p} \circ \tilde{q})(\tilde{v}) = \tilde{v}$. As the identity map $id\colon \tilde{G} \to \tilde{G}$ is a triangular covering map that fulfils $p = p \circ id$ and $id(\tilde{v}) = \tilde{v}$, we know by the uniqueness of the universal property of p that $\tilde{p} \circ \tilde{q} = id$, which implies that $\tilde{q}\colon \tilde{G} \to \bar{G}$ is an isomorphism. □

Theorem 4.7. *Every connected graph has a universal triangular cover, which is triangularly simply connected and unique up to isomorphism.*

Proof. By Lemma 4.4, the graph G has a triangularly simply connected triangular cover, which is universal by Lemma 4.5. Let $p\colon \tilde{G} \to G$ and $q\colon \bar{G} \to G$ be two universal triangular covering maps. By Lemma 4.6, the universal triangular covers are isomorphic. □

4.4 Clique Convergence and the Universal Triangular Covering Map

In this section, we show that the universal triangular cover of a clique convergent graph is clique convergent, by applying the theory we developed before.

Theorem A. *If a connected graph is clique convergent, so is its universal triangular cover.*

Proof. Let the clique operator be convergent on a graph G, i. e., there are $n \in \mathbb{N}_0$ and $\ell \in \mathbb{N}$ such that $k^n G \cong k^{n+\ell} G$, and let $p\colon \tilde{G} \to G$ be a universal triangular covering map. As the corresponding maps p_{k^n} and $p_{k^{n+\ell}}$, that were defined in Section 3.2.3, are triangular covering maps and as $k^n\tilde{G}$ and $k^{n+\ell}\tilde{G}$ are triangularly simply connected by Lemma 4.2, they are universal triangular covering maps. As the universal triangular cover is unique up to isomorphism by Lemma 4.6, $k^n\tilde{G} \cong k^{n+\ell}\tilde{G}$ and $\tilde{G}$ is clique convergent. □

4.5 Group Actions, Quotient Graphs, and Galois Covers

This section gives a sufficient condition under which a graph with a clique convergent universal triangular cover is clique convergent itself. Here, triangular covering maps are approached from a different angle, as the covered graph is obtained from the cover by factoring out a group of symmetries. Before we get to this, we take a look at the relationships between group actions and the clique graph operator.

Remark 4.8. *If G is a Γ-graph, so is kG with respect to the induced action $\gamma Q = \{\gamma v \mid v \in Q\}$. Note that in some literature, e. g. in [LN00], this action is denoted as the natural action of the group $\Gamma_k \leq \mathrm{Aut}(kG)$, which is isomorphic to Γ. For a second Γ-graph H and a Γ-isomorphism $\phi\colon G \to H$, the map* $\boldsymbol{\phi_k}\colon kG \to kH, Q \mapsto \{\phi(v) \mid v \in Q\}$ *is a Γ-isomorphism.*

For each vertex $v \in V(G)$ of a Γ-graph G, we denote the orbit of v under the action of Γ by $\boldsymbol{\Gamma v}$. These orbits form the vertex set of the **quotient graph** $\boldsymbol{G/\Gamma}$, two of which are adjacent if they contain adjacent representatives.

We call a triangular covering map $p\colon \tilde{G} \to G$ **Galois with $\boldsymbol{\Gamma}$** if Γ is some group acting on $\tilde{G}$ such that the fibres of p are exactly the orbits of the action, which implies $\tilde{G}/\Gamma \cong G$.

Proposition 4.9. *[LN00, Prop. 3.2] Let $p\colon \tilde{G} \to G$ be Galois with some group Γ. Then, $p_k\colon k\tilde{G} \to kG$ is also Galois with Γ. Consequently, $k^n\tilde{G}/\Gamma \cong k^nG$ for every $n \in \mathbb{N}$.*

Lemma 4.10. *If a triangular covering map $p\colon \tilde{G} \to G$ is Galois with some group Γ and if there are $n \in \mathbb{N}_0$ and $r \in \mathbb{N}$ such that $k^n\tilde{G}$ and $k^{n+r}\tilde{G}$ are Γ-isomorphic, the quotient graphs k^nG and $k^{n+r}G$ are isomorphic and G is clique convergent.*

Proof. As $k^n\tilde{G}$ and $k^{n+\ell}\tilde{G}$ are Γ-isomorphic, we have $k^n\tilde{G}/\Gamma \cong k^{n+\ell}\tilde{G}/\Gamma$. By Proposition 4.9, we conclude $k^nG \cong k^n\tilde{G}/\Gamma \cong k^{n+\ell}\tilde{G}/\Gamma \cong k^{n+\ell}G$. □

In order to apply the previous lemma to universal triangular covers, we need those covers to be Galois. We show in the next lemma, that this is always the case.

Lemma 4.11. *A universal triangular covering map $p\colon \tilde{G} \to G$ is Galois with $\Gamma := \{\gamma \in \operatorname{Aut}(\tilde{G}) \mid p \circ \gamma = p\}$, which is called the* ***deck transformation group*** *of p. Moreover, it holds that $k^n\tilde{G}/\Gamma \cong k^nG$.*

Proof. As each $\gamma \in \Gamma$ fulfils $p \circ \gamma = p$, the group Γ acts on every fibre of p individually. Thus, it suffices to show that for each pair of vertices $\tilde{v}, \tilde{w}$ with $p(\tilde{v}) = p(\tilde{w})$ there is a $\gamma \in \Gamma$ such that $\gamma(\tilde{v}) = \tilde{w}$. If we apply Lemma 4.5 with $q = p$, we get a triangular covering map $\tilde{q}_{\tilde{v},\tilde{w}}$ which maps $\tilde{v}$ to $\tilde{w}$ and which is an isomorphism by Lemma 4.6, thus $\gamma = \tilde{q}_{\tilde{v},\tilde{w}}$ fulfils the condition. As p is a Galois covering map, by [LN00, Proposition 3.2] so is p_{k^n}. Consequently, it holds that $k^n\tilde{G}/\Gamma \cong k^nG$. □

Thus, we receive the following result:

Theorem B. *Let $p\colon \tilde{G} \to G$ be a universal triangular covering map and let Γ be the corresponding deck transformation group. If there are $n \in \mathbb{N}_0$ and $r \in \mathbb{N}$ such that $k^n\tilde{G}$ and $k^{n+r}\tilde{G}$ are Γ-isomorphic, the quotient graphs k^nG and $k^{n+r}G$ are isomorphic and G is clique convergent.*

5 Literature on Locally Cyclic Graphs and Whitney Triangulations

The local structure of a graph has a lot of influence on clique dynamics, as cliques are subsets of closed neighbourhoods of vertices. Thus, it seems natural to investigate graph classes which arise by restricting the isomorphy types of vertex neighbourhoods. Then, the classification of clique dynamics for such a class is based on global properties. A graph is called locally H if each open vertex neighbourhood induces a graph isomorphic to H, and it is called locally cyclic if each open neighbourhood is a cycle.

As the closed neighbourhoods of locally cyclic graphs are planar, those graphs can be viewed as triangulations of surfaces. In the next section, the link between locally cyclic graphs and Whitney triangulations of surfaces is established. The subsequent sections deal with criteria for clique convergence based on vertex degrees and with those based on topological properties. They stem from early work by Escalate [Esc73] on the one hand as well as more recent cooperation between Larrión, Neumann-Lara, Pizaña and, lately, Villarroel-Flores on the other hand, see [LN99], [LN00], [LNP02], [LNP03], [Piz03], [LNP06], and [LPV13].

5.1 Locally Cyclic Graphs and Whitney Triangulations of Surfaces

In the following, a surface is a connected Hausdorff space (i. e., for each pair of distinct points x and y there are neighbourhoods U of x and V of y such that $U \cap V = \emptyset$) in which any point has a neighbourhood homeomorphic to an open set in the closed upper half-plane. A surface triangulation is defined analogously to a plane graph in which the euclidean plane is replaced by any surface and in which all regions are bounded by triangles.

A **Whitney triangulation** is a triangulation of a surface in which every triangle of the graph bounds a region of the triangulated surface. For example, the octahedron is a Whitney triangulation of the sphere, but the double tetrahedron is not, see Figure 5.1.

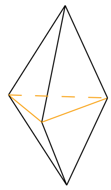

Figure 5.1: The double tetrahedron graph, in which the triangle does not bound a region.

As a finite graph is locally cyclic if and only if it is a Whitney triangulation of a closed surface (see [LNP02]), both concepts play vital roles in literature. Usually, papers about clique dynamics which deal with vertex degrees feature locally cyclic graphs and the ones which deal with topology feature Whitney triangulations, but this is not always the case.

5.2 Clique Dynamics Depending on Vertex Degrees

One of the first graphs which was known to be clique divergent is the unique (connected) 4-regular locally cyclic graph, also known as the octahedron graph. Its clique divergence was shown by Escalante [Esc73] through explicitly describing its sequence of iterated clique graphs and induction. Subsequently, the question for the clique dynamics of the unique (connected) 5-regular locally cyclic graph, i. e., the icosahedron graph, was asked, which was answered thirty years later by Pizaña [Piz03] using admissible relations, absolute saturation, and absolute freedom.

The clique divergence of some finite 6-regular locally cyclic graphs was shown in [LN99] and it was generalised to the divergence of all finite 6-regular locally cyclic graphs in [LN00]. While the first proof is done by direct construction and induction, the second uses triangular covering maps. Shortly after that, it was shown that all finite locally cyclic graphs with minimum degree $\delta \geq 7$, not just the regular ones, are clique convergent[LNP02]. This proof consists of a direct construction of the iterated clique graphs kG, k^2G and k^3G as well as the isomorphism between kG and k^3G and a generalisation to infinite locally cyclic graphs with minimum degree $\delta \geq 7$ is straight forward. Additionally, it is mentioned that the proof does not require the graph G to be locally cyclic as it suffices for each open vertex neighbourhood $N(v)$ to have a girth of at least 7, i. e., it cannot contain a cycle of length less than 7.

This thesis closes the gap between 6-regular locally cyclic graphs and locally cyclic graphs of minimum degree $\delta \geq 7$ and generalises to infinite graphs by giving a characterisation for the clique dynamics of locally cyclic graphs with minimum degree $\delta \geq 6$. A proof for triangularly simply connected graphs is done via direct construction and induction. For the other graphs, triangular covering maps are used in a way comparable to [LN00].

5.3 Clique Dynamics Depending on Topology

While the previous section focusses on the impact of the vertex degrees on the clique dynamics, this section deals with the connection between the topological structure of the triangulated surface and the clique dynamics.

In 2003, it was shown that each closed surface with the possible exceptions of the sphere, the Klein bottle, the projective plane, and the torus have a finite clique convergent Whitney triangulation [LNP03]. It is conjectured that at least for the sphere no such triangulation exists, not counting the tetrahedron which is seen more as a three-dimensional triangulation of a ball than a two-dimensional object. The proof was obtained by giving a construction for a triangulation of the surface that is achieved by cutting a hole into a torus and gluing the boundary of that hole to the boundary of the Möbius strip. It was then proven by computer verification that this graph is clique convergent and triangulations of the other surfaces in question were generated using triangular covering maps.

In contrast to the result above, every closed surface can be triangulated by a finite clique divergent locally cyclic graph [LNP06]. The proof works by constructing a triangulation for each possible genus of an orientable surface and shows clique divergence by giving an admissible relation from a rank divergent graph. The non-orientable graphs are dealt with by covering them with orientable ones using triangular covering maps.

Analogous versions of the two theorems above, which concern Whitney triangulations of compact surfaces with non-empty boundary were proven in [LPV13], but as they do not correspond to locally cyclic graphs (the open neighbourhoods of boundary vertices are paths and not cycles), the details are omitted here.

As the literature mainly deals with finite graphs, most papers focus on triangulations of compact surfaces. Consequently, the clique dynamics of infinite triangulations of non-compact surfaces is generally unknown and analogue proofs to the ones described above are likely to run into problems with generalising the tools to infinite graphs like those described in Chapter 3.

6 Preliminaries on Locally Cyclic Graphs

This chapter establishes definitions and tools for a non-recursive construction of the n-th iterated clique graphs of those graphs which can occur as the universal triangular cover of a locally cyclic graph with minimum degree $\delta \geq 6$. We know from Theorem 4.7 that these graphs have to be triangularly simply connected. By definition, the restrictions of triangular covering maps to closed neighbourhoods of vertices are isomorphisms, and thus, the universal covers of locally cyclic graphs with minimum degree $\delta \geq 6$ have to be locally cyclic of minimum degree $\delta \geq 6$, too. From now on, triangularly simply connected, locally cyclic graphs of minimum degree $\delta \geq 6$ are called **pikas** and throughout this chapter, the graph G will always be a pika.

In Section 6.1, the hexagonal plane and triangular-shaped graphs as well as some maps between them are introduced. In Section 6.2, straight walks in triangular-shaped graphs are described which are needed for proving some properties of triangular-shaped graphs in Section 6.3. The subsequent Section 6.4 describes the distance-d-neighbourhoods of vertices in pikas, Section 6.5 uses this description to characterise the possible neighbourhoods of triangular-shaped graphs, and Section 6.6 describes possible configurations of triangular-shaped subgraphs of pikas by extending isomorphisms to parts of the hexagonal plane. Section 6.1, Section 6.2, Section 6.3, Section 6.5, and Section 6.6 are based on joint work with Markus Baumeister [BL22].

6.1 The Hexagonal Grid and Triangular-Shaped Graphs

We start by defining the hexagonal grid and a class of its subgraphs called triangular-shaped graphs as well as certain maps between triangular-shaped graphs and other graphs. The triangular-shaped graphs will function as building blocks for the geometric construction of the iterated clique graphs of pikas in Chapter 7.

We define the coordinate set

$$\vec{\boldsymbol{D}}_{\boldsymbol{0}} := \{(1,-1,0),(1,0,-1),(-1,1,0),(0,1,-1),(-1,0,1),(0,-1,1)\}.$$

For $m \in \mathbb{Z}$, the **hexagonal grid of height** $\boldsymbol{m}$ is the graph $\boldsymbol{Hex_m} = (V_m, E_m)$ with

$$\boldsymbol{V_m} := \{(x_1,x_2,x_3) \in \mathbb{Z}^3 \mid x_1+x_2+x_3 = m\} \text{ and}$$
$$\boldsymbol{E_m} := \{\{x,y\} \subseteq V_m \mid x-y \in \vec{D}_0\}.$$

Defining the hexagonal grid as part of the hyperplane $x_1+x_2+x_3 = m$ instead of $\mathbb{R}^2$ has the advantage that some symmetries are given by coordinate permutations. Furthermore, it simplifies the next definition.

For $m \geq 0$, we define the **triangular-shaped graph of side length** $\boldsymbol{m}$, by $\boldsymbol{\Delta_m} := \mathrm{Hex}_m[V_m \cap \mathbb{N}_0^3]$ (called F_m in [BS95]). Figure 6.1 shows the smallest five of those subgraphs. Sometimes, a graph isomorphic to Δ_m will be called $\boldsymbol{\Delta_m}$**-shaped**.

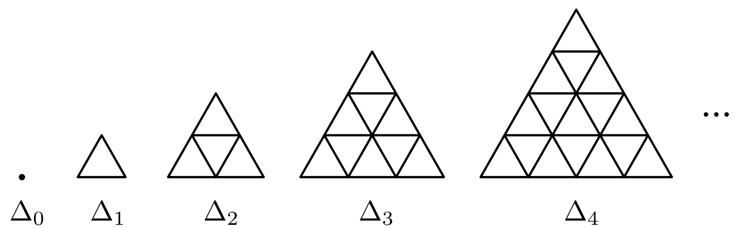

Figure 6.1: The triangular-shaped graphs Δ_m for $m \in \{0,\ldots,4\}$.

We see that triangular-shaped graphs are locally cyclic with boundary and the boundary $\partial\Delta_m$ is given by the vertices of degree less than six and the edges that only lie in a single three-cycle, see Figure 6.2.

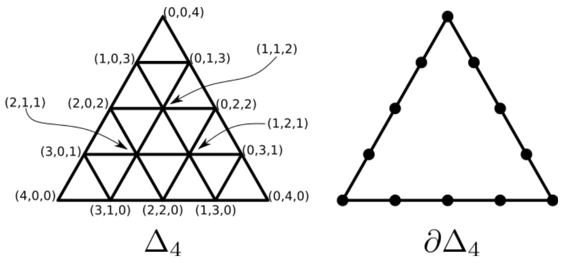

Figure 6.2: The triangular-shaped graph Δ_4 and its boundary $\partial\Delta_4$.

For a locally cyclic graph G, a **hexagonal chart** is a graph isomorphism $\mu\colon H \to F$ (also written $H \xrightarrow{\mu} F$) with subgraphs $H \subseteq \mathrm{Hex}_m$ and $F \subseteq G$. If $H = \Delta_m$, we call it a **standard chart**.

Since the symmetric group on three points acts on the hexagonal grid by coordinate permutations, every subgraph $F \cong \Delta_m$ with $m \geq 1$ has six standard charts.

For $(t_1, t_2, t_3) \in \mathbb{Z}^3$, we define the **triangle inclusion map**

$$\boldsymbol{\Delta_m^{t_1,t_2,t_3}}\colon \Delta_m \to \mathrm{Hex}_{m+t_1+t_2+t_3}, \qquad (a_1, a_2, a_3) \mapsto (a_1 + t_1, a_2 + t_2, a_3 + t_3).$$

The image of $\Delta_m^{t_1,t_2,t_3}$ is denoted by $\boldsymbol{\Delta_m + (t_1, t_2, t_3)}$. Furthermore, for a hexagonal chart $\mu\colon \Delta_m \to G$, the image of $\mu \circ \Delta_{m-t_1-t_2-t_3}^{t_1,t_2,t_3}$ is denoted by $\boldsymbol{\mu^{t_1,t_2,t_3}}$.

6.2 Straight Walks

Next, we define straight walks in locally cyclic graphs and list the long straight walks contained in a triangular-shaped graph, as this will be helpful to describe the way in which triangular-shaped subgraphs of certain locally cyclic graphs can intersect.

A monomorphism of locally cyclic graphs with boundary preserves vertex degrees of inner vertices. Furthermore, the number of incident faces on either side of a walk is preserved, too. We formalise this by the concept of **walk degrees**:

Let $\alpha = x_0 x_1 \ldots x_k$ be a walk in a locally cyclic graph G with boundary and consider a vertex x_i for $0 < i < k$.

- If x_i is an inner vertex, $N_G(x_i)$ is a cycle of length $\deg_G(x_i)$, and marking x_{i-1} and x_{i+1} splits the cycle into two paths of lengths l_1 and l_2, satisfying

$l_1 + l_2 = \deg_G(x_i)$. The **walk degree** $\boldsymbol{deg_G^\alpha(x_i)}$ is defined as $\{l_1, l_2\}$, as visualised in Figure 6.3.

- If x_i is a boundary vertex, $N_G(v)$ is a path graph containing a unique shortest walk from x_{i-1} to x_{i+1} with length l. The **walk degree** $\boldsymbol{deg_G^\alpha(x_i)}$ is defined as $\{l\}$.

The walk α is called **straight** if 3 is contained in $\deg_G^\alpha(x_i)$ for each $0 < i < k$. The concepts of walk degrees and straight walks are illustrated in Figure 6.3.

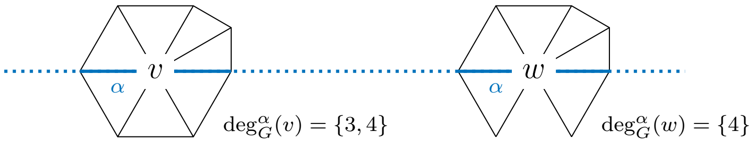

Figure 6.3: The walk degrees of the inner vertex v and the boundary vertex w. Since the walk degree $\deg_G^\alpha(w)$ does not contain 3, α is not straight.

As an important application, we construct the straight walks within Δ_m.

Remark 6.1. *For $m \geq 3$, up to symmetry (see Section 6.1), the maximal straight walks with length at least $m-2$ in Δ_m are the following three, which are also depicted in Figure 6.4.*

1. *For length m, we have $\alpha\colon \{0,\dots,m\} \to \mathbb{Z}^3$ with $t \mapsto (m-t, t, 0)$.*
2. *For length $m-1$, we have $\beta\colon \{0,\dots,m-1\} \to \mathbb{Z}^3$ with $t \mapsto (m-1-t, t, 1)$.*
3. *For length $m-2$, we have $\gamma\colon \{0,\dots,m-2\} \to \mathbb{Z}^3$ with $t \mapsto (m-2-t, t, 2)$.*

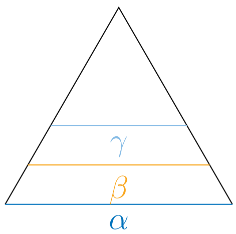

Figure 6.4: The maximal straight walks in Δ_m.

Proof. The boundary $\partial\Delta_m$ consists of six straight walks of length m, given by α and its images under coordinate permutations. Since any other straight walk contains an inner vertex and inner vertices have degree 6, the value of one of the three coordinates is constant along the walk. Without loss of generality (see Section 6.1), let this be the third coordinate. This way, we receive β and γ. $\square$

6.3 Properties of Triangular-Shaped Graphs

In this subsection, we prove some technicalities about triangular-shaped graphs and their relations.

Remark 6.2. *Let $m \geq 3$. The vertices of Δ_m with distance at least 1 to the boundary induce the graph $\Delta_m \setminus \partial\Delta_m \cong \Delta_{m-3}$. Thus, for $m \geq 6$, the vertices with distance at least 2 to the booundary induce $\Delta_m \setminus N_G[\partial\Delta_m] \cong \Delta_{m-6}$.*

We define some graphs and vertex sets for future reference. The set

$$\vec{\boldsymbol{E}} := V_1 \cap \mathbb{N}_0^3 = \{(1,0,0),(0,1,0),(0,0,1)\}$$

is the canonical basis, the graph

$$\boldsymbol{\nabla_1} := \mathrm{Hex}_2[(1,1,0),(0,1,1),(1,0,1)]$$

is the downward triangle of side length 1 in the centre of Δ_2, the graphs

$$\boldsymbol{\nabla_1} + \vec{\boldsymbol{e}} := \mathrm{Hex}_3[(1,1,0)+\vec{e},(0,1,1)+\vec{e},(1,0,1)+\vec{e}]$$

with $\vec{e} \in \vec{E}$ are the downward triangles of side length 1 inside Δ_3, the graph

$$\boldsymbol{\nabla_2} := \mathrm{Hex}_4[(2,2,0),(0,2,2),(2,0,2)]$$

is the downward triangle of side length 2 in the centre of Δ_4, the graph

$$\boldsymbol{\nabla_2'} := \mathrm{Hex}_1[(1,1,-1),(-1,1,1),(1,-1,1)]$$

is the downward triangle of side length 2 around Δ_1, and the graph

$$\boldsymbol{\nabla_3} := \mathrm{Hex}_3[(2,2,-1),(2,-1,2),(-1,2,2)]$$

is the result of rotating Δ_3 by $\pi/3$. The following two auxiliary lemmas discuss small special cases.

Lemma 6.3. *Let $m \geq 1$ and consider $\Delta_m \subseteq \mathrm{Hex}_m$. If $\Delta_{m-1} \cong S \subseteq \Delta_m$, either*

a) $S = \Delta_{m-1} + \vec{e}$ with $\vec{e} \in \vec{E}$, or

b) $m = 2$ and $S = \nabla_1$.

In particular, $\Delta_m \subseteq N_G[S]$.

