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ASYMPTOTIC NORMALITY OF THE COEFFICIENTS OF THE MORGAN-VOYCE POLYNOMIALS

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Abstract
We study arithmetic and asymptotic properties of polynomials provided by $Q_n(x) := x \sum_{k=1}^{n} k\, Q_{n-k}(x)$ with initial value $Q_0(x) = 1$. The coefficients satisfy a central limit theorem and a local limit theorem involving Fibonacci numbers. We apply the methods of Berry and Esseen, Harper, Bender, and Canfield.

Keywords Central limit theorem · Fibonacci numbers · Local limit theorem · Singularity analysis

Introduction and main results

Let g be a normalized arithmetic function such that $G(t) := \sum_{n=1}^{\infty} g(n)\, t^n$ is regular at $t = 0$. We are interested in the asymptotic properties of the double sequence $a(n, k)$ of coefficients of the polynomials $p_n(x)$ defined by

$$\sum_{n=0}^{\infty} p_n(x)\, t^n = \frac{1}{1 - x\, G(t)}. \tag{1.1}$$

If $g(n) = \frac{1}{n!}$, then the coefficients are asymptotically normal [1, 8] and calculate the number of partitions of an n-set having exactly k labeled blocks. Thus, they are equal to $k!\, S_{n,k}$, where $S_{