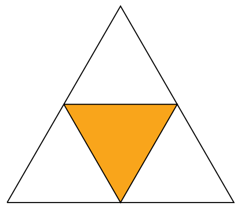

Figure 6.5: The special Δ_1-shaped subgraph of Δ_2.

Proof. Let $\Delta_{m-1}^{\vec{t}}\colon \Delta_{m-1} \to \mathrm{Hex}_m$ be a triangle inclusion map whose image lies in Δ_m. Thus, the component sum of $\vec{t}$ has to be 1 and each component has to be non-negative, leaving $\vec{t} \in V_1 \cap \mathbb{N}_0^3 = \vec{E}$.

It remains to consider those $\Delta_{m-1} \cong S \subseteq \Delta_m$ that are not the image of a triangle inclusion map. The boundary of such an S consists of three straight walks of length $m-1$. By Remark 6.1, there are six such walks along the boundary of Δ_m and three such walks in the interior (each given by all the vertices for which one fixed coordinate has value 1). Each of the boundary walks can only lie in one $\Delta_{m-1} \cong S \subseteq \Delta_m$. For each triangle inclusion map, two of the boundary walks of Δ_{m-1} are mapped to parts of boundary walks of Δ_m and one is mapped to an interior walk. As each of the straight boundary walks of Δ_m contains two subwalks of length $m-1$ and as each of them can only lie in the boundary of one subgraph isomorphic to Δ_{m-2}, the only remaining possibility is combining the three interior walks into a Δ_{m-1}. But this is only possible if the walks meet at their end points, which lie on the boundary and thus are $(1,1,0), (1,0,1)$ and $(0,1,1)$. Thus, this only happens if the component sum m is 2. □

Lemma 6.4. *Consider* $\Delta_m \subseteq \mathrm{Hex}_m$ *with* $m \geq 2$*. If* $\Delta_{m-2} \cong S \subseteq \Delta_m$*, either*

a) $S = \Delta_{m-2} + \vec{f}$ *with* $\vec{f} \in \vec{E} + \vec{E} = V_2 \cap \mathbb{N}_0^3$*,*

b) $m = 3$ *and* $S = \nabla_1 + \vec{e}$ *for some* $\vec{e} \in \vec{E}$*, or*

c) $m = 4$ *and* $S = \nabla_2$*.*

Proof. Let $\Delta_{m-2}^{\vec{t}}\colon \Delta_{m-2} \to \mathrm{Hex}_m$ be a triangle inclusion map whose image lies in Δ_m. Thus, each component of $\vec{t}$ has to be non-negative and thus $\vec{t} \in V_2 \cap \mathbb{N}_0^3 = \vec{E} + \vec{E}$. For $m = 2$, all $\Delta_0 \cong S \subseteq \Delta_m$ are possible images.

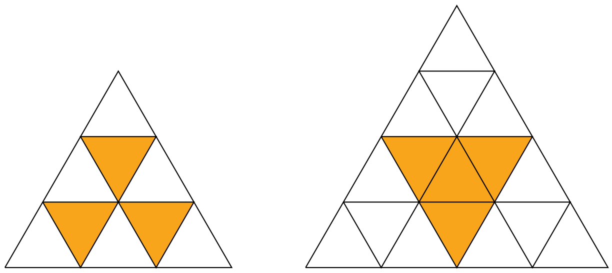

Figure 6.6: The special Δ_{m-2}-shaped subgraphs of Δ_m for $m = 3$ and $m = 4$.

It remains to consider those $\Delta_{m-2} \cong S \subseteq \Delta_m$ that are not the image of a triangle inclusion map. In this case, $m \geq 3$. The boundary of such an S consists of three straight walks of length $m - 2$. Remark 6.1 describes all walks of this kind inside Δ_m. Up to coordinate permutation, we only need to look at four of these walks:

1. The walk $\alpha_{|\{0,\dots,m-2\}}$ can only be the boundary of one Δ_{m-2} and it already is the boundary of $\Delta_{m-2}^{(2,0,0)}$.
2. The walk $\alpha_{|\{1,\dots,m-1\}}$ can only be the boundary of one Δ_{m-2} and it already is the boundary of $\Delta_{m-2}^{(1,1,0)}$.
3. The walk $\beta_{|\{0,\dots,m-2\}}$ is the boundary of $\Delta_{m-2}^{(1,0,1)}$, where the vertices each have a third component of at least 1. There can only be a Δ_{m-2} with third components of at most 1 if $m-2 \leq 1$, implying $m = 3$, as the case $m = 2$ was already treated. The triangular-shaped graph has corner vertices $(2, 0, 1)$, $(1, 1, 1)$, and $(2, 1, 0)$.
4. The walk γ is the boundary of $\Delta_{m-2}^{(0,0,2)}$ (whose third components are at least 2). There can only be a Δ_{m-2} with a lower third component if $m - 2 \leq 2$. The case $m = 3$ gives a triangular-shaped graph which is mapped to the one from (3.) by a suitable permutation of coordinates. The case $m = 4$ gives vertices $(2, 0, 2)$, $(0, 2, 2)$, and $(2, 2, 0)$.

Applying coordinate permutations to these Δ_{m-2} gives the desired results. □

6.4 Distance-d-Neighbourhoods in Pikas

We now show that the set of vertices of the pika G which are of distance at most d to a given vertex v induces a planar subgraph in which every bounded region is a triangle. Thus, let $\boldsymbol{N_G^d[v]} := \{w \in V(G) \mid \mathrm{dist}_G(v, w) \leq d\}$, especially, $N_G^0[v] = \{v\}$ and $N_G^1[v] = N_G[v]$.

Before we get to the main construction of this section, we prove an auxiliary lemma about triangles in the plane, which will be used to show planarity.

Lemma 6.5. *Let* $x, y, z \in \mathbb{R}^2$.

a) If $\operatorname{dist}(x,y) < \operatorname{dist}(x,z)$*, the two propositions* $\angle(xyz) \geq \pi/2$ *and* $\angle(zxy) \leq \arccos(\operatorname{dist}(x,y)/\operatorname{dist}(x,z))$ *are equivalent.*

b) If $\operatorname{dist}(x,y) < \operatorname{dist}(x,z)$ *and* $\angle(zxy) \leq \arccos(\operatorname{dist}(x,y)/\operatorname{dist}(x,z))$*, we have* $\operatorname{dist}(x,\overline{yz}) = \operatorname{dist}(x,y)$.

c) If $\operatorname{dist}(x,y) = \operatorname{dist}(x,z)$ *and* $\angle(zxy) \leq \pi/3$*, then* $\operatorname{dist}(x,\overline{yz}) \geq \sqrt{3}\operatorname{dist}(x,y)/2$.

Proof. a) Let $d_{xy} := \operatorname{dist}(x,y)$, $d_{xz} := \operatorname{dist}(x,z)$ and $d_{yz} := \operatorname{dist}(y,z)$. As the cosine is strictly decreasing on $[0,\pi]$ and by the law of the cosine, we have

$$\begin{aligned}\angle(xyz) \geq \pi/2 &\Leftrightarrow \cos(\angle(xyz)) \leq 0\\ &\Leftrightarrow \frac{d_{xy}^2 + d_{yz}^2 - d_{xz}^2}{2d_{xy}d_{yz}} \leq 0\\ &\Leftrightarrow \frac{2d_{xy}^2}{2d_{xy}d_{xz}} \leq \frac{d_{xy}^2 + d_{xz}^2 - d_{yz}^2}{2d_{xy}d_{xz}}\\ &\Leftrightarrow \frac{d_{xy}}{d_{xz}} \leq \cos(\angle(zxy))\\ &\Leftrightarrow \arccos\left(\frac{d_{xy}}{d_{xz}}\right) \geq \angle(zxy).\end{aligned}$$

b) By part (a), we know that $\angle(xyz) \geq \pi/2$, which implies that for each $a \in \overline{yz}$, we have $\angle(xaz) \geq \angle(xyz) \geq \pi/2$. Especially, for no point $a \in \overline{yz}$, we have $\operatorname{dist}(x,a) = \operatorname{dist}(x,yz)$ if yz is the (unbounded) line through y and z. Thus, $\operatorname{dist}(x,\overline{xz}) = \min(\operatorname{dist}(x,y),\operatorname{dist}(x,z)) = \operatorname{dist}(x,y)$.

c) As x, y, and z form an isosceles triangle with $\operatorname{dist}(x,y) = \operatorname{dist}(x,z)$, the distance $\operatorname{dist}(x,\overline{yz})$ is given by its height, which connects the middle point of $\overline{yz}$ to x and bisects the angle $\angle(zxy)$. By the right-angled triangle definition of the cosine as well as its monotonicity, $\angle(zxy) \leq \pi/3$ implies

$$\operatorname{dist}(x,\overline{yz}) = \cos\left(\frac{1}{2}\cdot\angle(zxy)\right)\operatorname{dist}(x,y) \geq \cos\left(\frac{\pi}{6}\right)\operatorname{dist}(x,y) = \frac{\sqrt{3}}{2}\operatorname{dist}(x,y).$$

□

Now, we get to the main lemma of this section.

Lemma 6.6. *Let $\mathcal{O} := (0,0)$ be the origin of $\mathbb{R}^2$. For a fixed vertex $v^* \in V(G)$, there is a sequence of plane graphs $(H_d)_{d\in\mathbb{N}_0}$ with $\mathcal{O} \in V(H_d)$ for each $d \in \mathbb{N}_0$ as well as a sequence of graph homomorphisms $(\psi_d)_{d\in\mathbb{N}_0}$ with $\psi_d \colon H_d \to G$ and $\psi_d(\mathcal{O}) = v^*$ for $d \in \mathbb{N}_0$, such that for $d \geq 1$ the following properties hold:*

i) The graph H_{d-1} is an induced subgraph of H_d.

ii) For each $v \in V(H_d \setminus H_{d-1})$, the graph theoretic and euclidean distances to the origin are given by $\operatorname{dist}_{H_d}(\mathcal{O}, v) = d$ and $|v| = 2^{d-1}$.

iii) The graph $H_d \setminus H_{d-1}$ is a cycle graph and the angle between each pair of consecutive vertices with respect to the origin is at most $\pi/3$.

iv) Each vertex $v \in V(H_{d-1})$ is of degree $\deg_{H_d}(v) = \deg_G(\psi_{d-1}(v))$ and its neighbourhood $N_{H_d}(v)$ induces a cycle graph.

v) Each vertex $v \in V(H_d \setminus H_{d-1})$ fulfils $3 \leq \deg_{H_d}(v) \leq 4$ and its neighbourhood $N_{H_d}(v)$ induces a path graph with both end vertices in $V(H_d \setminus H_{d-1})$.

vi) Each pair of adjacent vertices in $V(H_d \setminus H_{d-1})$ has exactly one common neighbour in H_d. That neighbour lies in $V(H_{d-1})$.

vii) For each $v \in V(H_{d-1})$, we have $\psi_d(v) = \psi_{d-1}(v)$.

viii) For each $v \in V(H_{d-1})$, the restriction $\psi_d|_{N_{H_d}[v]} \colon N_{H_d}[v] \to N_G[\psi_d(v)]$ is an isomorphism.

ix) For each $v \in V(H_d \setminus H_{d-1})$, the restriction $\psi_d|_{N_{H_d}[v]} \colon N_{H_d}[v] \to N_G[\psi_d(v)]$ is injective and the image $\psi_d(N_{H_d}[v])$ is an induced subgraph of $N_G[\psi_d(v)]$.

Proof. In order to prove our claim, for a given pika G and a fixed vertex $v^* \in V(G)$, we construct the sequences recursively. We start with the plane graph $H_0 = (\{\mathcal{O}\}, \emptyset)$ and the homomorphism $\psi_0 \colon H_0 \to G, \mathcal{O} \mapsto v^*$. Next, we construct H_1 and ψ_1 and check for the properties (i)-(ix).

As G is locally cyclic, $N_G[v^*]$ is planar. Let H_1 be the image graph of a planar embedding of $N_G[v^*]$ such that v^* is mapped to the origin, its neighbours are equidistantly mapped to the unit circle in cyclic order, and the edges are straight lines. Let ψ_1 be the inverse to that embedding, see Figure 6.7.

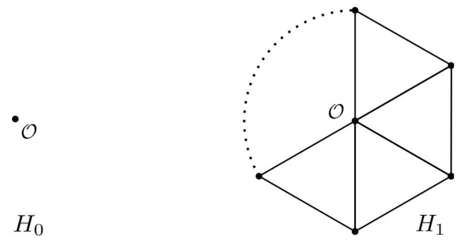

Figure 6.7: The graphs H_0 and H_1.

It is easy to see that H_1 is plane and that the pair of H_1 and ψ_1 fulfils all eight properties. We assume as an induction hypothesis that for some $d \geq 2$ the graphs $H_0, \dots H_{d-1}$ have already been constructed and fulfil the properties. We proceed by constructing H_d and showing that it is plane, proving (i)-(vi), constructing ψ_d, and proving (vii)-(ix).

By property (iii) of the induction hypothesis, the graph $H_{d-1} \setminus H_{d-2}$ forms a cycle of some length ℓ, whose vertices we denote by $v_1, \dots, v_\ell$ ordered anti-clockwise. All indices i in the following construction are meant to be integers modulo ℓ. First, we define vertices $v_{1,0}, \dots, v_{\ell,0}$ with $|v_{i,0}| = 2^{d-1}$ such that $\overline{\mathcal{O}v_{i,0}}$ bisects the angle $\angle(v_{i-1}\mathcal{O}v_i)$ or, equivalently, the polar angle of $v_{i,0}$ is the same as the polar angle of $(v_{i-1} + v_i)/2$. For each i, we define

$$r_i := \underbrace{\deg_G(\psi_{d-1}(v_i))}_{\substack{\geq 6 \\ \text{(as } G \text{ is a pika)}}} - \underbrace{\deg_{H_{d-1}}(v_i)}_{\substack{\in\{3,4\} \\ \text{(by property (v))}}} - 2 \geq 0.$$

We place r_i vertices $v_{i,1}, \dots v_{i,r_i}$ equidistantly and ordered by their polar angles on the circular arc of distance 2^{d-1} to the origin between the vertices $v_{i,0}$ and $v_{i+1,0}$.

With the three sets

$$\begin{aligned} R_d &:= \{v_{i,j} \mid i \in \mathbb{Z}/\ell\mathbb{Z}, 0 \leq j \leq r_i\}, \\ E_d &:= \{\overline{v_{i,j}v_{i,j+1}} \mid i \in \mathbb{Z}/\ell\mathbb{Z}, 0 \leq j < r_i\} \cup \{\overline{v_{i,r_i}v_{i+1,0}} \mid i \in \mathbb{Z}/\ell\mathbb{Z}\}, \text{ and} \\ E_{d-1,d} &:= \{\overline{v_i v_{i,j}} \mid i \in \mathbb{Z}/\ell\mathbb{Z}, 0 \leq j \leq r_i\} \cup \{\overline{v_{i-1}v_{i,0}} \mid i \in \mathbb{Z}/\ell\mathbb{Z}\}, \end{aligned}$$

we define H_d via

$$\begin{aligned} V(H_d) &:= V(H_{d-1}) \cup R_d \text{ and} \\ E(H_d) &:= E(H_{d-1}) \cup E_d \cup E_{d-1,d}. \end{aligned}$$

The construction of H_d is shown in Figure 6.8.

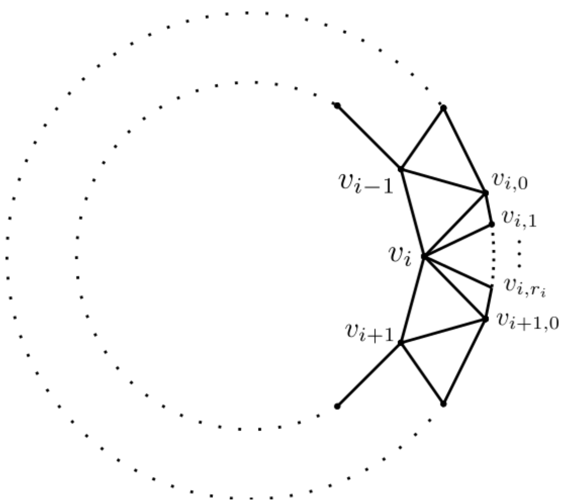

Figure 6.8: The construction of H_d from H_{d-1}.

For H_d to be plane, we need to show that two edges $e, f \in E(H_d)$ do not intersect unexpectedly. As all edges are line segments, edges with a common end vertex cannot intersect a second time. Thus, we can assume without loss of generality that e and f have pairwise distinct end vertices. Moreover, as H_{d-1} is plane by the induction hypothesis, we can assume without loss of generality that $e \notin E(H_{d-1})$. We distinguish the following cases:

Case 1: $e, f \in E_d$. As the polar angles of the points of e and f lie between the polar angles of their respective endpoints and as the vertices of R_d are cyclically ordered by their polar angles, e and f do not intersect.

Case 2: $e, f \in E_{d-1,d}$. Let the end vertices of e and f which lie in $V(H_{d-1})$ be denoted by v_i and v_g, respectively. The polar angles of all points of edges in $E_{d-1,d}$ which are incident to v_i lie between the polar angles of $(v_{i-1} + v_1)/2$ and $(v_i + v_{i+1})/2$. As the analogous proposition holds for the edges from $E_{d-1,d}$ incident to v_g, the respective ranges of possible polar angles can only intersect in a single point, which is in R_d and thus would imply e and f having a common end vertex in contradiction to the assumption.

Case 3: $e \in E_{d-1,d}$, $f \in E_d$. Analogously to the previous case, e and f cannot intersect if v_i is the end vertex of e which belongs to $V(H_{d-1})$ and if one of the end vertices of f is not adjacent to v_i. Thus, we consider the case that the end vertices of f are both adjacent to v_i. Let f' and f'' be the edges between v_i and the end vertices of f. As the vertices $v_{i,0}, \dots, v_{i,r_i}$, and $v_{i+1,0}$ are ordered by their polar angles, the line segment e cannot lie between f' and f''. Thus either $e \in \{f', f''\}$ and e and f intersect as expected or $e \notin \{f', f''\}$ and e and f do not intersect.

AS we assumed, that they do not intersect in end vertices, they do not intersect at all.

<u>Case 4:</u> $e \in E_{d-1,d}$ and $f \in E(H_{d-1})$. Let v_i be the end point of e which lies in $V(H_{d-1})$. As all points of f are of distance at most 2^{d-2} to the origin, it suffices to show that all points of $e \setminus \{v_i\}$ have a strictly larger distance to the origin.

We consider the triangle between the origin and the end points of e. For the minimum distance between $\mathcal{O}$ and any point on $\overline{v_i, v_{i,j}}$ to be 2^{d-2}, Lemma 6.5(b) requires $\angle v_i \mathcal{O} v_{i,j} \leq \arccos(\operatorname{dist}_{\mathbb{R}^2}(\mathcal{O}, v_i)/\operatorname{dist}_{\mathbb{R}^2}(\mathcal{O}, v_{i,j}))$. As $\angle(v_i \mathcal{O} v_{i,j}) \leq \angle(v_i \mathcal{O} v_{i+1,0})$ and $\operatorname{dist}_{\mathbb{R}^2}(\mathcal{O}, v_{i,j}) = 2^{d-2} = \operatorname{dist}_{\mathbb{R}^2}(\mathcal{O}, v_{i+1,j})$, it suffices to show that $\angle v_i \mathcal{O} v_{i+1,0} \leq \arccos(\operatorname{dist}_{\mathbb{R}^2}(\mathcal{O}, v_i)/\operatorname{dist}_{\mathbb{R}^2}(\mathcal{O}, v_{i+1,0}))$.

This is the case as the induction hypothesis implies $\angle(v_i \mathcal{O} v_{i+1}) \leq \pi/3$ and as $\overline{\mathcal{O} v_{i+1,0}}$ bisects $\angle(v_i \mathcal{O} v_{i+1})$ by definition, we get

$$\angle(v_i \mathcal{O} v_{i+1,0}) = \frac{1}{2}\angle(v_i \mathcal{O} v_{i+1}) < \frac{\pi}{3} = \arccos\left(\frac{1}{2}\right) = \arccos\left(\frac{\operatorname{dist}_{\mathbb{R}^2}(\mathcal{O}, v_i)}{\operatorname{dist}_{\mathbb{R}^2}(\mathcal{O}, v_{i,j})}\right).$$

<u>Case 5:</u> $e \in E_d$, $f \in E(H_{d-1})$. By applying Lemma 6.5(c) to the isosceles triangle with base e and tip $\mathcal{O}$, we receive

$$\operatorname{dist}(\mathcal{O}, e) \geq \frac{\sqrt{3} \cdot 2^{d-1}}{2} > 2^{d-2},$$

but the origin has a distance of at most 2^{d-2} to all points of f. Thus, e and f cannot intersect.

Hence, we have finished the proof that H_d is plane and move on to properties (i)-(vi).

i) By the construction of H_d from H_{d-1}, we keep all vertices and edges. As the additional edges are each incident to at least one new vertex, H_{d-1} is an induced subgraph of H_d.

ii) By the induction hypothesis, for each $v \in V(H_{d-1} \setminus H_{d-2})$, we have $\operatorname{dist}_{H_{d-1}}(\mathcal{O}, v) = d - 1$ and $|v| = 2^{d-2}$. The vertices in R_d have each a neighbour in R_{d-1} but none in $H_{d-1} \setminus R_{d-1}$. Thus, they have distance d to the origin with respect to H_d. By construction, they also have euclidean distance 2^{d-1} to the origin.

iii) It follows from the construction and from (ii) that $H_d \setminus H_{d-1}$ is a cycle with vertex set $R_d = \{v \in V(H_d) \mid |v| = 2^{d-1}\}$ and edge set E_d. Furthermore, by property (iii) of the induction hypothesis, we have $\angle(v_{i-1}\mathcal{O}v_i) \leq \pi/3$ for each $i \in \mathbb{Z}/\ell\mathbb{Z}$. Thus, for each $j \in \{0, \ldots, r_i - 1\}$, we have

$$\angle(v_{i,j}\mathcal{O}v_{i,j+1}) \leq \angle(v_{i,0}\mathcal{O}v_{i+1}) = \frac{\angle(v_{i-1}\mathcal{O}v_i) + \angle(v_i\mathcal{O}v_{i+1})}{2} \leq \pi/3$$

and, analogously, $\angle(v_{i,r_i}\mathcal{O}v_{i+1,0}) \leq \pi/3$.

iv) By the induction hypothesis, the neighbourhood $N_{H_{d-1}}(v)$ of any vertex $v \in V(H_{d-2})$ induces a cycle graph and so does $N_{H_d}(v)$, as the vertices of R_d have no neighbours in H_{d-2}.

For each i, we know by the induction hypothesis that the vertex $v_i \in R_{d-1}$ has degree $\deg_{H_{d-1}}(v_i) \in \{3, 4\}$ and that $N_{H_{d-1}}(v_i)$ is a path graph with both end vertices in $H_{d-1} \setminus H_{d-2}$, thus the end vertices are v_{i-1} and v_{i+1}. By the construction, the additional neighbours of v_i in H_d form the induced path $v_{i,0} \ldots v_{i,r_i}v_{i+1,0}$ and together with the edges $v_{i-1}v_{i,0}$ and $v_{i+1,0}v_{i+1}$ we receive a cycle of length $\deg_{H_{d-1}}(v_i) + 2 + r_i = \deg_G(\psi_{d-1}(v_i))$.

v) By construction, we can see that for each i, the neighbourhood $N_{H_d}(v_{i,0})$ induces the path $v_{i-1,r_{i-1}}v_{i-1}v_iv_{i+1,0}$, if $r_i = 0$ and $v_{i-1,r_{i-1}}v_{i-1}v_iv_{i,1}$, otherwise, so $\deg_{H_d}(v_{i,0}) = 4$. Either way, the end vertices lie in $H_d \setminus H_{d-1}$. Furthermore, for each i and for each $j \in \{1, \ldots, r_i\}$ the neighbourhood $N_{H_d}(v_{i,j})$ induces the path $v_{i,j-1}v_iv_{i+1,0}$, if $j = r_i$ and $v_{i,j-1}v_iv_{i,j+1}$, otherwise, so $\deg_{H_d}(v_{i,0}) = 3$. Again, the end vertices lie in $H_d \setminus H_{d-1}$.

vi) By construction, for each i with $r_i \geq 1$ and each $j \in \{0, \ldots, r_i - 1\}$, the adjacent vertices $v_{i,j}$ and $v_{i,j+1}$ have the unique common neighbour $v_i \in V(H_{d-1})$. Furthermore, for each i, the adjacent vertices v_{i,r_i} and $v_{i+1,0}$ have the unique common neighbour v_i.

Now, we can define ψ_d in two steps using the auxiliary graph $H'_d := H_d[V(H_{d-1}) \cup \{v_{i,0} \mid i \in \mathbb{Z}/\ell\mathbb{Z}\}]$. The first step constructs a homomorphism $\psi'_d \colon H'_d \to G$ and the second uses ψ'_d to construct ψ_d.

Step 1: For each $v \in V(H_{d-1})$, $\psi'_d(v) := \psi_{d-1}(v)$. By property (vi) of the induction hypothesis, for each i, the vertices v_{i-1} and v_i have exactly one common neighbour w in $V(H_{d-1})$. By the construction of H_d, they have exactly one additional neighbour $v_{i,0}$ in $V(H_d)$. As G is locally cyclic, the pair of adjacent vertices v_{i-1} and v_i has exactly two common neighbours. As ψ_{d-1} is a homomorphism, one of those neighbours is $\psi'_d(w) = \psi_{d-1}(w)$. We define $\psi'_d(v_{i,0})$ to be the other common neighbour. For ψ'_d to be a homomorphism, it remains to show that the added edges $v_i v_{i,0}$ and $v_i v_{i+1,0}$ are mapped to edges for each $i \in \mathbb{Z}/\ell\mathbb{Z}$. But this follows directly from the construction.

Step 2: For each $v \in V(H'_d)$, $\psi_d(v) := \psi'_d(v)$. As G is locally cyclic, for each i the neighbourhood $N_G(\psi_{d-1}(v_i))$ induces a cycle of length $\deg_G(\psi_{d-1}(v_i))$. By (iv), so does $N_{H_d}(v_i)$. As ψ'_d is a homomorphism, it maps the neighbourhood $N_{H'_d}(v_i)$, which is a path, to a walk in $N_G(\psi_{d-1}(v_i))$. By property (ix) of the induction hypothesis as well as by the choice of $\psi'_d(v_{i,0})$ and $\psi'_d(v_{i+1,0})$ this walk is a path of length $\deg_{H'_d}(v_i) = \deg_{H_{d-1}}(v_i) + 2 = \deg_G(\psi_{d-1}(v_i)) - r_i$ and we can chose the images $\psi_d(v_{i,1}), \dots, \psi_d(v_{i,r_i})$ to be the vertices of the walk of length r_i which is formed by the complement of $\psi'_d(N_{H'_d}(v_i))$ in $N_G(\psi_{d-1}(v_i))$. By the construction, ψ_d is a homomorphism. Now we can proof the remaining properties (vii)-(ix).

vii) By the construction, for each $v \in V(H_{d-1})$, we have $\psi_d(v) = \psi'_d(v) = \psi_{d-1}(v)$.

viii) For each i, the restriction $\psi_d|_{N_{H_d}[v_i]} \colon N_{H_d}[v_i] \to N_G[\psi_d(v_i)]$ is a bijective homomorphism by the construction. As the neighbourhoods $N_{H_d}(v_i)$ and $N_G(\psi_d(v_i))$ induce cycles, non-edges are mapped to non-edges and thus $\psi_d|_{N_{H_d}[v_i]}$ is an isomorphism.

ix) To prove that for each i and each $j \in \{1, \dots, r_i\}$, the restriction $\psi_d|_{N_{H_d}[v_{i,j}]} \colon N_{H_d}[v_{i,j}] \to N_G[\psi_d(v_{i,j})]$ is injective, we look at the explicit constructions of $N_{H_d}(v_{i,j})$ in the proof of (v). As for $j \in \{1, \dots, r_i\}$ the neighbourhood $N_{H_d}[v_{i,j}]$ lies completely inside the neighbourhood $N_{H_d}[v_i]$, by property (viii) the map $\psi_d|_{N_{H_d}[v_{i,j}]}$ is a restriction of an isomorphism and, thus, injective and $\psi_d(N_{H_d}[v_{i,j}])$ is an induced subgraph of $N_G[\psi_d(v_{i,j})]$. Thus, it remains to look at the neighbourhood of $v_{i,0}$. If $r_i \geq 1$, $N_{H_d}(v_{i,0})$ consists of the path $v_{i-1,r_{i-1}} v_{i-1} v_i v_{i,1}$. As ψ_d is a homomorphism, it maps adjacent vertices to adjacent, hence distinct, vertices. Thus, we need to show that the pairs of non-adjacent vertices from $\{v_{i-1,r_{i-1}}, v_{i-1}, v_i, v_{i,1}\}$ are neither mapped to the same nor to adjacent vertices. Thus, we need to show that

$$\psi_d(v_i) \neq \psi_d(v_{i-1,r_{i-1}}) \neq \psi_d(v_{i,1}) \neq \psi_d(v_{i-1})$$

as well as

$$\psi_d(v_i)\psi_d(v_{i-1,r_{i-1}}),\ \psi_d(v_{i-1,r_{i-1}})\psi_d(v_{i,1}),\ \psi_d(v_{i,1})\psi_d(v_{i-1}) \notin E(H_d).$$

Assuming $\psi_d(v_{i,1}) = \psi_d(v_{i-1})$ or $\psi_d(v_{i-1,r_{i-1}}) = \psi_d(v_{i,1})$, we get that

$\psi_d(v_{i,1}) \in N_G[\psi_d(v_{i-1})]$ but $v_{i,1} \notin N_{H_d}[v_{i-1}]$, thus $\psi_d|_{N_{H_d}[v_{i-1}]}$ would not be isomorphic and, thus, would be contradicting property (viii). The proof of $\psi_d(v_i) \neq \psi_d(v_{i-1,r_{i-1}})$ is analogous.

Assuming $\psi_d(v_{i,1})\psi_d(v_{i-1}) \in E(H_d)$, the vertices $\psi_d(v_{i,1}), \psi_d(v_i), \psi_d(v_{i-1})$, and $\psi_d(v_{i,0})$ would form a complete subgraph in G, in contradiction to G being locally cyclic with minimum degree $\delta \geq 6$. The proof of $\psi_d(v_i)\psi_d(v_{i-1,r_{i-1}}) \notin E(H_d)$ is analogous. Assuming $\psi_d(v_{i-1,r_{i-1}})\psi_d(v_{i,1}) \in E(H_d)$, the neighbourhood $N_G(\psi_d(v_{i,0}))$ would contain a cycle of length 4, which also contradicts G being locally cyclic with minimum degree $\delta \geq 6$.

The proof for $r_i = 0$ is analogous.

As H_d and ψ_d fulfil properties (i)-(ix), the claim holds by induction. □

We remark that for the graphs H_d from Lemma 6.6, $N^i_{H_d}[\mathcal{O}] = H_i$ for each $d \in \mathbb{N}_0$ and each $i \in \{0, \dots, d\}$.

Theorem 6.7. *For each vertex v^* of a pika G, the distance-d-neighbourhood $N^d_G[v^*]$ is a locally cyclic graph with boundary and that boundary is a cycle. Furthermore, $N^d_G[v^*]$ can be embedded into the plane such that the boundary borders the unbounded region. Especially, every bounded region is a triangle.*

Proof. As the sequence $(H_d)_{d\in\mathbb{N}_0}$ is an increasing sequence of induced subgraphs, the union graph $H = \bigcup_{d\in\mathbb{N}_0} H_d$ with $V(H) = \bigcup_{d\in\mathbb{N}_0} V(H_d)$ and $E(H) = \bigcup_{d\in\mathbb{N}_0} E(H_d)$ also contains each H_d with $d \in \mathbb{N}_0$ as an induced subgraph.

Analogously, we can define $\psi\colon H \to G$ via $\psi(v) = \psi_d(v)$ if $v \in V(H_d)$. As $\psi|_{H_d} = \psi_d$ for each $d \in \mathbb{N}_0$, property (viii) implies that ψ is a local isomorphism and, thus, a triangular covering map. As G is triangularly simply connected, by Lemma 4.5 it is its own universal cover and the corresponding universal triangular covering map is the identity. By the universal property, there is a unique triangular covering map $\tau\colon G \to H$ such that $\psi \circ \tau = id$. Thus, ψ is an isomorphism and so is each ψ_d when restricted to its image. As the H_d are plane and as only the boundary of the unbounded region is a cycle of length more than three, the first part of the theorem is shown.

As each edge in a planar graph belongs to at most two regions and as every inner edge of a locally cyclic graph belongs to exactly two triangles, the only edges that can belong to regions with more that three edges, are boundary edges. As the boundary graph is a cycle bounding the unbounded face, and as the two end vertices of a boundary edge have one common neighbour, the other region belonging to the boundary edge is a triangle. Thus, all bounded regions are triangles. □

From this, we can deduce the following corollary.

Corollary 6.8. *For each $r \geq 6$, there is a unique r-regular pika up to isomorphism.*

Proof. As the construction of the sequence $(H_d)_{d \in \mathbb{N}_0}$ only depends on degrees, it yields the same union graph H no matter with which r-regular pika we start. Thus, all r-regular pikas are isomorphic and we have proven uniqueness. Furthermore, we can do the construction of the sequence $(H_d)_{d \in \mathbb{N}_0}$ without any particular pika in mind and just choose every vertex degree as r. This way, we prove existence. □

6.5 Neighbourhoods of Triangular-Shaped Subgraphs in Pikas

To understand the structure of a pika G properly, we make sure that the boundary vertices of a Δ_m-shaped subgraph are not connected by walks in an unexpected way.

Lemma 6.9. *For each subgraph $S \cong \Delta_m$ of G, the following statements hold:*

a) Any edge incident to two boundary vertices of S is an edge of S. Consequently, S is an induced subgraph of G.

b) Let $v_0v_1v_2$ be a walk with $v_0, v_2 \in \partial S$ and $v_1 \notin S$. Then, either $v_0 = v_2$ or $\{v_0, v_1, v_2\}$ is a face (i.e. v_0v_2 is a boundary edge).

c) Let $v_0v_1v_2v_3$ be a path (especially v_0, v_1, v_2, and v_3 are distinct) with $v_0, v_3 \in \partial S$ and $v_1, v_2 \notin S$ such that neither $\{v_0, v_1, v_3\}$ nor $\{v_0, v_2, v_3\}$ are faces. Then, there exists a boundary vertex $v \in \partial S$ such that $\{v, v_1, v_2\}$ is a face.

Proof. In all cases, we start with a walk $v_0v_1 \dots v_k$ with $v_0, v_k \in \partial S$ and $v_1, \dots, v_{k-1} \notin S$ such that none of the edges v_iv_{i+1}, for $0 \leq i < k$, lies in S. In the first case, we aim for a contradiction, in the second and third cases, we show the claims directly.

As both the walk $v_0 \dots v_k$ and S have finite diameter and as G is locally finite, they both lie in the distance-d-neighbourhood of some common vertex for some $d \in \mathbb{N}_0$. By Theorem 6.7, this distance-d-neighbourhood has a planar embedding, such that every bounded region is a triangle. Henceforth, we identify that distance-d-neighbourhood with its embedding. Thus, all following calculations take place in the plane.

There are two walks along ∂S that connect v_k to v_0. By [Tho81, Corollary 1.2], out of three walks in the plane with the same start and end vertices, two of them bound a triangulated disc containing the third one. As $v_0 \dots v_k$ does not lie in S, it cannot be the middle one of those walks, and together with one of the walks along ∂S, it bounds a disc containing the other walk along ∂S. The walk which lies "inside" will be denoted by $v_k = s_0s_1 \dots s_m = v_0$, the other one by $v_kv_{k+1} \dots v_r = v_0$. The following construction is depicted in Figure 6.9.

For each $i \in \{0, \dots, k\}$, we denote the number of faces which are incident to v_i and contained in the triangulated disc with boundary walk $v_0 \dots v_k s_1 \dots s_m$ by α_i (thus, α_0 and α_k are the numbers of faces between the walk $v_0 \dots v_k$ and ∂S at v_0 and v_k, respectively). Analogously, for each $j \in \{0, \dots, r\}$, we denote the number of faces incident to v_j and contained in the disc with boundary $v_0 \dots v_k v_{k+1} \dots v_r$ by β_j. Consequently, for $i \in \{1, \dots, k-1\}$, we get $\alpha_i = \beta_i$.

We note that $\alpha_i = 0$ implies that the vertices which come before and after v_i in the walk $v_0 \dots v_k s1 \dots s_m$ coincide. Thus, $\alpha_0 \geq 1$ and $\alpha_k \geq 1$ as otherwise $v_1 \in S$ or $v_k \in S$.

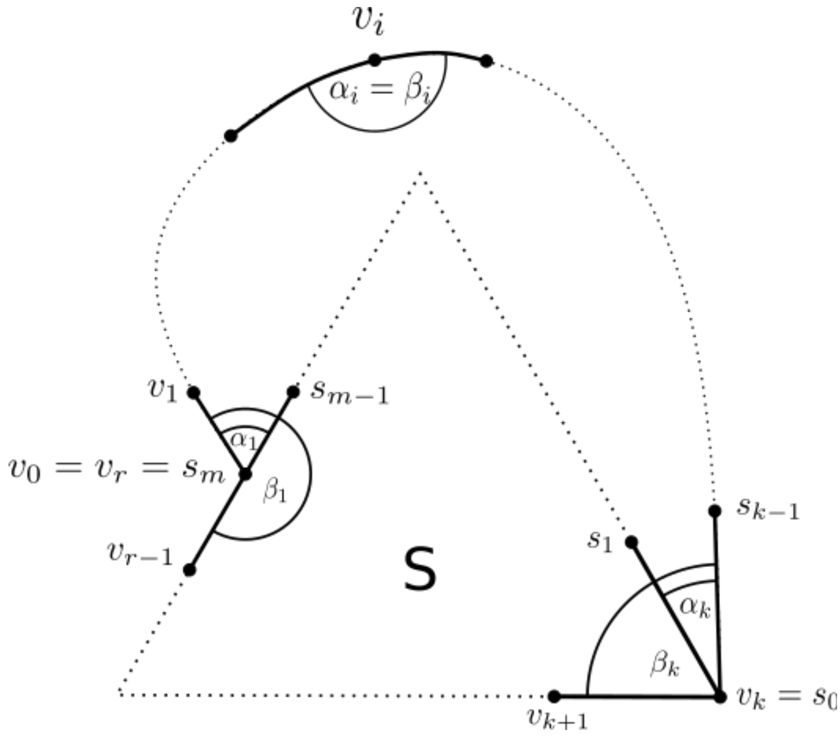

Figure 6.9: Illustration of a walk starting at a boundary vertex of $\Delta_m \cong S \subseteq G$ and ending at a corner vertex

Since $v_0 \dots v_r$ bounds a disc in the plane, Lemma 4.1.5 from [Bau20] is applicable and gives

$$6 = \sum_{v \text{ inner vertex}} (6 - \deg(v)) + \sum_{j=0}^{r-1} (3 - \beta_j).$$

Since $\deg(v) \geq 6$ for all vertices in G, we obtain

$$6 \leq \sum_{j=0}^{r-1} (3 - \beta_j) = (3 - \beta_0) + \sum_{i=1}^{k-1} (3 - \alpha_i) + (3 - \beta_k) + \sum_{j=k+1}^{r-1} (3 - \beta_j).$$

If a vertex v_j with $k < j < r$ lies in a corner of S, we have $\beta_j = 1$; otherwise, $\beta_j = 3$. analyse β_0 and β_k. Furthermore, if v_0 is a corner vertex of S, we have $\beta_0 = 1 + \alpha_0$; otherwise, we have $\beta_0 = 3 + \alpha_0$ and analogously, if v_k is a corner vertex of S, we have $\beta_k = 1 + \alpha_k$; otherwise, we have $\beta_k = 3 + \alpha_k$. Let c' be the number of corner vertices in $\{v_0, v_k\}$, let c'' be the number of corner vertices in $\{v_{k+1}, \dots, v_{r-1}\}$ and let $c = c' + c''$. We can rewrite the inequality above in the following way

$$
\begin{aligned}
6 &\le \sum_{j=0}^{r-1}(3-\beta_j) = (3-\beta_0) + \sum_{i=1}^{k-1}(3-\alpha_i) + (3-\beta_k) + \sum_{j=k+1}^{r-1}(3-\beta_j) \\
\Leftrightarrow 6 &\le (\alpha_0 - \beta_0) + \sum_{i=0}^{k}(3-\alpha_i) + (\alpha_k - \beta_k) + \sum_{j=k+1}^{r-1}(3-\beta_j) \\
\Leftrightarrow 6 &\le \sum_{i=0}^{k}(3-\alpha_i) - 6 + 2c' + \sum_{j=k+1}^{r-1}(3-\beta_j) \\
\Leftrightarrow 6 &\le \sum_{i=0}^{k}(3-\alpha_i) - 6 + 2c' + 2c'' \\
\Leftrightarrow 6 &\le \sum_{i=0}^{k}(3-\alpha_i) - 6 + 2c.
\end{aligned}
$$

As S is triangular-shaped, $c \in \{0, 1, 2, 3\}$. With the previous inequality, we proceed through the three cases of the lemma. Recall that $\alpha_0 \ge 1$ and $\alpha_k \ge 1$.

a) For the walk v_0v_1 we obtain $6 \le 2c - \alpha_0 - \alpha_1$, which has no solutions. Thus, there can be no edge between v_0 and v_1 that does not already lie in S. As each inner vertex of S has a cyclic neighbourhood with respect to S, it cannot have any additional neighbours. Thus, S is an induced subgraph.

b) For the walk $v_0v_1v_2$, we obtain $3 \le 2c - \alpha_0 - \alpha_1 - \alpha_2$. If $\alpha_1 = 0$, we obtain $v_0 = v_2$. Otherwise, the only possible solution is $c = 3$ and $\alpha_0 = \alpha_1 = \alpha_2 = 1$. This already implies that $\{v_0, v_1, v_2\}$ is a face of G.

c) For the path $v_0v_1v_2v_3$, we obtain $2c \ge \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3$. Since v_0, v_1, v_2, and v_3 are distinct, we have $\alpha_1 \ge 1$ and $\alpha_2 \ge 1$; thus $c \in \{2, 3\}$.

   Now, $\alpha_1 = 1$ implies that $\{v_0, v_1, v_2\}$ is a face. In particular, $v_0v_2v_3$ is a walk. With part (b) of this lemma, we conclude that $\{v_0, v_2, v_3\}$ is a face, in contradiction to our assumption. The same argument applies if $\alpha_2 = 1$. Thus, both of them have to be at least 2.

   Then, the only solution is $c = 3$ with $\alpha_0 = \alpha_3 = 1$ and $\alpha_1 = \alpha_2 = 2$. Since

$\alpha_0 = 1$, the triple $\{v_0, s_{m-1}, v_1\}$ forms a face. Since $\alpha_1 = 2$, there are two faces in the disc bounded by $v_0 \dots v_r$ which contain v_1. As one of them is $\{v_0, s_{m-1}, v_1\}$, the other has to be $\{v_1, s_{m-1}, v_2\}$. For $v := s_{m-1}$, this was the claim which needed to be shown. □

This helps proving two additional auxiliary lemmas.

Lemma 6.10. *Let $m \geq 1$ and let $S \cong \Delta_m$ be a subgraph of a pika G. Furthermore, let $S = S_1 \cup S_2 \cup S_3$ with $S_i \cong \Delta_{m-1}$ for each $i \in \{1, 2, 3\}$. Then, $N_G[S_1] \cap N_G[S_2] \cap N_G[S_3] \subseteq S$.*

Proof. Let $\mu\colon \Delta_m \to S$ be an isomorphism. First, we remark that $S = S_1 \cup S_2 \cup S_3$ implies that S_1, S_2, and S_3 are $\mu(\Delta_{m-1} + (1, 0, 0))$, $\mu(\Delta_{m-1} + (0, 1, 0))$, and $\mu(\Delta_{m-1} + (0, 0, 1))$, as those are the only triangular-shaped subgraphs of side length $m - 1$, whose union contains all three corners of Δ_m.

Now, assume to the contrary of the claim that there is an $x \in (N_G[S_1] \cap N_G[S_2] \cap N_G[S_3]) \backslash S$. Since $S_i \subseteq S$, we conclude $x \in N_G[S] \backslash S$. Without loss of generality, x is adjacent to $\mu(t, m - t, 0)$ for some $m/2 \leq t \leq m$ (otherwise permute the coordinates accordingly).

<u>Case 1:</u> $t < m$. Since $(t, m - t, 0)$ is not in $\Delta_{m-1} + (0, 0, 1)$, the vertex x has to be adjacent to a second boundary vertex of S, which lies in $\mu(\Delta_{m-1} + (0, 0, 1))$, say $\mu(s, 0, m - s)$ for some $0 \leq s < m$, see Figure 6.10.

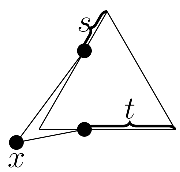

Figure 6.10: Illustration of common neighbour x.

By Lemma 6.9(b), the vertices $(t, m - t, 0)$ and $(s, 0, m - s)$ have to be adjacent. This is only possible for $t = s = m - 1$. But both faces incident to the edge $\{(m - 1, 1, 0), (m - 1, 0, 1)\}$ already lie in S (for $m > 1$), contradicting $x \notin S$. Note, that this case cannot occur for $m = 1$.

<u>Case 2:</u> $t = m$. Since x is only adjacent to corner vertices (otherwise we would be in the case $t < m$), it has to be adjacent to $\mu(0, m, 0)$ and $\mu(0, 0, m)$ as well (to lie in each $N_G[S_i]$). By Lemma 6.9(b), this implies the adjacency of the corner

vertices, i. e., $m = 1$. But then, the neighbourhood of x contains a cycle of length 3, in contradiction to neighbourhoods being cycles of length at least 6.

In both cases we reached a contradiction, hence the assumption is false. □

Lemma 6.11. *Let $S \cong \Delta_m$ be a subgraph of a pika G. Then, $N_G[S] \setminus S$ is a cycle and vertices in $N_G[S] \setminus S$ are incident to at most three faces in $N_G[S]$. Furthermore, $N_G[S] \setminus S$ has at least $3m + 6$ vertices and the inner vertices of $N_G[S]$ are exactly the vertices of S.*

Proof. If $m = 0$, this directly follows from the definition of locally cyclic. For $m \geq 1$, we enumerate the boundary vertices of S in cyclic order by $b_0 b_2 \ldots b_{3m-1}$ starting with b_0 being a corner vertex and we consider the indices to be integers modulo $3m$. For each adjacent pair (b_i, b_{i+1}), there is exactly one common neighbour $n_{i,i+1}$ of b_i and b_{i+1} such that the edges $b_i n_{i,i+1}$ and $b_{i+1} n_{i,i+1}$ do not belong to S. By Lemma 6.9 (a), $n_{i,i+1} \notin V(S)$.

With this notation, either $n_{i-1,i}$ and $n_{i,i+1}$ are either adjacent or there are vertices $x_{i,1}, \ldots, x_{i,r_i}$ such that $N_G(b_i) \setminus S$ is the path $n_{i-1,i} x_{i,1} \ldots x_{i,r_i} n_{i,i+1}$ (there are no further edges between these vertices since the neighbourhood of b_i is a cycle). If b_i is a corner of S, r_i is at least 2, otherwise, it might be 0, which would indicate that $n_{i-1,i}$ and $n_{i,i+1}$ are adjacent. None of those $x_{i,j}$ lies in S, since Lemma 6.9 (a) would imply that $b_i x_{i,j}$ were a boundary edge of S, in contradiction to our assumption. This situation is illustrated in Figure 6.11.

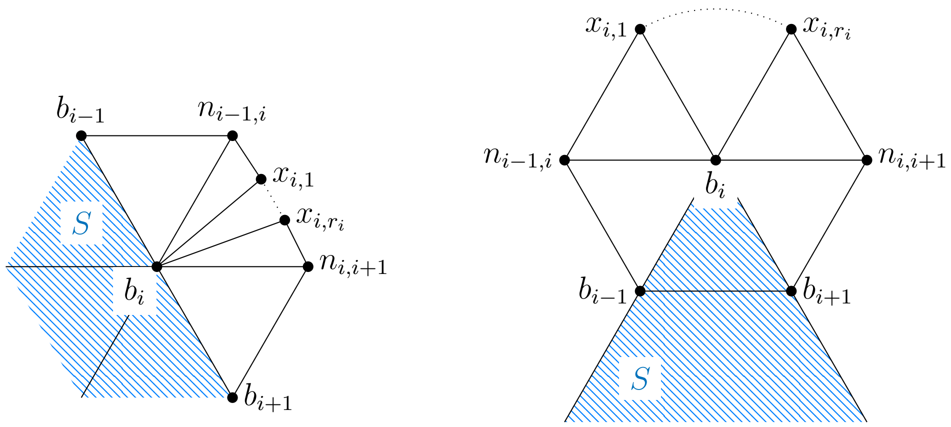

Figure 6.11: Parts of $N_G[S]$ at a side (left) and at a corner (right) of $S \cong \Delta_m$.

By Lemma 6.9(c), any edge between vertices in $N_G[S] \setminus S$ already lies in one $N_G(b_i) \setminus S$. Combining these paths gives the desired cycle.

By Lemma 6.9(a), S is an induced subgraph. Thus each neighbour of a vertex in ∂S that is not the end vertex of an edge from S lies in $N_G[S] \setminus S$. As G has a minimum degree of at least 6, each of the 3 vertices at he corners of S has at least 4 neighbours in G/S and each of the remaining $3m-3$ vertices in ∂S has at least 2 neighbours in G/S. As by Lemma 6.9(b) only ∂S-neighbouring vertices of ∂S can share a neighbour, we are left with at least $3 \cdot 4 + (3m-3) \cdot 2 - 3m = 3m+6$ elements in $N_G[S] \setminus S$. □

6.6 Chart Extensions

In this section, we discuss the possible ways in which triangular-shaped subgraphs of a pika G can lie in the closed neighbourhoods of other triangular-shaped subgraphs of the same side length. These descriptions will be used in the next chapter, as the construction of n-th iterated clique graphs uses this kind of configuration.

We prove that for $m \geq 3$, even though the closed neighbourhood $N_G[S]$ of some subgraph $S \cong \Delta_m$ of G is not necessarily isomorphic to a subgraph of the hexagonal grid, the union of all Δ_m-shaped subgraphs in $N_G[S]$ is. This way, all occurring configurations are subconfigurations of those from the hexagonal grid. We start with a description of the closed neighbourhood $N_{\mathrm{Hex}_m}[\Delta_m]$ and its Δ_m-shaped subgraphs.

Remark 6.12. *For $m \geq 3$, the closed neighbourhood of Δ_m in the hexagonal grid is*

$$N_{\mathrm{Hex}_m}[\Delta_m] = (\Delta_{m+3} \setminus \Lambda_{m+3}) + (-1,-1,-1)$$

with $\Lambda_{m+3} = \{(m+3,0,0),(0,m+3,0),(0,0,m+3)\}$ as can be seen in Figure 6.12.

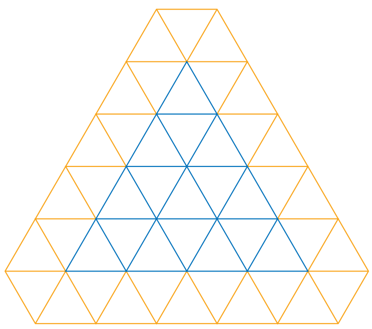

Figure 6.12: The closed neighbourhood $N_{\mathrm{Hex}_m}[\Delta_m]$ (in orange) of Δ_m (in blue).

Analogously to Remark 6.1, we can identify the straight walks in $N_{\mathrm{Hex}_m}[\Delta_m]$ *and build boundary paths for* Δ_m*-shaped subgraphs from them. This way, for* $m \geq 3$*, we receive the following* Δ_m*-shaped subgraphs, which are depicted in Figure 6.13:*

$$\{T \subseteq N_{\mathrm{Hex}_m}[\Delta_m] \mid T \cong \Delta_m\} = \begin{cases} \{\Delta_m\} \cup \{\Delta_m + \vec{d} \mid \vec{d} \in \vec{D}_0\}, & \text{if } m \geq 4, \\ \{\Delta_m\} \cup \{\Delta_m + \vec{d} \mid \vec{d} \in \vec{D}_0\} \cup \{\nabla_3\}, & \text{if } m = 3. \end{cases}$$

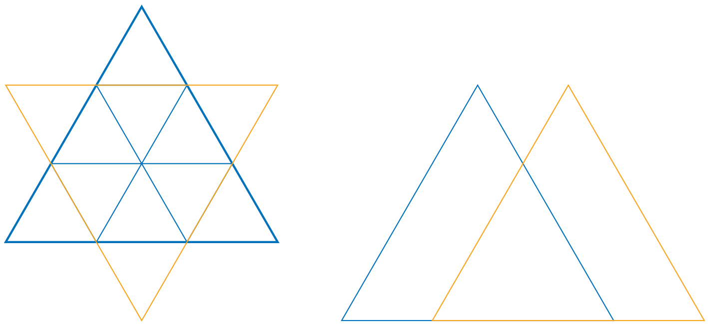

Figure 6.13: The graph Δ_3 (in blue) together with ∇_3 (in orange) on the left and the boundary of Δ_m together with the boundary of $\Delta_m + (-1, 1, 0)$ on the right.

For each $\Delta_m \cong T \subseteq N_{\mathrm{Hex}_m}[\Delta_m]$*, we have* $\Delta_m \subseteq N_{\mathrm{Hex}_m}[T]$*, so lying in each others closed neighbourhood is an equivalence relation on the* Δ_m*-shaped subgraphs of* Hex_m*.*

Next we show that for all Δ_m-shaped subgraphs S and T of G such that $T \subseteq N_G[S]$ and for a given isomorphism $\mu\colon \Delta_m \to S$ there is a graph $\Delta_m \subseteq H \subseteq \mathrm{Hex}_m$ and a hexagonal chart $\mu_T\colon H \to S \cup T$ extending μ. For this, we need two auxiliary lemmas that describe the way S and T intersect.

Lemma 6.13. *For* $m \geq 3$*, let* S *and* T *be* Δ_m*-shaped subgraphs of* G *and let* $S \neq T \subseteq N_G[S]$*. For each vertex* $v \in \partial S \cap \partial T$*, we have* $N_S(v) \cap N_T(v) \neq \emptyset$*. Furthermore,* $N_{S \cup T}(v)$ *cannot induce a cycle with respect to* $S \cup T$*.*

Proof. First, we observe that the only boundary vertices of a Δ_m-shaped graph with $m \geq 3$ which are not adjacent to an inner vertex are its corners. The distance between a corner and the closest inner vertex is 2. Let $v \in \partial S \cap \partial T$. We distinguish the following two cases:

Case 1: The vertex v is not a corner of T. Let w be an inner vertex of T which is adjacent to v. As $T \subseteq N_G[S]$, the vertex w is also an inner vertex of $N_G[S]$ and, by Lemma 6.11, it lies in S. Consequently, $w \in N_S(v) \cap N_T(v) \neq \emptyset$. Furthermore, if $N_{S\cup T}(v)$ would induce a cycle, it would be of length at least 6 as G is a pika. As the boundary vertices of Δ_m-shaped graphs are of degree 2 (for corners) or 4 (for non-corners), this would imply that $\deg_S(v) = \deg_T(v) = 4$ and $N_S(v)$ and $N_T(v)$ would both induce paths of length 3 which intersect only in their start and end vertices. As those share only one T-neighbour with v, they are thus boundary vertices of T. As w is an inner vertex of T it cannot be in $N_S(v) \cap N_T(v) \neq \emptyset$ contradicting what we have shown above. We conclude that $N_{S\cup T}(v)$ does not induce a cycle.

Case 2: The vertex v is a corner of T. Then, T has an inner vertex x of T-distance two to v. By the same argument as above, x lies in S. Let w be the middle vertex of a shortest walk from v to x. By Lemma 6.9 (b), either w lies in S and thus $w \in N_S(v) \cap N_T(v) \neq \emptyset$, or v and x are S-adjacent and thus $x \in N_S(v) \cap N_T(v) \neq \emptyset$. As $\deg_T(v) = 2$ and as $N_S(v) \cap N_T(v) \neq \emptyset$, we have $\deg_{S\cup T}(v) \leq 2 + 4 - 1 = 5$ and thus $N_{S\cup T}(v)$ cannot induce a cycle of length at least 6. ☐

Lemma 6.14. *For $m \geq 3$, let S and T be Δ_m-shaped subgraphs of G such that $S \neq T \subseteq N_G[S]$. The union subgraph $S \cup T$ of G is a locally cyclic graph with boundary, inner vertices of $S \cup T$ have degree* 6*, and boundary vertices have degree at most* 6*.*

Proof. We begin by checking that vertex neighbourhoods have the correct size and that they induce cycles or path graphs.

Each inner vertex v from S is an inner vertex of $S \cup T$ and it has degree 6 as S already contains all G-neighbours of v. The same holds for inner vertices of T.

Each boundary vertex of S that does not belong to T is a boundary vertex of $S \cup T$ and its $S \cup T$-degree equals its S-degree. As the boundary vertices of S have S-degree 2 or 4, those vertices fulfil the claim. The same holds for boundary vertices of T that do not lie in S.

It only remains to consider vertices which are boundary vertices of both S and T. Thus, let $v \in \partial S \cap \partial T$. As $N_S(v)$ and $N_T(v)$ are both path graphs of length 1 or 3 which lie in the cyclic neighbourhood $N_G(v)$, either they form a cycle of length 6, a path of length at most 5 or two paths. By Lemma 6.13, the first and third option are not possible. Thus, v is a boundary vertex of $S \cup T$ with degree at most 6. □

Now we have the prerequisites for showing that two Δ_m-shaped graphs have a common hexagonal chart.

Lemma 6.15. *Let $S \cong \Delta_m$ be a subgraph of G together with a standard chart $\mu\colon \Delta_m \to S$ and let $S \subseteq U \subseteq N_G[S]$ be locally cyclic with boundary such that its inner vertices have degree 6 and its boundary vertices have degree at most 6. Then, there is a subgraph $\Delta_m \subseteq H_U \subseteq \mathrm{Hex}_m$ and a hexagonal chart $\mu_U\colon H \to U$ that extends μ.*

Proof. We start by applying Lemma 6.11 to both $S \subseteq G$ and $\Delta_m \subseteq \mathrm{Hex}_m$. For the vertices from ∂S and $N_G[S] \setminus S$ we keep the notation from Lemma 6.11, so the vertices from ∂S are called b_i, the common neighbour of b_i and b_{i+1} is called $n_{i,i+1}$ and the additional neighbours of b_i are called $x_{i,1}, \dots, x_{i,r_i}$. The corresponding vertices from $\partial\Delta_m$ and $N_{\mathrm{Hex}_m}[\Delta_m] \setminus \Delta_m$, however, will be called v_i, $y_{i,i+1}$ and $z_{i,1}, \dots, z_{i,s_i}$, respectively. We choose the indices in a way that $\mu(v_i) = b_i$ for all i. Note that only for the corners of Δ_m we have $s_i > 0$ so, for most i, the only neighbours of v_i in $N_{\mathrm{Hex}_m}[\Delta_m] \setminus \Delta_m$ are $n_{i-1,i}$ and $n_{i,i+1}$. Now, we define a map $\overline{\mu}\colon U \to \mathrm{Hex}_m$. For each $x \in V(S)$, we define $\overline{\mu}(x) := \mu^{-1}(x)$. For each i, we look at $N_U[b_i]$. If $N_U[b_i]$ induces a cycle, this cycle is of length 6 and thus it is isomorphic to $N_{\mathrm{Hex}_m}[v_i]$. As $N_S[b_i]$ and $N_{\Delta_m}[v_i]$ induce path graphs of the same length, the isomorphism $\overline{\mu}|_{N_S[b_i]}$ can be uniquely extended to $N_U[b_i]$ analogously to the proof of Lemma 6.6. If $N_U[b_i]$ induces a path, this path is of length at most 5 while $N_{\mathrm{Hex}_m}[v_i]$ induces a cycle of length 6. Again, $N_U[b_i]$ contains $N_S[b_i]$ as a subpath which is mapped to $N_{\Delta_m}[v_i]$. Thus, $\overline{\mu}|_{N_S[b_i]}$ can be uniquely extended to $N_U[b_i]$. We remark, that in both cases $\overline{\mu}(n_{i,i+1}) = y_{i,i+1}$. As the only intersections of the $N_U[b_i]$ outside S are the vertices $n_{i,i+1}$ this is sufficient to show the well-definedness of the extension and it is easy to see, that $\overline{\mu}$ is a homomorphism. As $N_{\mathrm{Hex}_m}[\Delta_m] \setminus \Delta_m$ forms a cycle by Lemma 6.11, this homomorphism is injective and thus the restriction to its image H is an isomorphism. The inverse of this restriction will be called μ and is an isomorphism as well. This concludes the proof. □

By combining Lemma 6.14 and Lemma 6.15, we obtain the following corollary.

Corollary 6.16. *For each pair of Δ_m-shaped subgraphs S and T of a pika G such that $T \subseteq N_G[S]$, there is a subgraph $\Delta_m \subseteq H \subseteq \mathrm{Hex}_m$ and a hexagonal chart $\mu_T \colon H \to S \cup T$ extending a given standard chart $\mu \colon \Delta_m \to S$.*

Together with Remark 6.12, this yields the following corollary.

Corollary 6.17. *For each pair of Δ_m-shaped subgraphs S and T of a pika G, we have that $T \subseteq N_G[S]$ if and only if $S \subseteq N_G[T]$.*

After we have shown that we can extend a given standard chart of S to one Δ_m-shaped subgraph $T \subseteq N_G[S]$, we show next that it is possible to extend it to all such T simultaneously

Lemma 6.18. *For a Δ_m-shaped subgraph S of a pika G, for $\hat{S} := \bigcup\{T \cong \Delta_m \mid T \subseteq N_G[S]\}$, and for a standard chart $\mu \colon \Delta_m \to S$ there is a subgraph $\Delta_m \subseteq H \subseteq \mathrm{Hex}_m$ and a hexagonal chart $\hat{\mu} \colon H \to \hat{S}$ extending μ. Furthermore, for each $T \cong \Delta_m$ with $T \subseteq N_G[S]$,*

$$\hat{\mu}^{-1}(T) \in \begin{cases} \{\Delta_m\} \cup \{\Delta_m + \vec{d} \mid \vec{d} \in \vec{D}_0\}, & \textit{if } m \geq 4, \\ \{\Delta_m\} \cup \{\Delta_m + \vec{d} \mid \vec{d} \in \vec{D}_0\} \cup \{\nabla_3\}, & \textit{if } m = 3. \end{cases}$$

Proof. Let $H = \bigcup_{\Delta_m \cong T \subseteq N_G[S]} \mu_T^{-1}(T)$. We define $\hat{\mu} \colon H \to \hat{S}$ by $\hat{\mu}(x) = \mu_T(x)$ for each $x \in V(T)$ and $\Delta_m \cong T \subseteq N_G[S]$. In order to proof that $\hat{\mu}$ is well-defined, we need to show that for any $T_1, T_2 \in \{T \cong \Delta_m \mid T \subseteq N_G[S]\}$ such that the preimages H_1 of μ_{T_1} and H_2 of μ_{T_2} intersect non-trivially, we have$\mu_{T_1}(x) = \mu_{T_2}(x)$ for each $x \in V(H_1) \cap V(H_2)$. By Remark 6.12, we have

$$H_1, H_2 \in \begin{cases} \{\Delta_m\} \cup \{\Delta_m + \vec{d} \mid \vec{d} \in \vec{D}_0\}, & \text{if } m \geq 4, \\ \{\Delta_m\} \cup \{\Delta_m + \vec{d} \mid \vec{d} \in \vec{D}_0\} \cup \{\nabla_3\}, & \text{if } m = 3. \end{cases}$$

If H_1 or H_2 are Δ_m or if they only intersect in vertices of Δ_m, the claim follows immediately. Up to symmetry we are left with $H_1 = \Delta_m + (-1, 1, 0)$ and $H_2 \in \{\Delta_m + (-1, 0, 1), \nabla_3\}$.

If $m = 3$ and $H_2 = \nabla_3$, we have $V((H_1 \cap H_2) \setminus \Delta_3) = \{(-1, 2, 2)\}$ as can be seen in Figure 6.14. As the vertex $(-1, 2, 2)$ is incident to $(0, 1, 2)$ and $(0, 2, 1)$ with respect to both H_1 and H_2, and as they have a second common neighbour $(1, 1, 1)$

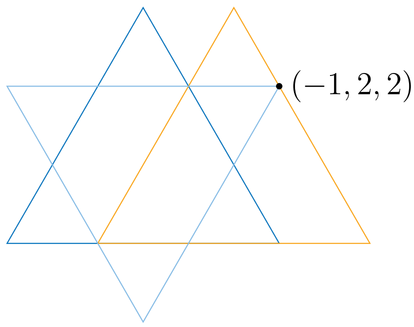

Figure 6.14: The boundaries of Δ_3, $\Delta_3 + (-1, 1, 0)$, and ∇_3.

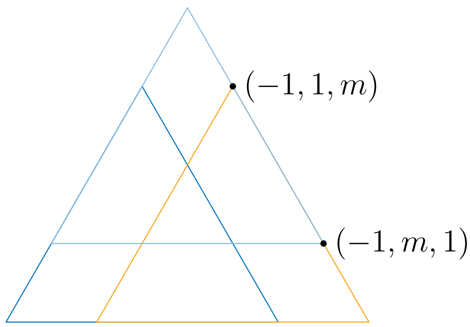

Figure 6.15: The boundaries of Δ_m, $\Delta_m + (-1, 1, 0)$, and $\Delta_m + (-1, 0, 1)$.

in Δ_3, which is mapped to the same vertex by μ, the hexagonal charts μ_{H_1} and μ_{H_2} map $(-1, 2, 2)$ to the same vertex.

If $m \geq 3$ and $H_2 = \Delta_m + (-1, 0, 1)$, we have $V((H_1 \cap H_2) \setminus \Delta_m) = \{(-1, m, 1), (-1, m-1, 2), \ldots, (-1, 1, m)\}$ as can be seen in Figure 6.15. For the vertices $(-1, m-1, 2), (-1, m-2, 3), \ldots, (-1, 2, m-1)$ the same argumentation as in the previous case applies. As the vertex $(-1, 1, m)$ is incident to $(0, 1, m-1)$ and $(-1, 2, m-1)$ with respect to both H_1 and H_2, and as they have a second common neighbour $(0, 2, m-2)$ in H_1 and H_2, which is mapped to the same vertex by μ_{H_1} and μ_{H_2} by the previous argument, the hexagonal charts μ_{H_1} and μ_{H_2} map $(-1, 1, m)$ to the same vertex. The argumentation for $(-1, m, 1)$ is analogous.

Like in the corresponding part in the proof of Lemma 6.15, we see that $\hat{\mu}$ is injective. As $\hat{\mu}$ is a surjective homomorphism by construction, it is a hexagonal chart.

It follows directly from the construction that for each $\Delta_m \cong T \subseteq N_G[S]$,

$$\hat{\mu}^{-1}(T) \in \begin{cases} \{\Delta_m\} \cup \{\Delta_m + \vec{d} \mid \vec{d} \in \vec{D}_0\}, & \text{if } m \geq 4, \\ \{\Delta_m\} \cup \{\Delta_m + \vec{d} \mid \vec{d} \in \vec{D}_0\} \cup \{\nabla_3\}, & \text{if } m = 3. \end{cases} \qquad \square$$

7 Construction of the Geometric Clique Graph of a Pika

This chapter gives a non-recursive geometric construction for the n-th iterated clique graph k^nG of a pika. In Section 7.1, the geometric clique graph G_n is defined via triangular-shaped graphs and in Section 7.2, two classes of cliques of G_n are constructed. Section 7.3 shows that any clique of G_n is one of those constructed in Section 7.1. Finally, Section 7.4 shows that the clique intersections in G_n match the edge definitions in G_{n+1}, which implies $G_n \cong k^nG$ by induction. Throughout the chapter, G will always refer to a pika and G_n will be the geometric clique graph defined in Definition 7.1. The chapter is based on joint work with Markus Baumeister [BL22].

7.1 The Geometric Clique Graph G_n

For a pika G, we construct a graph sequence $(G_n)_{n\in\mathbb{N}_0}$ in a geometric way.

Definition 7.1. *Let G be a pika. For a non-negative integer n, the* ***geometric clique graph G_n*** *has the following form:*

1. *Its vertex set $V(G_n)$ consists of the subgraphs of G isomorphic to triangular-shaped graphs Δ_m with $m \leq n$, where $m \equiv_2 n$.*

2. *Its edge set $E(G_n)$ is defined as follows:*

   i) *Two subgraphs (of G) $S_1 \cong \Delta_m$ and $S_2 \cong \Delta_m$ are adjacent (in G_n) if $S_1 \subseteq N_G[S_2]$ (which is equivalent to $S_2 \subseteq N_G[S_1]$ by Corollary 6.17), see Figure 7.1.*

   ii) *Two subgraphs $S_1 \cong \Delta_m$ and $S_2 \cong \Delta_{m-2}$ (with $m \geq 2$) are adjacent if $S_2 \subseteq S_1$, see Figure 7.2.*

iii) Two subgraphs $S_1 \cong \Delta_m$ and $S_2 \cong \Delta_{m-4}$ (with $m \geq 4$) are adjacent if $S_2 \subseteq S_1$ and S_2 does not contain any vertex of ∂S_1, i. e., $S_2 \cap \partial S_1 = \emptyset$, see Figure 7.3.

iv) Two subgraphs $S_1 \cong \Delta_m$ and $S_2 \cong \Delta_{m-6}$ (with $m \geq 6$) are adjacent if $S_2 \subseteq S_1$ and S_2 does not contain any vertex with distance at most 1 from the boundary of ∂S_1, i. e., $S_2 \cap N_G[\partial S_1] = \emptyset$, see Figure 7.4.

*A subgraph $S \cong \Delta_m$ of G is said to be of **level m** in G_n.*

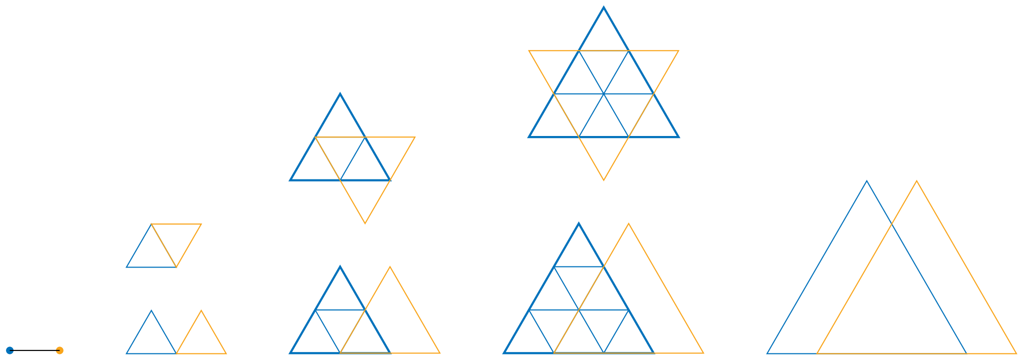

Figure 7.1: Configurations of Δ_m-shaped graphs connected by an edge of type (i) for $m = 0$, $m = 1$, $m = 2$, $m = 3$, and $m \geq 4$.

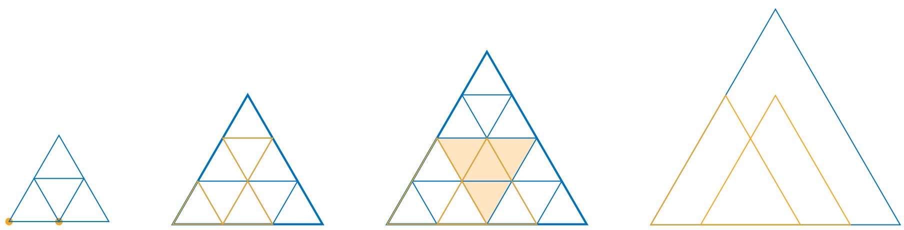

Figure 7.2: Configurations of Δ_m-shaped graphs connected by an edge of type (ii) for $m = 2$, $m = 3$, $m = 4$, and $m \geq 5$.

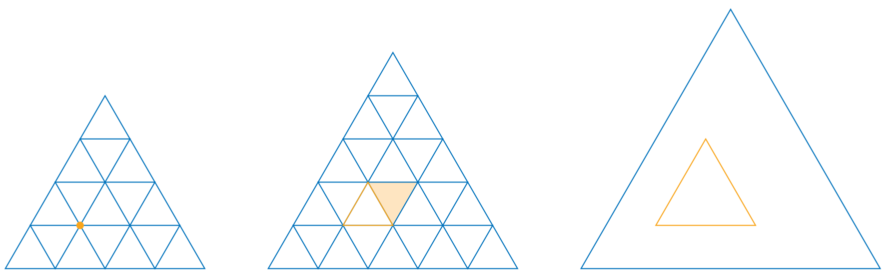

Figure 7.3: Configurations of Δ_m-shaped graphs connected by an edge of type (iii) for $m = 4$, $m = 5$, and $m \geq 6$.

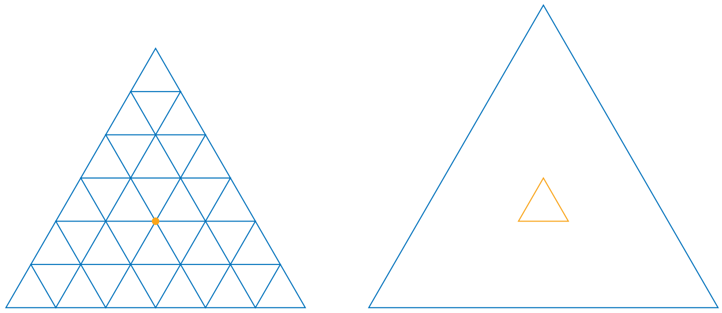

Figure 7.4: Configurations of Δ_m-shaped graphs connected by an edge of type (iv) for $m = 6$ and $m \geq 7$.

For two adjacent $S_1, S_2 \in V(G_n)$ with $S_1 \cong \Delta_{m_1}$ and $S_2 \cong \Delta_{m_2}$, we call S_2 a neighbour of S_1 of **type** $m_2 - m_1$.

Clearly, $G_0 = G$. Thus we aim for proving $G_n \cong k^n G$ by induction.

7.2 Clique Construction of the Geometric Clique Graph

In this section, we construct two classes of cliques of G_n; those that contain three Δ_m within a fixed Δ_{m+1}-shaped subgraph (Lemma 7.2), and those that contain all Δ_1-shaped subgraphs incident to a fixed vertex (Lemma 7.3).

In the the next lemma, we use the following notation from Section 6.1: for a hexagonal chart $\mu\colon \Delta_{m+1} \to S$ and $(t_1, t_2, t_3) \in \mathbb{Z}^3$, we denote the image of $\mu \circ \Delta^{t_1,t_2,t_3}_{m+1-t_1-t_2-t_3}$ by μ^{t_1,t_2,t_3}.

Lemma 7.2. *Let G be a pika and $\Delta_{m+1} \xrightarrow{\mu} S \subseteq G$ a hexagonal chart with $m \leq n$ and $m \equiv_2 n$. The common neighbourhood $N^{\cap}_{G_n}[\mu^{1,0,0}, \mu^{0,1,0}, \mu^{0,0,1}]$ forms a clique in G_n.*

Proof. By Definition 7.1, the $\mu^{\vec{e}}$ with $\vec{e} \in \vec{E}$ are vertices of G_n. They are all contained in S and fulfil $S \subseteq N_G\left[\mu^{\vec{e}}\right]$. Thus, by Definition 7.1, the $\mu^{\vec{e}}$ are pairwise adjacent to each other. The first step of the proof is finding all elements in the common neighbourhood $N^{\cap}_{G_n}[\mu^{1,0,0}, \mu^{0,1,0}, \mu^{0,0,1}]$. By the definition of G_n, $\mu^{1,0,0}, \mu^{0,1,0}$, and $\mu^{0,0,1}$ can only have Δ_{m+s}-shaped neighbours for some $s \in \{-6, -4, -2, 0, 2, 4, 6\}$. We list the elements of the common neighbourhood by side length.

1. The Δ_{m-k}-shaped elements of $N_{G_n}^{\cap}\left[\mu^{1,0,0},\mu^{0,1,0},\mu^{0,0,1}\right] \setminus \{\mu^{1,0,0},\mu^{0,1,0},\mu^{0,0,1}\}$ for $k \in \{2,4,6\}$ and $k \leq m$:

   By Definition 7.1, those elements are subgraphs of $\mu^{1,0,0} \cap \mu^{0,1,0} \cap \mu^{0,0,1}$. For $m \in \{0,1\}$, we have $\mu^{1,0,0}\cap\mu^{0,1,0}\cap\mu^{0,0,1} = \emptyset$, thus the common neighbourhood does not contain such elements. For $m \geq 2$, we have $\mu^{1,0,0} \cap \mu^{0,1,0} \cap \mu^{0,0,1} = \mu^{1,1,1} \cong \Delta_{m-2}$. We distinguish between the possible values of k.

   a) For $k = 2$, the intersection $\mu^{1,1,1}$ itself is the unique such element of the common neighbourhood.

   b) For $k = 4$ and for any such element T, we have $T \subseteq \mu^{\vec{e}} \setminus \partial\mu^{\vec{e}}$ for the three $\vec{e} \in \vec{E}$. Thus, by Remark 6.2,
   $\Delta_{m-4} \cong T \subseteq \mu^{1,1,1} \setminus \partial\mu^{1,1,1} \cong \Delta_{m-5}$, which is impossible.

   c) For $k = 6$ and for any such element T, we have $T \subseteq \mu^{\vec{e}} \setminus \partial N_G\left[\mu^{\vec{e}}\right]$ for the three $\vec{e} \in \vec{E}$. Thus, by Remark 6.2, $\Delta_{m-6} \cong T \subseteq \mu^{1,1,1} \setminus \partial N_G\left[\mu^{1,1,1}\right] \cong \Delta_{m-8}$, which is impossible.

   We conclude that for $m \geq 2$, the subgraph $\mu^{1,0,0} \cap \mu^{0,1,0} \cap \mu^{0,0,1} = \mu^{1,1,1} \cong \Delta_{m-2}$ is the unique element of the common neighbourhood of a level which is lower than m.

2. The Δ_m-shaped elements of $N_{G_n}^{\cap}\left[\mu^{1,0,0},\mu^{0,1,0},\mu^{0,0,1}\right] \setminus \{\mu^{1,0,0},\mu^{0,1,0},\mu^{0,0,1}\}$:

   By Definition 7.1, such elements are subgraphs of $N_G\left[\mu^{1,0,0}\right] \cap N_G\left[\mu^{0,1,0}\right] \cap N_G\left[\mu^{0,0,1}\right]$. Since by Lemma 6.10, $S = N_G\left[\mu^{1,0,0}\right] \cap N_G\left[\mu^{0,1,0}\right] \cap N_G\left[\mu^{0,0,1}\right]$, Lemma 6.3 shows that a fourth Δ_m-shaped subgraph T of S can only appear if $m = 1$ (in which case it is $\mu(\nabla_1)$). In particular, such a T is never in $N_{G_n}^{\cap}\left[\mu^{1,0,0},\mu^{0,1,0},\mu^{0,0,1}\right]$ if $\mu^{1,1,1}$ is.

3. The Δ_{m+k}-shaped elements of $N_{G_n}^{\cap}\left[\mu^{1,0,0},\mu^{0,1,0},\mu^{0,0,1}\right] \setminus \{\mu^{1,0,0},\mu^{0,1,0},\mu^{0,0,1}\}$ for $k \in \{2,4,6\}$:

   By Definition 7.1, we have $\Delta_{m+1} \cong S = \mu^{1,0,0} \cup \mu^{0,1,0} \cup \mu^{0,0,1}$ is a subgraph of each such element of the common neighbourhood. Again, we distinguish between the possible values of k.

   a) For $k = 2$, we employ Lemma 6.3 to describe how S can lie inside each possible such element of the common neighbourhood. We see that each of those elements lies in $N_G[S]$ and thus they pairwise lie in each others closed neighbourhoods. By the Definition 7.1, all of these possible elements are pairwise adjacent.

Consider the adjacency of these elements of the common neighbourhood to the smaller ones, which we already described: If $m = 1$, the additional $\mu(\nabla_1)$ from (2) lies in S and is thus adjacent to all common neighbours of level $m+2 = 3$. If $m \geq 2$, the intersection $\mu^{1,1,1} \cong \Delta_{m-2}$ from (1a) has distance 1 to ∂S. Thus, it also has a distance 1 or more to each ∂T and it is therefore adjacent to all of them.

b) For $k = 4$, each of the subgraphs $\mu^{1,0,0}, \mu^{0,1,0}$, and $\mu^{0,0,1}$ needs to have distance 1 to the boundary of T. Thus, S also has this distance. By Remark 6.2, this uniquely defines a subgraph $T \cong \Delta_{m+4}$ with $N_G[S] \subseteq T$.

Since the Δ_{m+2}-shaped elements from (3a) lie in $N_G[S]$ (Lemma 6.3), each of them is adjacent to this T. For $m = 1$, the additional $\mu(\nabla_1)$ from (2) lies in S and has distance 1 from the boundary of T. For $m \geq 2$, the intersection $\mu^{1,1,1}$ from (1a) has distance 1 from ∂S. Since S has distance 1 from ∂T, the total distance between $\mu^{1,1,1}$ and ∂T is 2, showing their adjacency.

c) For $k = 6$, by Remark 6.2, there is only one embedding $\Delta_m \to \Delta_{m+6}$ with distance 2 to the boundary. Thus, there is no such element adjacent to $\mu^{1,0,0}, \mu^{0,1,0}$, and $\mu^{0,0,1}$ simultaneously.

Finally, we conclude that $N^{\cap}_{G_n}[\mu^{1,0,0}, \mu^{0,1,0}, \mu^{0,0,1}]$ is a clique of G_n. □

After having covered the triangle case, we now investigate the vertex case.

Lemma 7.3. *Let G be a pika and v a vertex in G. For odd n, the common neighbourhood $N^{\cap}_{G_n}[T \cong \Delta_1 \mid v \subseteq \Delta_1]$ of all Δ_1 containing v forms a clique in G_n.*

Proof. Clearly, all these Δ_1 are pairwise adjacent in G_n, since they share v. Thus, they lie in a clique which then lies in the common neighbourhood $N^{\cap}_{G_n}[T \cong \Delta_1 \mid v \subseteq \Delta_1]$. As G is locally cyclic, $\{T \cong \Delta_1 \mid v \subseteq \Delta_1\}$ consists of the faces which each contain v as well as one edge from the cycle $N(v)$ and it has $\deg_G(v)$ elements.

By Definition 7.1, all elements from $N^{\cap}_{G_n}[T \cong \Delta_1 \mid v \subseteq \Delta_1]$ are isomorphic to Δ_{1+k} for some $k \in \{0, 2, 4, 6\}$. We treat the possibilities for k separately.

1. For $k = 0$, if there were a Δ_1-shaped element in $N^{\cap}_{G_n}[T \cong \Delta_1 \mid v \subseteq \Delta_1]$ which would not contain v, all of its vertices would lie in $N_G(v)$ (each of its vertices can only lie in two faces and the number of faces is at least 6). Thus, $N_G(v)$ contains a three-cycle, in contradiction to being at least a 6-cycle. Thus, there can't be an additional element in $N^{\cap}_{G_n}[T \cong \Delta_1 \mid v \subseteq \Delta_1]$ isomorphic to Δ_1.

2. For $k = 2$, any Δ_3-shaped element in $N^{\cap}_{G_n}[T \cong \Delta_1 \mid v \subseteq \Delta_1]$ is one of the two Δ_3-shaped subgraphs of G, which contain v as its middle vertex. They exist, if and only if v has degree 6, and are adjacent to each other in G_n.

3. For $k = 4$, by Definition 7.1 and Remark 6.2, every Δ_1-shaped element of G_n which is adjacent to a given Δ_5 has to lie within the central Δ_2. By Lemma 6.3, there are four of these Δ_1. Since $\deg(v) \geq 6$, no Δ_5-shaped subgraph can be adjacent to all the faces in $\{T \cong \Delta_1 \mid v \subseteq \Delta_1\}$.

4. For $k = 6$, by Definition 7.1 and Remark 6.2, a Δ_7-shaped element of G_n is only adjacent to one Δ_1. Since $\deg(v) \geq 6$, no Δ_7-shaped subgraph can be adjacent to all the faces in $\{T \cong \Delta_1 \mid v \subseteq \Delta_1\}$.

Thus, all the elements in $N^{\cap}_{G_n}[T \cong \Delta_1 \mid v \subseteq \Delta_1]$ are pairwise adjacent, and we obtain a clique. □

The previous two lemmas suggest a correspondence between the cliques of G_n we constructed and the vertices of G_{n+1}.

Remark 7.4. *For every pika G and every $n \in \mathbb{N}_0$, there is a map*

$$C_{n+1} \colon V(G_{n+1}) \to \{\textit{cliques of } G_n\},$$

$$S \mapsto \textit{the clique from} \begin{cases} \textit{Lemma 7.3}, & \textit{if } S \textit{ is of level } 0, \\ \textit{Lemma 7.2}, & \textit{otherwise.} \end{cases}$$

The following remark gives an explicit description of C_{n+1}.

Remark 7.5. *a) Let $\Delta_m \overset{\mu}{\twoheadrightarrow} S \in V(G_{n+1})$, for $m \geq 1$. If $m = 1$ let $\hat{\mu}\colon \nabla_2' \to G$ be the hexagonal chart extending μ. It exists and is unique, as each pair of vertices of S has one common neighbour outside S.*

It follows from the proof of Lemma 7.2 that an explicit description of $C_{n+1}(S)$ is given through

$$C_{n+1}(S) = \underbrace{M_{m-1}}_{|\cdot|=3} \cup \underbrace{M_{m+1}}_{\substack{|\cdot|\leq 3\\ |\cdot|=0,\ \text{if } n=m,}} \cup \underbrace{M_{m+3}}_{\substack{|\cdot|\leq 1\\ |\cdot|=0,\ \text{if } n\leq m+2,}} \cup \begin{cases} \emptyset, & \text{if } m=1 \text{ and } n \leq 1,\\ \{\hat{\mu}(\nabla_2')\}, & \text{if } m=1 \text{ and } n\geq 2,\\ \{\mu(\nabla_1)\}, & \text{if } m=2,\\ \{S \setminus \partial S\}, & \text{if } m\geq 3. \end{cases}$$

- *M_{m-1} consists of the elements $\Delta_{m-1} \cong \mu^{\vec{e}}$ for $\vec{e} \in \vec{E}$.*
- *M_{m+1} consists of the elements $\Delta_{m+1} \overset{\nu}{\twoheadrightarrow} T$ fulfilling $\mu = \nu \circ \Delta_{m-1}^{\vec{e}}$ for an $\vec{e} \in \vec{E}$.*
- *M_{m+3} consists of the element $\Delta_{m+3} \cong T$ enclosing S with distance 1, i. e., $S = T \setminus \partial T$.*

b) For $\Delta_0 \cong S \in V(G_{n+1})$, we denote the vertex of S by v. By the proof of Lemma 7.3, an explicit description of $C_{n+1}(S)$ is given through

$$C_{n+1}(S) = \underbrace{\{T \in V(G_n) \mid T \cong \Delta_1,\ S \subseteq T\}}_{|\cdot|=\deg_G(v)} \cup \underbrace{\{T \in V(G_n) \mid T \cong \Delta_3,\ S \subseteq T \setminus \partial T\}}_{\substack{|\cdot|=0,\ \text{if } \deg_G(v)\geq 7 \text{ or } n\leq 2,\\ |\cdot|=2,\ \text{if } \deg_G(v)=6 \text{ and } n\geq 3,}}.$$

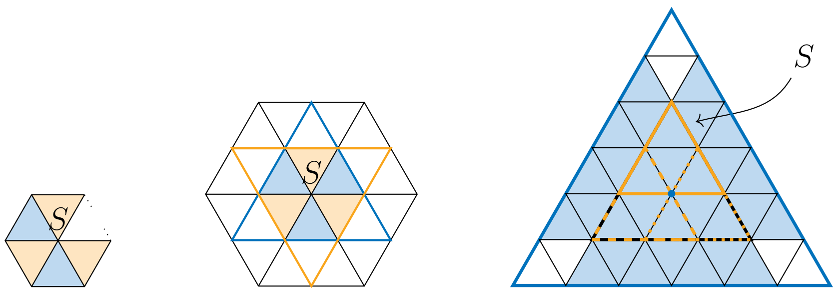

Figure 7.5: $C_{n+1}(S)$ for $S \cong \Delta_0$, $n = 0$; for $S \cong \Delta_0$, $\deg_G(S) = 6$, $n \geq 2$; and for $S \cong \Delta_3$, $n \geq 6$.

In conclusion, for each $m \geq 0$ and $S \cong \Delta_m$ the elements of $C_{n+1}(S)$ can only be isomorphic to Δ_{m-3}, Δ_{m-1}, Δ_{m+1}, or Δ_{m+3}.

7.3 The Full Clique Description of the Geometric Clique Graph

In this section, we show that the cliques from Section 7.2 are all the cliques of the geometric clique graph G_n, which implies that the map C_{n+1} from Remark 7.4 is a bijection. For $m \geq 3$, we employ the chart extensions from Section 6.6. The smaller cases have to be argued separately.

7.3.1 Exceptional (Small) Cases

In this subsection, we discuss the cliques which only contain elements of levels smaller than 3.

Lemma 7.6. *Let Q be a clique of G_n in which every vertex is of level 0 or 2. Then Q is one of the cliques described in Lemma 7.2.*

Proof. We start with the case where all vertices of Q are of level 0, i. e., they are isomorphic to Δ_0. In this case, they form a clique of G, i. e., a triangular-shaped graph $S \cong \Delta_1$. So, Q is constructed from S by Lemma 7.2.

For the remainder, we assume that Q contains a vertex of level 2, i. e., a subgraph $S \cong \Delta_2$ of G. Thus, Q lies in the closed neighbourhood $N_{G_n}[S]$. We visualise the neighbourhood in Figure 7.6.

It is easy to see that all the depicted Δ_2-shaped subgraphs exist. We label the Δ_0-shaped subgraphs with their preimage under a standard chart of S. Since it is not necessarily possible to extend this chart to all the Δ_2-shaped subgraphs in the neighbourhood, we label those in a new labelling scheme. We place every label inside the central face of the subgraph. Two different Δ_2-shaped subgraphs are adjacent if and only if their central faces have face-distance at most 2.

We describe all the cliques of $N_{G_n}[S]$ which contain S using the labels in Figure 7.6.

1. If a corner-vertex of S, like $(2, 0, 0)$, is contained in the clique, the common neighbourhood of this vertex and S with respect to G_n is a clique, which by Lemma 7.2 is constructed from the Δ_1 in S containing the corner-vertex.

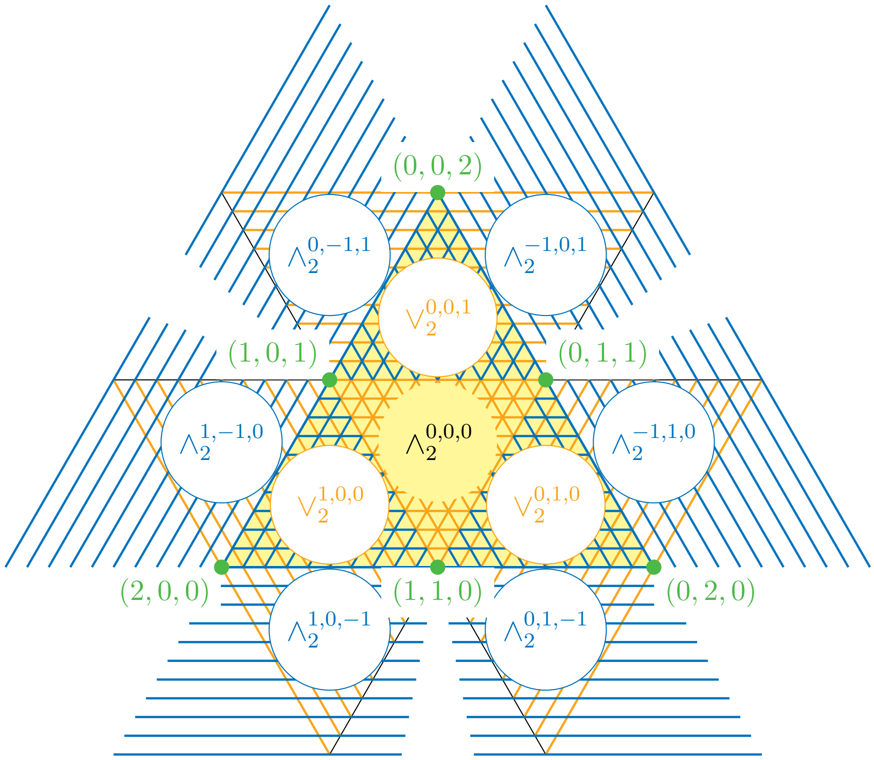

Figure 7.6: Neighbourhood of an $S \cong \Delta_2$ in G_n, Δ_2-shaped subgraphs are labelled in their middle face with the symbols $\wedge$ for 'upward' and $\vee$ for 'downward' facing.

2. Assume no corner-vertex of S is contained in the clique. For all three corner-vertices, there must be an element in the clique which is not adjacent to it; otherwise, the clique would not be maximal. From the remaining elements in $N_{G_n}(S)$, the three middle vertices are each adjacent to exactly two corner-vertices, the other elements are each adjacent to exactly one corner-vertex. Thus, to exclude the corner-vertices, either the three middle vertices are in Q or at least one other element is in Q.

   a) In the first case, the clique is constructed from the three middle-vertices using Lemma 7.2.

   b) In the second case, there is an element not adjacent to two of the corner vertices. Without loss of generality, these corner-vertices are $(0, 2, 0)$ and $(0, 0, 2)$ and the element not adjacent to them is $\wedge_2^{1,0,-1}$ or $\vee_2^{1,0,0}$, which will be called S_1. Additionally, in Q there needs to be an element not adjacent to $(2, 0, 0)$ called S_2, which is adjacent to S_1 since they both lie in Q.

   Thus, S_2 can be neither $\wedge_2^{-1,1,0}$ nor $\wedge_2^{-1,0,1}$ nor $\wedge_2^{0,-1,1}$ since those are not adjacent to each of the possible S_1. Picking S_2 to be $\vee_2^{0,1,0}$ is only possible if S_1 is $\vee_2^{1,0,0}$. In this case, $N^{\cap}_{G_n}[S, S_1, S_2]$ is a clique containing the middle-vertices and we are in Case (2a). The same happens if we choose S_2 to be $\vee_2^{0,0,1}$.

If the degree of $(1, 1, 0)$ is at least 7, there is no other possibility for S_2, but if the degree of $(1, 1, 0)$ is 6, the vertices $\wedge_2^{1,0,-1}$ and $\wedge_2^{0,1,-1}$ are adjacent, and if $S_1 = \wedge_2^{1,0,-1}$ we can choose S_2 to be $\wedge_2^{0,1,-1}$. In this case, S, S_1, and S_2 are contained in a common $T \cong \Delta_3$, from which Q is constructed by Lemma 7.2. □

Lemma 7.7. *Let Q be a clique of G_n, in which every vertex is of level* 1*. Then, Q is one of the cliques described in Lemma 7.2 or in Lemma 7.3.*

Proof. If Q is not given as the common neighbourhood of the set of faces incident to a given vertex like in Lemma 7.3, the intersection of the elements of Q is empty and Q has at least three elements. Furthermore, there are two elements of Q that do not intersect in an edge, as otherwise, for any three element subset of Q there would be a vertex v in the intersection of the three elements and the neighbourhood of v would contain a three-cycle.

Thus, we choose two elements S and T from Q which intersect in a vertex v but not in an edge. Since the common intersection of all elements of Q is empty, there must be an element $U \in Q$ not containing v, but intersecting S and T in at least one vertex each, which we will call s and t, see Figure 7.7.

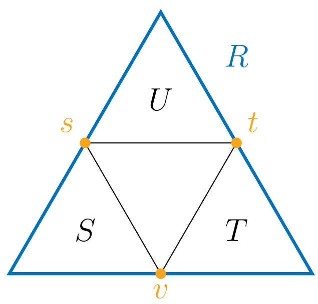

Figure 7.7: The clique of G_n containing S, T, and U is constructed from their union $R \cong \Delta_2$ using Lemma 7.2.

Those two vertices are distinct since S, T, and U do not have a common vertex, and they are connected by an edge from U. As s and t also lie in the G-neighbourhood of v, the edge st also lies in this neighbourhood. Since, by assumption, the third vertex of U is not v, it is the other common neighbour of s and t. This way, we proved that Q is constructed from the union of S, T, and U, which is Δ_2-shaped, using Lemma 7.2, as it is depicted in Figure 7.7. $\square$

7.3.2 The Generic (Large) Case

Up to now we only investigated cliques lying in the lower levels of G_n. The cliques left to discuss are those containing a Δ_m with $m \geq 3$. In this generic case, we describe the neighbourhood $N_{G_n}[S]$ of a $S \cong \Delta_m$ explicitly by using triangle inclusion maps. Then, we classify the cliques there explicitly.

We can describe the adjacency conditions of Definition 7.1 combinatorially with triangle inclusion maps. In addition to the aforementioned set

$$\vec{D}_0 \quad = \{(1,-1,0),(1,0,-1),(-1,1,0),(0,1,-1),(-1,0,1),(0,-1,1)\},$$

we define the following sets of coordinates:

$$\begin{aligned}
\boldsymbol{\vec{D}_{-2}} &:= \{(2,0,0),(1,1,0),(0,2,0),(0,1,1),(0,0,2),(1,0,1)\},\\
\boldsymbol{\vec{D}_{-4}} &:= \{(2,1,1),(1,2,1),(1,1,2)\}, \text{ and}\\
\boldsymbol{\vec{D}_{-6}} &:= \{(2,2,2)\}.
\end{aligned}$$

Lemma 7.8. *Let $\mu\colon H \to F \subseteq G$ be a hexagonal chart of the pika G. Let $\vec{s}, \vec{t} \in \mathbb{Z}^3$ as well as $k \in \{0,2,4,6\}$ and $m \geq k$ be such that $\Delta_m + \vec{s}$ and $\Delta_{m-k} + \vec{t}$ are subsets of H. Further, let $S := \mu(\Delta_m + \vec{s}) \subseteq F$ and $T := \mu(\Delta_{m-k} + \vec{t}) \subseteq F$. Then S and T are adjacent in the clique graph G_n for all $n \geq m$ with $n \equiv_2 m$ if and only if $\vec{t} - \vec{s} \in \vec{D}_{-k}$.*

Proof. Since μ is an isomorphism, S and T are adjacent in G_n if and only if $\Delta_m + \vec{s}$ and $\Delta_{m-k} + \vec{t}$ are connected by an edge of the n-th iterated geometric clique graph $(\mathrm{Hex}_{m+|\vec{s}|})_n$ of the hexagonal grid. Therefore, it is sufficient to prove the claim for $G = \mathrm{Hex}_m$. Since

$$\mathrm{Hex}_{m+|\vec{s}|} \to \mathrm{Hex}_m, \qquad a \mapsto a - \vec{s}$$

is an isomorphism between hexagonal grids, we can assume without loss of generality that $S = \Delta_m$ and $T = \Delta_{m-k} + (\vec{t} - \vec{s})$ with corners $(m-k,0,0) + \vec{t} - \vec{s}$, $(0,m-k,0) + \vec{t} - \vec{s}$, and $(0,0,m-k) + \vec{t} - \vec{s}$. Now, we distinguish with respect to k

<u>Case 1:</u> $k = 0$. T is adjacent to S if the corners of T lie in the neighbourhood $N_G[S]$. A vertex $(v_1, v_2, v_3) \in \mathrm{Hex}_m$ lies in $N_G[\Delta_m]$ if and only if $-1 \leq v_1, v_2, v_3 \leq m + 1$. Since the components of $\vec{t} - \vec{s}$ sum to 0, this is equivalent to $\vec{t} - \vec{s} \in \vec{D}_0$.

<u>Case 2:</u> $k = 2$. $T \subseteq S$ if and only if the corners of T lie in S. Equivalently, all components of $\vec{t} - \vec{s}$ have to be non-negative. Since the components sum to 2, this is equivalent to $\vec{t} - \vec{s} \in \vec{D}_{-2}$.

<u>Case 3:</u> $k = 4$. The corners of $T \subseteq S$ do not lie on the boundary of S if and only if all components of $\vec{t} - \vec{s}$ are at least 1. Since the components sum to 4, this is equivalent to $\vec{t} - \vec{s} \in \vec{D}_{-4}$.

<u>Case 4:</u> $k = 6$. The corners of $T \subseteq S$ have distance 2 from the boundary of S if and only if all components of $\vec{t} - \vec{s}$ are at least 2. Since the components sum to 6, this is equivalent to $\vec{t} - \vec{s} \in \vec{D}_{-6}$. □

From every clique, we can choose an element S of maximal level m. Then, we describe the clique as a clique of the lower-level neighbourhood $N_{G_m}[S] = N_{G_n}[S] \cap V(G_m)$. To describe $N_{G_m}[S]$ combinatorially, we introduce the **local hexagonal graph $\boldsymbol{LHG}$**: its vertices are

$$\boldsymbol{V_{LHG}} := \left\{ v_0^{0,0,0} \right\} \cup \left\{ v_r^{\vec{d}} \mid r \in \{0, -2, -4, -6\}, \vec{d} \in \vec{D}_r \right\}$$

and its edges are given by

$$\boldsymbol{E_{LHG}} := \left\{ (v_r^{\vec{x}}, v_{r-k}^{\vec{y}}) \mid \vec{y} - \vec{x} \in \vec{D}_{-k} \text{ for a } k \in \{0, 2, 4, 6\} \right\}.$$

For a vertex $S \in V(G_n)$ of a given level m, the **lower level neighbourhood** of S is defined as set $N_{G_m}[S] \subseteq V(G_n)$, which consists of S and all its neighbours that have a level of at most m.

Lemma 7.9. *Let $S \cong \Delta_m$ be a vertex in G_n with $m \geq 3$. The lower level neighbourhood of S in G_n is isomorphic to an induced subgraph of the local hexagonal graph.*

Proof. We prove the claim by giving a graph monomorphism $\phi\colon N_{G_n}[S] \cap G_m \to LHG$ that maps non-edges to non-edges. We start with a standard chart $\mu\colon \Delta_m \to S$. By Lemma 6.18, we can extend it to a hexagonal chart $\hat{\mu}\colon E \to G$ such that all adjacent $T \cong \Delta_m$ are contained in the image. We distinguish between different values of m. For the generic case $m \geq 6$, we have the following vertices of smaller level which are adjacent to S:

1. The inclusions of Δ_{m-2}-shaped subgraphs into S are all described by triangle inclusion maps since $m > 4$ (Lemma 6.4).

2. The inclusions of Δ_{m-4}-shaped subgraphs into the central Δ_{m-3}-shaped subgraph $S \setminus \partial S$ (compare Remark 6.2) are all described by triangle inclusion maps since $m - 3 > 2$ (Lemma 6.3).

3. The inclusion of Δ_{m-6} into the central Δ_{m-6}-shaped subgraph $S \setminus N_G[\partial S]$ (compare Remark 6.2) is unique and it is the identity, which is a triangle inclusion map.

Thus, all adjacent triangular-shaped graphs of smaller level are given by triangle inclusion maps. By Lemma 7.8, the elements of $N_{G_n}[S] \cap V(G_m)$ are given by $\mu(\Delta_{m-k} + \vec{d})$ with $\vec{d} \in \vec{D}_{-k}$.

Therefore, the map $\phi\colon N_{G_n}[S] \cap G_m \to LHG$ defined by $\phi(\mu(\Delta_{m-k} + \vec{d})) = v_{-k}^{\vec{d}}$ with $\vec{d} \in \vec{D}_{-k}$ is a monomorphism mapping non-edges to non-edges, but it is not necessarily an isomorphism since not all of the $\vec{D}_0$-translated neighbours of S need to be present.

We continue with the case that $m = 5$. The lower level neighbours are illustrated in Figure 7.8. Since $5 > 4$, all neighbours of level $m-2$ are given by triangle inclusion maps (Lemma 6.4). For level $m - 4 = 1$, we need to consider inclusions of Δ_1 into $\mu(\Delta_2)$ (Remark 6.2). By Lemma 6.3, one exceptional case occurs: a graph $T \cong \Delta_1$ with vertices $\mu(2,1,2)$, $\mu(2,2,1)$, and $\mu(1,2,2)$, i. e., $T = \mu(\nabla_1 + (1,1,1))$. However, there is no neighbour of level $m-6$ since $m - 6 = -1$. Thus, we define ϕ as in the generic case, but we map T to $v_{-6}^{2,2,2}$. Since Lemma 7.8 shows which triangle inclusion maps correspond to edges of G_n, it remains to show that the edges of the local hexagonal graph correctly describe the adjacencies of T.

- By definition, $T = \mu(\nabla_1 + (1,1,1))$ is adjacent to $S = \mu(\Delta_5)$ in G_n but not to any of the Δ_5-shaped neighbours of S. As $v_0^{0,0,0}$ and $v_{-6}^{2,2,2}$ are adjacent in LHG but $v_{-6}^{2,2,2}$ is not adjacent to any other $v_0^{\vec{d}}$ with $\vec{d} \in \vec{D}_0$, the adjacencies and non-adjecencies between $\mu(\nabla_1 + (1,1,1))$ and the Δ_5-shaped subgraphs are preserved by ϕ.

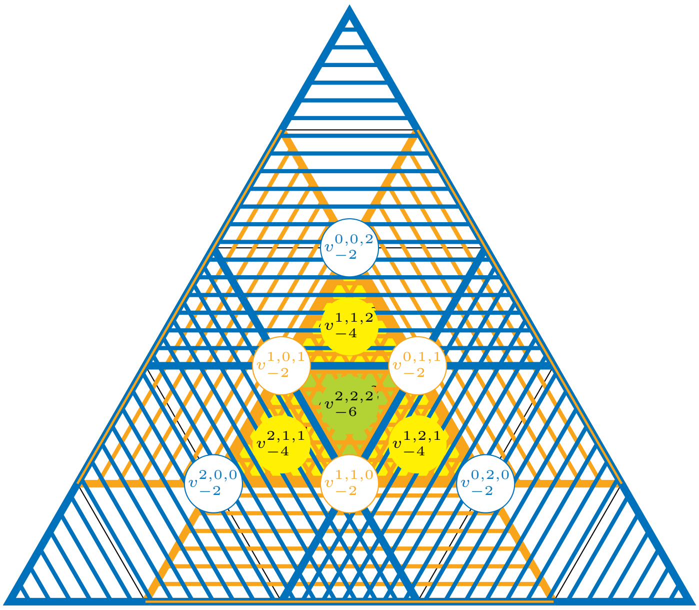

Figure 7.8: Adjacencies for $m = 5$.

- Furthermore, T is adjacent to the three triangular-shaped graphs $\mu(\Delta_3 + (0,1,1)), \mu(\Delta_3 + (1,0,1))$, and $\mu(\Delta_3 + (1,1,0))$ (see the exceptional case in Lemma 6.4), exactly as in the local hexagonal graph.

- Since T is adjacent to $\mu(\Delta_1 + (2,1,1))$, $\mu(\Delta_1 + (1,2,1))$, and $\mu(\Delta_1 + (1,1,2))$, the description of the local hexagonal graph is correct again.

Next, we move on to $m = 4$. Again, the only difference to the generic case is the designated preimage of $v_{-6}^{2,2,2}$, which is $\mu(\nabla_2)$ with corners $\mu(2,2,0)$, $\mu(0,2,2)$, and $\mu(2,0,2)$. The subgraph $\mu(\nabla_2)$ is G_n-adjacent to S but not to any Δ_4-shaped neighbour of S, so ϕ maps edges to edges and non-edges to non-edges. As can be seen in Figure 7.9, we check the adjacencies to the levels 0 and 2. For both, ϕ maps edges to edges and non-edges to non-edges.

Finally, we deal with $m = 3$ with several differences to the generic case:

1. There an additional Δ_3-shaped subgraph adjacent to S which is given by $T := \hat{\mu}(\nabla_3)$. We define $\phi(T) := v_{-6}^{2,2,2}$.

2. There are the three Δ_1-shaped subgraphs from Lemma 6.4 adjacent to S, given by $T_{1,0,0} := \mu(\nabla_1 + (1,0,0)), T_{0,1,0} := \mu(\nabla_1 + (0,1,0))$, and $T_{0,0,1} := \mu(\nabla_1 + (0,0,1))$, and we map them by $\phi(T_{1,0,0}) = v_{-4}^{2,1,1}$, $\phi(T_{0,1,0}) = v_{-4}^{1,2,1}$, and $\phi(T_{0,0,1}) = v_{-4}^{1,1,2}$.

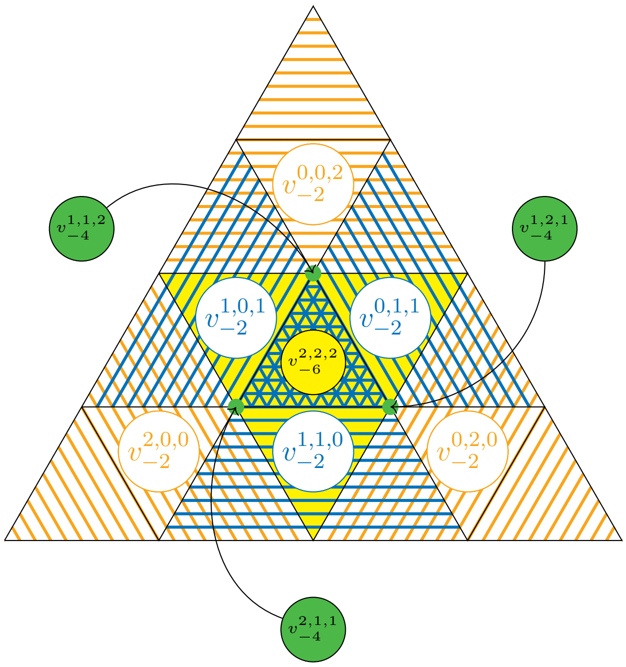

Figure 7.9: Adjacencies for $m = 4$.

We can see in Figure 7.10 that the adjacencies and non-adjacencies between T, $T_{1,0,0}$, $T_{0,1,0}$, and $T_{0,0,1}$ and the elements of level 1 are preserved by ϕ. Furthermore, T is not adjacent to any existing level 3 element given by $\hat{\mu}(\Delta_3 + \vec{d})$ for $\vec{d} \in \vec{D}_0$ and $T_{\vec{e_1}}$ is adjacent to $\hat{\mu}(\Delta_3 + \vec{e_1} - \vec{e_2})$ for $e_1, e_2 \in \vec{E}$ with $e_2 \neq e_1$. Again, ϕ preserves adjacencies and non-adjacencies. $\square$

We describe the cliques of the local hexagonal graph.

Lemma 7.10. *Let Q be a clique in the local hexagonal graph with $v_0^{0,0,0} \in Q$. Then, one of the following three cases holds:*

1. $$Q = \boldsymbol{Q_{-6}} := \{v_0^{0,0,0}, v_{-2}^{1,1,0}, v_{-2}^{0,1,1}, v_{-2}^{1,0,1}, v_{-4}^{2,1,1}, v_{-4}^{1,2,1}, v_{-4}^{1,1,2}, v_{-6}^{2,2,2}\} = N_{LHG}^{\cap}\left[v_{-4}^{2,1,1}, v_{-4}^{1,2,1}, v_{-4}^{1,1,2}\right] = N_{LHG}\left[v_{-6}^{2,2,2}\right],$$
2. $$Q = \boldsymbol{Q_{-4}^{\vec{e}}} := \{v_0^{-(1,0,0)+\vec{e}}, v_0^{-(0,1,0)+\vec{e}}, v_0^{-(0,0,1)+\vec{e}}, v_{-2}^{(1,0,0)+\vec{e}}, v_{-2}^{(0,1,0)+\vec{e}}, v_{-2}^{(0,0,1)+\vec{e}}, v_{-4}^{(1,1,1)+\vec{e}}\} = N_{LHG}^{\cap}\left[v_{-2}^{(1,0,0)+\vec{e}}, v_{-2}^{(0,1,0)+\vec{e}}, v_{-2}^{(0,0,1)+\vec{e}}\right] = N_{LHG}\left[v_{-2}^{2\vec{e}}\right] \text{ for an } \vec{e} \in \vec{E},$$
3. $$Q = \boldsymbol{Q_{-2}^{\vec{e}}} := \{v_0^{(1,0,0)-\vec{e}}, v_0^{(0,1,0)-\vec{e}}, v_0^{(0,0,1)-\vec{e}}, v_{-2}^{(1,1,1)-\vec{e}}\} = N_{LHG}^{\cap}\left[v_0^{(1,0,0)-\vec{e}}, v_0^{(0,1,0)-\vec{e}}, v_0^{(0,0,1)-\vec{e}}\right] \text{ for an } \vec{e} \in \vec{E}.$$

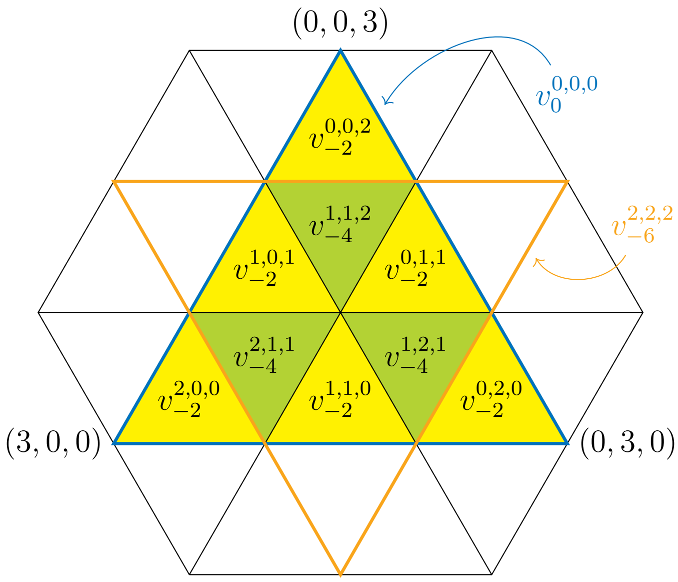

Figure 7.10: Adjacencies for $m = 3$.

Proof. By the definition of the local hexagonal graph, the given sets form complete subgraphs. Furthermore, they are represented as common neighbourhoods of triangular-shaped graphs or as closed neighbourhoods of vertices in the claimed way. Thus, they are also maximal. It remains to show that there cannot be any other cliques.

If $v_{-6}^{2,2,2} \in Q$, we get $Q = N_{LHG}\left[v_{-6}^{2,2,2}\right]$ since this neighbourhood already forms a clique. Thus, the first case of the lemma holds. Otherwise, Q contains an element not incident to $v_{-6}^{2,2,2}$. Those elements are either given by $v_{-2}^{2\vec{e}}$ for an $\vec{e} \in \vec{E}$ or by $v_0^{\vec{e_2}-\vec{e_1}}$ for $\vec{e_1}, \vec{e_2} \in \vec{E}$ with $\vec{e_1} \neq \vec{e_2}$.

If $v_{-2}^{2\vec{e}} \in Q$, we get $Q = N_{LHG}\left[v_{-2}^{2\vec{e}}\right]$ since this neighbourhood already forms a clique. Thus, the second case of the lemma holds.

Finally, we assume $v_0^{\vec{e_2}-\vec{e_1}} \in Q$, but $v_{-2}^{2\vec{e_2}} \notin Q$ (the other two vertices $v_{-2}^{2\vec{e}}$ with $\vec{e} \in \vec{E}$ are not adjacent to $v_0^{\vec{e_1}-\vec{e_2}} \in Q$ anyway). For reasons of symmetry, we can choose $e_1 = (1,0,0)$ and $e_2 = (0,1,0)$. Thus, we have $v_0^{-1,1,0} \in Q$, but $v_{-2}^{0,2,0} \notin Q$. The set of neighbours of $v_0^{-1,1,0}$ is

$$\left\{v_0^{0,0,0}, v_0^{0,1,-1}, v_0^{-1,0,1}, v_{-2}^{0,1,1}, v_{-2}^{1,1,0}, v_{-2}^{0,2,0}, v_{-4}^{1,2,1}\right\}.$$

Only $v_0^{-1,0,1}$ is not adjacent to $v_{-2}^{0,2,0}$, so it has to lie in Q since Q is maximal. Then, $Q = N_{LHG}^{\cap}\left[v_0^{-1,1,0}, v_0^{-1,0,1}, v_0^{0,0,0}\right] = Q_{-2}^{1,0,0}$, as described in the third case of the lemma. □

The following lemma describes how we can find the cliques of an induced subgraph using the cliques of the surrounding graph.

Lemma 7.11. *For a graph G and an induced subgraph H, every clique of H is given as the intersection of a (not necessarily unique) clique of G with H.*

Proof. Let Q be a clique of H. Then, Q is a complete subgraph of G. Therefore, there is at least one clique Q_G of G containing Q. Obviously, $Q \subseteq Q_G \cap H$. If there was an $x \in Q_G \cap H \setminus Q$, the union $Q \cup \{x\}$ were a complete subgraph of H since H is an induced subgraph, in contradiction to Q being chosen maximal. □

We apply this to the image of a lower level neighbourhood under the embedding given in Lemma 7.9. This way, we can classify all the cliques of G_n.

Theorem 7.12. *If Q is a clique of G_n containing a vertex Δ_m with $m \geq 3$, Q is given by the construction in Lemma 7.2 or Lemma 7.3.*

Proof. Let Q be a clique of G_n and let $m \geq 3$ be maximal such that Q contains a vertex $S \cong \Delta_m$. Thus, Q is a subset of the lower level neighbourhood of S. By Lemma 7.9, the lower level neighbourhood of S is isomorphic to an induced subgraph H of the local hexagonal graph containing $v_0^{0,0,0}$ and the monomorphism ϕ maps S to $v_0^{0,0,0}$. Thus, by Lemma 7.11, Q is isomorphic to the intersection of H with a clique Q_{LHG} of the local hexagonal graph.

Thus, Q_{LHG} is one of the cliques given in Lemma 7.10. For reasons of symmetry, we can restrict our investigation to the cliques Q_{-6}, $Q_{-4}^{1,0,0}$, and $Q_{-2}^{1,0,0}$.

1. If $Q_{LHG} = Q_{-6}$ and $m \geq 4$, the preimages of $v_{-4}^{2,1,1}, v_{-4}^{1,2,1}$, and $v_{-4}^{1,1,2}$ with respect to the monomorphism ϕ from the proof of Lemma 7.9 are Δ_{m-4}-shaped subgraphs of S. Thus, they do exist and Q is given by the construction of Lemma 7.2. If $m = 3$, the preimages of $v_{-4}^{2,1,1}, v_{-4}^{1,2,1}$, and $v_{-4}^{1,1,2}$ do exist, but they are not contained in a common Δ_2 and we cannot apply Lemma 7.2. Therefore, we look at the preimages of $v_{-2}^{1,1,0}, v_{-2}^{0,1,1}, v_{-2}^{1,0,1}, v_{-4}^{2,1,1}, v_{-4}^{1,2,1}$, and $v_{-4}^{1,1,2}$ which do exist, since they are Δ_1-shaped subgraphs of S. Furthermore, those preimages are the Δ_1-shaped subgraphs containing the middle vertex of S. Thus Q is constructed from this vertex by Lemma 7.3.

2. If $Q_{LHG} = Q_{-4}^{1,0,0}$, the preimages of $v_{-2}^{(1,0,0)+\vec{e}}, v_{-2}^{(0,1,0)+\vec{e}}$ and $v_{-2}^{(0,0,1)+\vec{e}}$ are Δ_{m-2}-shaped subgraphs of S. Thus, they exist and Q is given by the construction of Lemma 7.2.

3. If $Q_{LHG} = Q_{-2}^{1,0,0}$, either the preimages of $v_0^{(1,0,0)-\vec{e}}, v_0^{(0,1,0)-\vec{e}}, v_0^{(0,0,1)-\vec{e}}$ exist and Q is their common neighbourhood, or one of them does not exist. In the second case, without loss of generality, $\vec{e} = (1,0,0)$. Thus $v_0^{(1,0,0)-\vec{e}} = v_{0,0,0}$ and there is no preimage of $v_0^{(0,0,1)-\vec{e}} = v_0^{1,0,-1}$. The remaining elements of $Q_2^{\vec{e}}$ are at most $v_{0,0,0}, v_{-1,1,0}$ and $v_{-1}^{0,1,1}$ which also lie in $Q_{-4}^{0,1,0}$. Hence, we can also see Q as the intersection of LHG with $Q_{-4}^{0,1,0}$ and, by applying the second case, it is given by the construction of Lemma 7.2. □

As Lemma 7.6, Lemma 7.7, and Theorem 7.12 imply the surjectivity of the map C_{n+1} between the vertices of G_{n+1} and the cliques of G_n from Remark 7.4, we move on to injectivity.

Theorem 7.13. *The map*

$$C_{n+1}\colon V(G_{n+1}) \to \{\textit{cliques of } G_n\},$$

$$S \mapsto \textit{the clique from} \begin{cases} \textit{Lemma 7.3}, & \textit{if } S \textit{ is of level } 0, \\ \textit{Lemma 7.2}, & \textit{otherwise,} \end{cases}$$

is bijective.

Proof. The map C_{n+1} is surjective by Lemma 7.6, Lemma 7.7, and Theorem 7.12. For the injectivity, we discuss three cases. The cliques from Lemma 7.3 contain at least six Δ_1-shaped subgraphs of G and the cliques from Lemma 7.2 contain at most four of them. Thus, a clique which is constructed through Lemma 7.2 cannot be constructed through Lemma 7.3 and vice versa.

Furthermore, for an $S \cong \Delta_0$, the Δ_1-shaped elements of $C_{n+1}(S)$ (recall Section 7.2) have a unique common vertex, which is the vertex of S, and the preimage of $C_{n+1}(S)$ is this unique vertex. Finally, let $S \cong \Delta_m$ for an $m \geq 1$. We look for a $T \cong \Delta_{m'}$ such that $C_{n+1}(S) = C_{n+1}(T)$. To do this, we check the options for $T_1, T_2, T_3 \cong \Delta_{m'-1}$ like in Lemma 7.2. By Remark 7.5, for $m \geq 3$, the only other three triangular-shaped graphs in $C_{n+1}(S)$ of a common level are the elements of level $m+1$ if all three of them exist. But their union is not isomorphic to a Δ_{m+2}. For $m = 2$, there is an additional element of level 1, but it does not form a Δ_2

with two of the other three. For $m = 1$, there is an additional element of level 2 , but it does not form a Δ_3 with two of the other three. □

7.4 Clique Intersections of the Geometric Clique Graph

After having constructed all cliques of the geometric clique graph G_n (and proven that these cliques correspond to vertices of G_{n+1}), we need to show that two cliques $C_{n+1}(S_1)$ and $C_{n+1}(S_2)$ intersect if and only if the corresponding vertices S_1 and S_2 in G_{n+1} are connected by an edge. In Subsection 7.4.1 we discuss the case that S_1 and S_2 are on the same level and in Subsection 7.4.2 the case that they are on different levels.

7.4.1 Intersections of Same Level Cliques

Lemma 7.14. *For $S_1, S_2 \in V(G_{n+1})$ with $S_1 \cong S_2 \cong \Delta_m$ for some $m \geq 0$, the cliques $C_{n+1}(S_1)$ and $C_{n+1}(S_2)$ do not intersect in a vertex $T \cong \Delta_{m+3}$. Furthermore, if $m \geq 4$, they do not intersect in an element $T \cong \Delta_{m-3}$.*

Proof. From Remark 7.5, we see that if a clique $C_{n+1}(S)$ contains an element $T \cong \Delta_{m+3}$, the clique is uniquely defined by this element since $S = T \setminus \partial T$. Furthermore, for $m \geq 4$ the clique is also uniquely defined by an element $T \cong \Delta_{m-3}$ since then $T = S \setminus \partial S$, which has only one solution S. In either way, the vertex T cannot lie in two distinct cliques of G_n defined triangular-shaped graphs of the same level. □

Lemma 7.15. *For $S_1, S_2 \in V(G_{n+1})$ with $S_1 \cong S_2 \cong \Delta_0$ but $S_1 \neq S_2$, the cliques $C_{n+1}(S_1)$ and $C_{n+1}(S_2)$ intersect non-trivially if and only if S_1 and S_2 are adjacent in G_{n+1}, i. e., they are adjacent as vertices of G.*

Proof. At first, we suppose that S_1 and S_2 are adjacent in G. Since G is locally cyclic, they have two common neighbours, especially, there is a $\Delta_1 \cong T \subseteq G$ with $S_1 \subseteq T$ and $S_2 \subseteq T$. Thus, T lies in both $C_{n+1}(S_1)$ and $C_{n+1}(S_2)$. Conversely, suppose there is a $T \in C_{n+1}(S_1) \cap C_{n+1}(S_2)$. By Lemma 7.14, $T \cong \Delta_1$. Furthermore, by Remark 7.5, S_1 and S_2 are both vertices of T and thus adjacent. □

Lemma 7.16. *For $S_1, S_2 \in V(G_{n+1})$ with $S_1 \cong S_2 \cong \Delta_m$ for some $m \geq 1$ but $S_1 \neq S_2$, if the cliques $C_{n+1}(S_1)$ and $C_{n+1}(S_2)$ intersect in a $T_1 \cong \Delta_{m+1}$, they also intersect in a $T_2 \cong \Delta_{m-1}$.*

Proof. If $T \in C_{n+1}(S_1) \cap C_{n+1}(S_2)$ for a $\Delta_{m+1} \overset{\nu}{\twoheadrightarrow} T$, both $S_1 \subseteq T$ and $S_2 \subseteq T$. If $m + 1 \geq 3$, it follows from Remark 7.5 that $S_1 = \nu^{\vec{e}}$ and $S_2 = \nu^{\vec{f}}$ for $\vec{e}, \vec{f} \in \vec{E}$ and $\vec{e} \neq \vec{f}$, implying $S_1 \cap S_2 = \nu^{\vec{e}+\vec{f}} \cong \Delta_{m-1}$.

If $m + 1 = 2$, either the situation is the same as in the foregoing case or, without loss of generality, $S_1 = \mu^{\vec{e}}$ for an $\vec{e} \in \vec{E}$ and $S_2 = \hat{\mu}(\nabla_2')$. Even in this case, S_1 and S_2 intersect in a vertex. □

Consequently, if $m \geq 1$, we only have to investigate whether two cliques which are images of two triangular-shaped graphs from level m intersect in a Δ_{m-1}-shaped vertex of G_n.

Lemma 7.17. *For $S_1, S_2 \in V(G_{n+1})$ with $S_1 \cong S_2 \cong \Delta_m$ for an $m \geq 1$ but $S_1 \neq S_2$, the cliques $C_{n+1}(S_1)$ and $C_{n+1}(S_2)$ intersect non-trivially if and only if S_1 and S_2 are adjacent in G_{n+1}, i. e., $S_1 \subseteq N_G[S_2]$.*

Proof. If there is a $T \in C_{n+1}(S_1) \cap C_{n+1}(S_2)$, by Lemma 7.14 and Lemma 7.16, we can choose $T \cong \Delta_{m-1}$. Thus, we have $S_1 \subseteq N_G[T] \subseteq N_G[S_2]$, where $S_1 \subseteq N_G[T]$ follows from Lemma 6.3 and $N_G[T] \subseteq N_G[S_2]$ follows from $T \subseteq S_2$.

Conversely, suppose $S_1 \subseteq N_G[S_2]$. We distinguish between the values of m.

1) If $m = 1$, S_1 is one of the additional faces in $N_G[S_2]$. Thus S_1 and S_2 intersect in at least one vertex, which lies in both $C_{n+1}(S_1)$ and $C_{n+1}(S_2)$.

2) If $m = 2$, there is a $\Delta_1 \cong T \subseteq S_1 \cap S_2$. Thus, by Remark 7.5, $T \in C_{n+1}(S_1) \cap C_{n+1}(S_2)$.

3) If $m \geq 4$, let $\mu\colon \Delta_m \to S_1$ be a standard chart. By Lemma 6.18, there is an extension $\hat{\mu}\colon E \to \hat{S}_1$ such that $S_2 = \hat{\mu}(\Delta_m + \vec{t})$ for a $\vec{t} \in \vec{D}_0$. Therefore, $S_1 \cap S_2 \cong \hat{\mu}^{-1}(S_1 \cap S_2) = \Delta_m \cap (\Delta_m + \vec{t}) \cong \Delta_{m-1}$. Thus, by Remark 7.5, $S_1 \cap S_2 \in C_{n+1}(S_1) \cap C_{n+1}(S_2)$, so they intersect non-trivially.

4) If $m = 3$, by Lemma 6.18, we are either in the same situation as for $m \geq 4$, which proves the claim, or $S_2 = \hat{\mu}(\nabla_3)$. In this case, $C_{n+1}(S_1)$ and $C_{n+1}(S_2)$ share the vertex $\mu(1, 1, 1)$, which is the midpoint of both S_1 and S_2. □

Thus, for $S_1 \cong S_2 \in V(G_{n+1})$, intersection in G_n and adjacency in G_{n+1} are equivalent.

7.4.2 Intersections of Different Level Cliques

Lemma 7.18. *For $S_1, S_2 \in V(G_{n+1})$ with $S_1 \cong \Delta_m$ and $S_2 \cong \Delta_{m+2}$ for an $m \geq 0$, the cliques $C_{n+1}(S_1)$ and $C_{n+1}(S_2)$ intersect non-trivially if and only if S_1 and S_2 are adjacent in G_{n+1}, i. e., $S_1 \subseteq S_2$.*

Proof. At first, we suppose that $S_1 \subseteq S_2$. We choose a standard chart $\mu\colon \Delta_{m+2} \to S_2$. In the following we again denote the image of $\mu \circ \Delta^{t_1,t_2,t_3}_{m+2-t_1-t_2-t_3}$ for some $(t_1, t_2, t_3) \in \mathbb{Z}^3$ by μ^{t_1,t_2,t_3}.

By Lemma 6.4, there are three cases. In the first case we have $S_1 = \mu^{\vec{e}+\vec{f}}$ for some $\vec{e}, \vec{f} \in \vec{E}$ and $T := \mu^{\vec{e}}$ fulfils $S_1 \subseteq T \subseteq S_2$. By Remark 7.5, T lies in both cliques $C_{n+1}(S_1)$ and $C_{n+1}(S_2)$. In the second case, we have $m = 1$ and $S_1 = \mu(\nabla_1 + \vec{e})$ for some $\vec{e} \in \vec{E}$. By Remark 7.5, $C_{n+1}(S_1)$ and $C_{n+1}(S_2)$ both contain the middle vertex $\mu(1, 1, 1)$ of S_2. In the third case, we have $m = 2$ and the triangular-shaped graph S_1 is $\mu(\nabla_2)$, which contains the unique $S_2 \setminus \partial S_2 := S \cong \Delta_1$, which has distance 1 to ∂S_2 and thus lies in both $C_{n+1}(S_1)$ and $C_{n+1}(S_2)$. In all three cases we have shown that $C_{n+1}(S_1)$ and $C_{n+1}(S_2)$ intersect non-trivially.

Conversely, we now suppose that $C_{n+1}(S_1)$ and $C_{n+1}(S_2)$ intersect non-trivially. We distinguish between the possibilities for an element in the intersection. Any element $T \in C_{n+1}(S_1) \cap C_{n+1}(S_2)$ is isomorphic to Δ_{m-1}, Δ_{m+1}, or Δ_{m+3}.

- If $T \cong \Delta_{m-1}$ (i.e. $m \geq 1$), $T = S_2 \setminus \partial S_2$ by Remark 7.5. All the Δ_m-shaped graphs which contain T are subgraphs of S_2; thus $S_1 \subseteq S_2$.
- If $T \cong \Delta_{m+1}$, by Remark 7.5, we have $T = \mu^{\vec{e}}$ for some $\vec{e} \in \vec{E}$ and $S_1 = \mu^{\vec{e}+\vec{f}}$ or $T = \mu(\nabla_1)$ for some $\vec{f} \in \vec{E}$ and S_1 is a vertex of T. In both cases $S_1 \subseteq T \subseteq S_2$, which proves the claim.
- If $T \cong \Delta_{m+3}$, $S_1 = T \setminus \partial T$ by Remark 7.5. This subgraph is contained in every Δ_{m+2}-shaped subgraph of T; thus $S_1 \subseteq S_2$. □

Lemma 7.19. *For $S_1, S_2 \in V(G_{n+1})$ with $S_1 \cong \Delta_m$ and $S_2 \cong \Delta_{m+4}$ for an $m \geq 0$, the cliques $C_{n+1}(S_1)$ and $C_{n+1}(S_2)$ intersect non-trivially if and only if S_1 and S_2 are adjacent in G_{n+1}, i. e., $S_1 \subseteq S_2 \setminus \partial S_2$.*

Proof. If $S_1 \subseteq S_2 \setminus \partial S_2$, the element $S_2 \setminus \partial S_2 \cong \Delta_{m+1}$ lies in both cliques by Remark 7.5.

Conversely, if the cliques intersect, by Remark 7.5, each element T of the intersection is isomorphic to Δ_{m+1} or Δ_{m+3} and all possible T fulfil $S_1 \subseteq T \subseteq S_2$. Furthermore, either $T \cong \Delta_{m+1}$ and the distance of T and ∂S_2 is 1 or $T \cong \Delta_{m+3}$ and the distance of S_1 and ∂T is 1. Thus, the distance between S_1 and ∂S_2 is 1 and $S_1 \subseteq S_2 \setminus \partial S_2$. □

Lemma 7.20. *For $S_1, S_2 \in V(G_{n+1})$ with $S_1 \cong \Delta_m$ and $S_2 \cong \Delta_{m+6}$ for an $m \geq 0$, the cliques $C_{n+1}(S_1)$ and $C_{n+1}(S_2)$ intersect non-trivially if and only if S_1 and S_2 are adjacent in G_{n+1}, i. e., $S_1 \subseteq S_2 \setminus N_G[\partial S_2]$.*

Proof. If $S_1 \subseteq S_2 \setminus N_G[\partial S_2]$, the only $T \cong \Delta_{m+3}$ such that $S_1 \subseteq T \setminus \partial T$ and $T \subseteq S_2 \setminus \partial S_2$ is $T = S_2 \setminus \partial S_2$, by Remark 7.5, and this T exists. Thus, T lies in both $C_{n+1}(S_1)$ and $C_{n+1}(S_2)$.

Conversely, if the cliques intersect, by Remark 7.5 they intersect in a $T \cong \Delta_{m+3}$. The boundary of T has distance 1 to both S_1 and the boundary of S_2; therefore, S_1 has distance 2 to the boundary of S_2. □

We deduce from Remark 7.5 that $C_{n+1}(S_1)$ and $C_{n+1}(S_2)$ can only intersect if the levels of S_1 and S_2 differ by 0, 2, 4, or 6. Thus, we conclude from the preceding lemmas and Theorem 7.13 another lemma, which directly implies Theorem C.

Lemma 7.21. *For every $n \in \mathbb{N}_0$ and every pika G, the map $C_{n+1}\colon G_{n+1} \to kG_n$ from Theorem 7.13 is an isomorphism.*

Theorem C. *For each $n \in \mathbb{N}_0$ and each pika G, its n-th geometric clique graph G_n is isomorphic to its n-th iterated clique graph k^nG.*

8 Clique Dynamics of Pikas

In this chapter, we finally prove our main theorem for pikas. We start by proving a sufficient criterion for convergence and proceed to show that the criterion is actually also necessary by giving a divergent parameter for the other case. While Section 8.1 is based on joint work with Markus Baumeister [BL22], Section 8.2 is based on joint work with Martin Winter [LW23].

8.1 The Convergence Criterion for Pikas

From the geometric construction of the clique graphs of pikas, we immediately obtain a criterion for clique convergence.

Theorem 8.1. *Let G be a triangularly simply connected locally cyclic graph with minimum degree $\delta \geq 6$. If there is an $m \in \mathbb{N}_0$ such that Δ_m cannot be embedded into G, the clique operator is convergent on G.*

Proof. If Δ_m cannot be embedded into G, this means $m \geq 2$ and $G_{m-2} = G_m$ since the graphs G_m and G_{m-2} can only differ in vertices isomorphic to Δ_m, which would be subgraphs of G. By Theorem C, this implies $k^m G \cong k^{m-2} G$, which is the definition of the clique operator being convergent on G. □

8.2 The Divergence Criterion for Pikas

Throughout this section, we assume that G is a pika. We can then apply Theorem C and investigate the dynamics of the sequence of geometric clique graphs $(G_n)_{n \in \mathbb{N}_0}$ in place of $(k^n G)_{n \in \mathbb{N}_0}$.

The section is devoted to proving that if G contains arbitrarily large triangular-shaped subgraphs, then G is clique divergent. For this, we identify a graph invariant that is both finite and unbounded for the sequence G_n as $n \to \infty$, as long as G contains arbitrarily large triangular-shaped subgraphs. It turns out that a suitable graph invariant can be built from measuring distances between vertices of certain degrees. Curiously, the degree 26 plays a special role, and the following notation comes in handy:

$$\mathbf{DEG_{26}}(\boldsymbol{H}) := \{v \in V(H) \mid \deg_H(v) = 26\}$$
$$\overline{\mathbf{DEG}}_{\mathbf{26}}(\boldsymbol{H}) := \{v \in V(H) \mid \deg_H(v) \neq 26\}$$

The corresponding graph invariant is the following:

$$\boldsymbol{D(H)} := \max_{v \in V(H)} \operatorname{dist}_H\Big(v, \overline{\mathrm{DEG}}_{26}(H)\Big). \tag{8.1}$$

The significance of the number 26 stems from the observation that most vertices of G_n have G_n-degree ≤ 26; and they have G_n-degree *exactly* 26 only in very special circumstances that can characterised by the existence of certain triangular-shaped subgraphs in G. This is proven in Lemma 8.2 and Lemma 8.3. Finitude and divergence of $D(G_n)$ as $n \to \infty$ are proven afterwards in Lemma 8.4 and Lemma 8.5.

In the following, we generally consider G_n only for even $n \in 2\mathbb{N}$, as this cuts down on the cases we need to investigate, and is still sufficient to show that $D(G_n)$ is unbounded. Note that each $S \in V(G_n)$ is then of even side length $m \in \{0, 2, 4, 6, \ldots\}$.

Lemma 8.2. *Let $S \in V(G_n)$ be a triangular-shaped graph of side length $m \geq 6$. Then $\deg_{G_n}(S) \leq 26$, with equality if and only if S has a neighbour of type $+6$.*

Lemma 8.2 actually holds unchanged for $m \geq 2$. Since we do not need these cases to prove Theorem A, and since verifying them requires a distinct case analysis (because of "twisted adjacencies", cf. Figure 8.2), we do not include them here.

Proof of Lemma 8.2. Figure 8.1 shows all potential configurations of S and a G_n-neighbour of S according to Definition 7.1 (here we need $m \geq 6$, as there are exceptional "twisted adjacencies" for smaller m, see Figure 8.2). In total this amounts to a degree of at most 26. In particular, if just one of the neighbours is missing, say the neighbour of type $+6$, the G_n-degree of S must be less than 26.

Conversely, one can verify that if S has a neighbour of type +6, say $T \in N_{G_n}(S)$, then all other neighbours of types $-6, -4, -2, 0, +2$, and $+4$ can be found as subgraphs of T. Therefore, all 26 neighbours are present and the degree is 26. $\square$

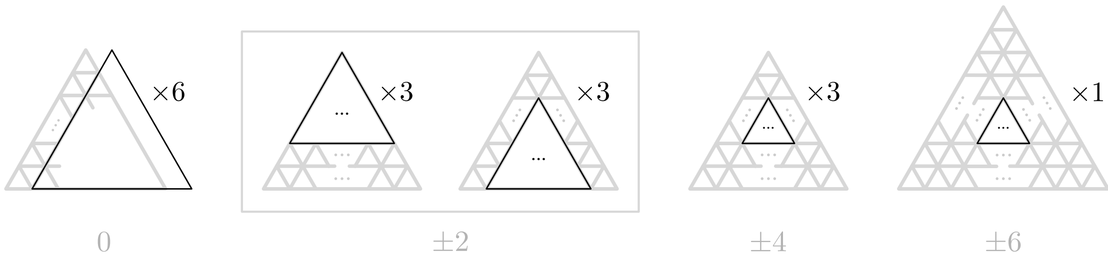

Figure 8.1: The 26 possible ways in which a triangular-shaped graph $S \in V(G_n)$ of side length $m \geq 6$ can be G_n-adjacent to another triangular-shaped graph $T \in V(G_n)$ of side length $m + s$, where $s \in \{-6, -4, -2, 0, +2, +4, +6\}$. Two configurations may differ merely by a symmetry (one of the six "reflections" and "rotations" of a triangular-shaped graph), and we always show only a single configuration with the multiplication factor next to it indicating the number of equivalent configuration related by symmetry. Note that for the types ± 2, ± 4 and ± 6, the configurations must be accounted for twice in the G_n-degree of S: once with S being the larger graph (in grey), and once with S being the smaller graph (in black). Then $26 = 6 + 2 \cdot (3 + 3 + 3 + 1)$.

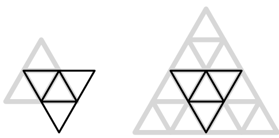

Figure 8.2: For $m \in \{2, 4\}$, there also exist the following "twisted adjacencies".

For $m = 0$ only one direction holds, which is also sufficient for our purpose.

Lemma 8.3. *Let $n \in 2\mathbb{N}$ and $s \in V(G_n)$ be a triangular-shaped graph of side length $m = 0$ (that is, s is a vertex of G). If s has no G_n-neighbour of type +6, then* $\deg_{G_n}(s) \neq 26$.

Proof. Clearly, s has no neighbours of type $-6, -4$ or -2. The G_n-neighbours of type 0 are exactly the vertices that are also adjacent to s in G, that is, there are *exactly* $\deg_G(s)$ many. The potential neighbours of type +4 and +6 are shown in Figure 8.3, which amount to *at most* eight neighbours of these types. Note that these can exist only if $\deg_G(s) = 6$.

It remains to count the neighbours of type +2, which will turn out at *exactly* $2 \deg_G(s)$, independent of the specifics of G. Observe first that there can be two

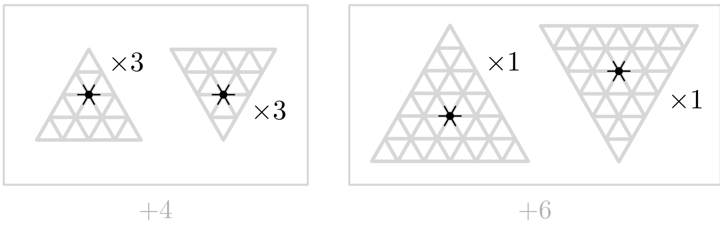

Figure 8.3: The eight possible neighbours of a triangular-shaped graph of side length $m = 0$ of type +4 and +6. See the caption of Figure 8.1 for an explanation of the multiplicities.

types of neighbours $T \in N_{G_n}(s)$ of type +2 distinguished by the T-degree of s, which is either two or four (cf. Figure 8.4). We shall say that these neighbours are of type $+2_2$ and $+2_4$, respectively.

In the following, an *r-chain* is an inclusion chain $s \subseteq \Delta \subseteq T$, where Δ is an s-incident triangle in G, and T is a neighbour of s of type $+2_r$. The following information can be read from Figure 8.4: a neighbour of s of type $+2_r$ can be extended to an r-chain in exactly n_r ways (where $n_2 = 1$ and $n_4 = 3$). Likewise, an s-incident triangle can be extended to an r-chain in exactly n_r ways as well. By double counting, we find that for both $r = 2$ and $r = 4$ the number of r chains equals both n_r times the number of s-incident Δ and n_r times the number of neighbours of s of type $+2_r$. Through division by n_r and as the number of s-incident Δ is $\deg(s)$, we obtain that the number of neighbours of s of type +2 is *exactly* $2\deg_G(s)$.

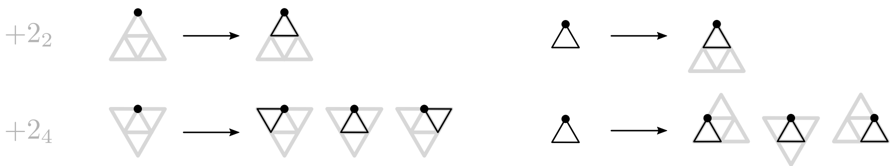

Figure 8.4: Row $+2_r$ shows the ways in which an inclusion $s \subseteq T$ (left; T being a G_n-neighbour of s of type $+2_r$) or an inclusion $s \subseteq \Delta$ (right; Δ being an triangle in G) extends to an r-chain in $n_r = r - 1$ ways.

Taking together all of the above, we count

$$\deg_{G_n}(s) \begin{cases} = \deg_G(s) + 2\deg_G(s) = 3\deg_G(s) & \text{if } \deg_G(s) \neq 6, \\ \leq 6 + 2 \cdot 6 + 8 = 26 & \text{if } \deg_G(s) = 6. \end{cases}$$

Since $26 \not\equiv 0 \pmod 3$, if $\deg_G(s) \neq 6$ we obtain $\deg_{G_n}(s) \neq 26$ right away. If $\deg_G(s) = 6$ and if there is no G_n-neighbour of type +6, then the maximal amount of 26 neighbours cannot have been attained, and $\deg_{G_n}(s) \neq 26$ as well. □

It remains to show that if G contains arbitrarily large triangular-shaped subgraphs, then the graph invariant $D(G_n)$ is both finite and unbounded as $n \to \infty$. We first prove finitude of $D(G_n)$ if $n \in 2\mathbb{N}$ (in particular, $n \geq 2$, as $D(G_0) = D(G)$ might be infinite).

Lemma 8.4. *If $n \in 2\mathbb{N}$, then each $S \in V(G_n)$ has a distance to $\overline{\mathrm{DEG}}_{26}(G_n)$ of at most $n/6 + 1$. That is, $D(G_n) \leq n/6 + 1$.*

Proof. Suppose $S \cong \Delta_m$ with $m \in 2\mathbb{N}$. We distinguish two cases.

Case 1: there is a $T \in V(G_n)$ of side length $\mu \geq 6$ and $\mathrm{dist}_{G_n}(S, T) \leq 2$. We then fix a maximally long path $T_0 T_1 \ldots T_\ell$ in G_n with $T_0 := T$ and $T_i \cong \Delta_{\mu+6i}$ (i. e., T_i and T_{i+1} are adjacent of type ± 6; see Figure 8.5). Since the path is maximal, T_ℓ has no G_n-neighbour of type $+6$, and since T_ℓ is of side length $\mu + 6\ell \geq \mu \geq 6$, we have $T_\ell \in \overline{\mathrm{DEG}}_{26}(G_n)$ by Lemma 8.2. As a vertex of G_n, T_ℓ is of side length at most n, and hence $\mu + 6\ell \leq n \Longrightarrow \ell \leq n/6 - \mu/6 \leq n/6 - 1$. We conclude

$$\begin{aligned} \mathrm{dist}_{G_n}(S, \overline{\mathrm{DEG}}_{26}(G_n)) &\leq \mathrm{dist}_{G_n}(S, T) + \mathrm{dist}_{G_n}(T, \overline{\mathrm{DEG}}_{26}(G_n)) \\ &\leq 2 + (n/6 - 1) = n/6 + 1. \end{aligned}$$

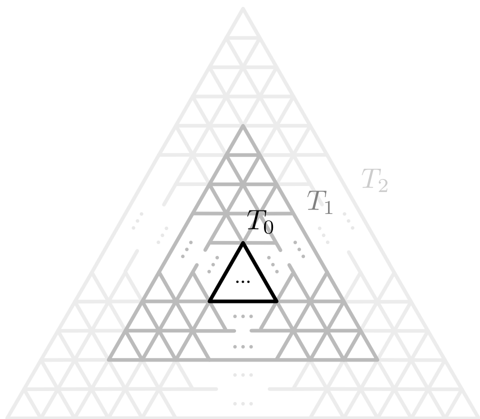

Figure 8.5: Initial segment $T_0 T_1 T_2 \ldots$ of an increasing path of triangular-shaped subgraphs of G where T_i and T_{i+1} are adjacent of type ± 6.

Case 2: there is *no* $T \in V(G_n)$ of side length $\mu \geq 6$ and $\mathrm{dist}_{G_n}(S, T) \leq 2$. Then we can conclude two things: First, $m < 6$ (otherwise, choose $T := S$) and so there is an $s \in N_{G_n}(S)$ of side length zero. Second, s has no neighbour of type $+6$ (otherwise, set T to be this neighbour). But then s cannot have degree 26 by Lemma 8.3, and therefore

$$\mathrm{dist}_{G_n}(S, \overline{\mathrm{DEG}}_{26}(G_n)) \leq \mathrm{dist}_{G_n}(S, s) = 1 \leq n/6 + 1.$$

□

Finally, we show that $D(G_n)$ is unbounded as $n \to \infty$, assuming that there are arbitrarily large triangular-shaped subgraphs of G.

Lemma 8.5. *If G contains a triangular-shaped subgraph of side length $n \in 48\mathbb{N}$, then there exists $S' \in V(G_n)$ with distance to $\overline{\text{DEG}}_{26}(G_n)$ of more than $n/48$. That is, $D(G_n) > n/48$.*

Proof. Choose a triangular-shaped graph $S \in V(G_n)$ of side length $n \in 48\mathbb{N}$. Roughly, the idea is to define a set $\mathcal{M} \subseteq \text{DEG}_{26}(G_n)$ that contains "deep vertices", i. e., vertices that have no "short" G_n-paths that lead out of $\mathcal{M}$. We claim that the following set has all the necessary properties:

$$\mathcal{M} := \left\{ T \in V(G_n) \;\middle|\; \begin{array}{l} T \subseteq S, \\ T \text{ has side length } m \geq 6 \text{ and} \\ \text{dist}_G(T, \partial S) \geq 4 \end{array} \right\}.$$

The following observation will be used repeatedly and we shall abbreviate it by $(*)$: if $T \in V(G_n)$ is of side length $m \geq 6$ (e. g. if $T \in \mathcal{M}$) and if $T' \in N_{G_n}(T)$ is some G_n-neighbour, then $\text{dist}_G(T, v) \leq 4$ for all $v \in T'$. This can be verified by considering the configurations shown in Figure 8.1. The bound ≤ 4 is best possible as seen in Figure 8.6.

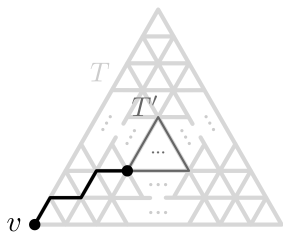

Figure 8.6: The "corner vertex" v of $T \in V(G_n)$ (light grey) has G-distance four to the neighbour $T' \in N_{G_n}(T)$ of type -6 (dark grey).

We first verify $\mathcal{M} \subseteq \text{DEG}_{26}(G_n)$. Fix $T \in \mathcal{M}$ and consider an embedding of S into the hexagonal lattice. In this embedding, $T \subseteq S$ has a neighbour T' of type $+6$ that, for all we know, might partially lie outside of S; though we now show that actually $T' \subseteq S$: In fact, for all $v \in V(T')$ holds

$$\text{dist}_G(T, \partial S) \geq 4 \geq \text{dist}_G(T, v),$$

where we used both $T \in \mathcal{M}$ and $(*)$. Thus $T' \subseteq S$ and T' also exists in G. Note that this argument shows that all G_n-neighbours of T are contained in S, as all neighbours which are not of type $+6$ are included in the type $+6$ neighbour. We denote the latter fact by $(**)$ as we reuse it below. For now, we conclude that since T has a G_n-neighbour of type $+6$, we have $T \in \textsc{deg}_{26}(G_n)$ by Lemma 8.2.

Next we identify a "deep vertex" in $\mathcal{M}$, that is, a vertex with distance to $V(G_n)\backslash\mathcal{M}$ of more than $n/48$. We claim that we can choose for this the "central" triangular-shaped subgraph $S' \cong \Delta_{n/2}$. By that we mean the triangular-shaped graph obtained from S by repeatedly deleting the boundary $n/6$ times. The resulting triangular-shaped subgraph has side length $n/2$ and $\mathrm{dist}_G(S', \partial S) = n/6$. Since $n \geq 48$, we have both $m_0 \coloneqq n/2 \geq 6$ and $\mathrm{dist}_G(S', \partial S) = n/6 \geq 4$, and therefore $S' \in \mathcal{M}$. It remains to show that we have $\ell \coloneqq \mathrm{dist}_{G_n}(S', V(G_n) \setminus \mathcal{M}) > n/48$. Let $S'_0 \dots S'_\ell$ be a path in G_n from $S'_0 \coloneqq S'$ to some $S'_\ell \notin \mathcal{M}$. Let $m_i \in \mathbb{N}_0$ be the side length of S'_i. Since $S'_{\ell-1} \in \mathcal{M}$, by $(**)$ we have $S'_\ell \subseteq S$. Thus, for S'_ℓ to be not in $\mathcal{M}$, only two reasons are left, and we verify that either implies $\ell \geq n/48$:

<u>Case 1:</u> $m_\ell < 6$. Since S'_{i-1} and S'_i are adjacent in G_n, they can differ in side length by at most six (via an adjacency of type ± 6). That is, $m_{i-1} - m_i \leq 6$, and thus

$$6\ell \geq m_0 - m_\ell > n/2 - 6 \implies \ell > n/12 - 1 \geq n/48.$$

<u>Case 2:</u> $\mathrm{dist}_G(S'_\ell, \partial S) < 4$. Note first that for all $i \in \{1, \dots, \ell\}$ holds

$$\mathrm{dist}_G(S'_{i-1}, \partial S) - \mathrm{dist}_G(S'_i, \partial S) \leq \mathrm{dist}_G(S'_{i-1}, S'_i) \overset{(*)}{\leq} 4.$$

It then follows

$$4\ell \geq \mathrm{dist}_G(S'_0, \partial S) - \mathrm{dist}_G(S'_\ell, \partial S) > n/6 - 4 \implies \ell > n/24 - 1 \geq n/48.$$

In both cases, the right-most inequality was obtained using $n \geq 48$. □

Since in our setting, we have $G_n \cong k^n G$, and since $D(\,\cdot\,)$ is a graph invariant, we have $D(k^n G) = D(G_n)$. We can then conclude

Corollary 8.6. *If G contains Δ_n as a subgraph for $n \in 48\mathbb{N}$, then*

$$D(k^n G) \in \left(\tfrac{n}{48}, \tfrac{n}{6} + 1\right],$$

where $D(\cdot)$ *is the graph invariant defined in* (8.1)*. In particular, if* G *contains arbitrarily large triangular-shaped subgraphs, then* $D(k^n G)$ *is unbounded as* $n \to \infty$*, and* G *is therefore clique divergent.*

Together with Theorem 8.1, we conclude the characterisation of convergent clique dynamics for triangularly simply connected locally cyclic graphs of minimum degree $\delta \geq 6$.

Theorem 8.7. *A triangularly simply connected locally cyclic graph of minimum degree* $\delta \geq 6$ *is clique divergent if and only if it contains arbitrarily large triangular-shaped subgraphs.*

9 Clique Dynamics of Locally Cyclic Graphs with $\delta \geq 6$

Up to this point, we only considered triangularly simply connected locally cyclic graphs. We generalise our result by applying the framework from Chapter 4 to locally cyclic graphs with minimum degree $\delta \geq 6$ and their universal triangular covers, which are the pikas.

As we saw in Theorem A, the clique convergence of a connected graph implies the clique convergence of its universal triangular cover. As pikas are triangularly simply connected, they are the universal triangular covers of their quotient graphs. Thus, if a pika is clique divergent, so are its quotients.

To obtain a similar result for the quotients of clique convergent pikas, we investigate the action of a group Γ on a pika G and its induced actions on k^nG and G_n.

This chapter is based on joint work with Markus Baumeister [BL22] and with Martin Winter [LW23].

Remark 9.1. *The action of a group Γ on a Γ-graph G which is also a pika induces an action on the triangular-shaped subgraphs of G, which turns the geometric clique graph G_n into a Γ-graph as well. Furthermore, we consider the action of Γ on k^nG from Remark 4.8, which is recursively defined by $\gamma Q = \{\gamma v \mid v \in Q\}$ for $n \geq 1$ and $Q \in k^nG$, in which γv is given by the (already defined) action of Γ on $k^{n-1}G$. If G is locally cyclic, triangularly simply connected and of minimum degree $\delta \geq 6$, then the isomorphism $\boldsymbol{\psi_n}\colon G_n \to k^nG$ provided by Theorem C is a Γ-isomorphism.*

By close inspection of Remark 7.5 describing the isomorphism $C_n\colon G_n \to kG_{n-1}$ from Remark 7.4, it can be seen that it is Γ-equivariant in the following way: The elements of the clique $C_n(S)$ for some $\Delta_m \overset{\mu}{\hookrightarrow} S \in V(G_n)$ are each defined by *hexagonal charts or by subgraph inclusions, which behave well towards the automorphisms induced by the elements from Γ. For example, the triangular-shaped graphs from M_{m-1} fulfil the following equivalences and similar calculations can be given for the other types of triangular-shaped graphs in the clique:*

$$\begin{aligned} T \in M_{m-1}(S) &\Leftrightarrow T = \mu^{\vec{e}} \text{ for some } \vec{e} \in \vec{E} \\ &\Leftrightarrow \gamma(T) = (\gamma \circ \mu)^{\vec{e}} \text{ for some } \vec{e} \in \vec{E} \\ &\Leftrightarrow \gamma(T) \in M_{m-1}(\gamma(S)). \end{aligned}$$

Thus, ψ_n is obtained from the following chain of Γ-isomorphisms:

$$\begin{aligned} G_n &\xrightarrow{C_n} kG_{n-1} \xrightarrow{(C_{n-1})_k} k(kG_{n-2}) = k^2 G_{n-2} \\ &\longrightarrow \cdots \longrightarrow k^{n-2}(kG_1) = k^{n-1}G_1 \xrightarrow{(C_1)_{k^{n-1}}} k^{n-1}(kG) = k^n G. \end{aligned}$$

Now, we can apply Theorem B to show that the quotients of clique convergent pikas are clique convergent.

Lemma 9.2. *Let G be a connected, locally cyclic graph with minimum degree $\delta \geq 6$. If its universal cover $\tilde{G}$ is clique convergent, so is G itself.*

Proof. Let Γ be the deck transformation group of the universal covering map $p \colon \tilde{G} \to G$. Since $\tilde{G}$ is clique convergent, by Theorem 8.7, there is an $m \in \mathbb{N}$ such that $\tilde{G}$ does not contain a Δ_m-shaped subgraph. As $V(\tilde{G}_m) = \{\Delta_{m'} \cong S \subseteq \tilde{G} \mid m' \equiv_2 m \text{ and } m' \leq m\}$, this implies $\tilde{G}_{m-2} = \tilde{G}_m$. As the above defined action of Γ is defined on the triangular-shaped graphs this implies that $\tilde{G}_{m-2}$ and $\tilde{G}_m$ are Γ-isomorphic. By Remark 9.1, additionally $k^{m-2}\tilde{G}$ and $\tilde{G}_{m-2}$ are Γ-isomorphic as well as $k^m\tilde{G}$ and $\tilde{G}_m$, so concatenating Γ-isomorphisms, we obtain that $k^{m-2}\tilde{G}$ and $k^m\tilde{G}$ are Γ-isomorphic. By Theorem B, we conclude that G is clique convergent. □

We conclude the classification of clique dynamics of locally cyclic graphs with minimum degree $\delta \geq 6$, by joining Theorem 8.7 with Theorem A for one implication and with Lemma 9.2 for the other one.

Theorem D. *A (not necessarily finite) connected locally cyclic graph of minimum degree $\delta \geq 6$ is clique divergent if and only if its universal triangular cover contains arbitrarily large triangular-shaped subgraphs.*

If the graph G is finite, the characterisation can be given in a nicer way. The first implication of the following corollary is [LN00, Theorem 1.1], but we prove it independently.

Corollary E. *A finite and connected locally cyclic graph with minimum degree $\delta \geq 6$ is clique divergent if and only if it is 6-regular.*

Proof. The universal triangular cover of a 6-regular finite connected locally cyclic graph G is a 6-regular pika. As the hexagonal plane is a 6-regular pika and as it is the unique one by Corollary 6.8, the hexagonal plane is the universal cover of every 6-regular such G. As by definition, every triangular-shaped graph is a subgraph of the hexagonal plane, G is clique divergent.

For the other implication, let $p\colon \tilde{G} \to G$ be the universal covering map and let d be the diameter G, which is finite, as G is finite and connected. As G is clique divergent, by Theorem D, there is a subgraph $\Delta_{3d+3} \cong S \subseteq \tilde{G}$. Let $\tilde{v}$ be the middle vertex of S, i. e., the vertex that corresponds to $(d+1, d+1, d+1) \in V(\Delta_{3d+3})$, and let $v = p(\tilde{v})$ be its image under p.

As for any $\tilde{w} \in V(S \setminus \partial S)$, the open neighbourhoods $N_S(\tilde{w})$ induces a cycle of length 6 and as $\tilde{G}$ is locally cyclic, $\tilde{w}$ cannot be adjacent to any vertex $\tilde{G} \setminus S$. Consequently, $\mathrm{dist}_{\tilde{G}}(\tilde{v}, \tilde{G} \setminus S) \geq \mathrm{dist}_S(\tilde{v}, \partial S) + 1 = d + 2$ and the distance-d-neighbourhood $N^d_{\tilde{G}}[\tilde{v}]$ is a subset of $S \setminus \partial S$ and consists of vertices of degree 6.

For each vertex $u \in V(G)$, there is a vertex $\tilde{u} \in V(\tilde{G})$ with $p(\tilde{u}) = u$ such that $\mathrm{dist}_{\tilde{G}}(\tilde{v}, \tilde{u}) \leq d$, which is found by lifting a shortest walk connecting v and u utilising the unique walk lifting property of p. Thus, $\tilde{u} \in N^d_{\tilde{G}}[\tilde{v}]$ and it has degree 6 by the previous paragraph. As triangular covering maps cannot change vertex degrees, the vertex u has degree 6, too. Consequently, G is 6-regular.

□

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Eidesstattliche Erklärung

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erklärt hiermit, dass diese Dissertation und die darin dargelegten Inhalte die eigenen sind und selbstständig, als Ergebnis der eigenen originären Forschung, generiert wurden.

Hiermit erkläre ich an Eides statt

1. Diese Arbeit wurde vollständig oder größtenteils in der Phase als Doktorand dieser Fakultät und Universität angefertigt;

2. Sofern irgendein Bestandteil dieser Dissertation zuvor für einen akademischen Abschluss oder eine andere Qualifikation an dieser oder einer anderen Institution verwendet wurde, wurde dies klar angezeigt;

3. Wenn immer andere eigene- oder Veröffentlichungen Dritter herangezogen wurden, wurden diese klar benannt;

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5. Alle wesentlichen Quellen von Unterstützung wurden benannt;

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   - M. Baumeister and A.M. Limbach. Clique dynamics of locally cyclic graphs with $\delta \geq 6$. Discrete Mathematics, 345(7):112873, 2022

   - A.M. Limbach and M. Winter. Characterising Clique Convergence for Locally Cyclic Graphs of Minimum Degree $\delta \geq 6$. arXiv:2305.00503v1

Aachen, 23.10.23

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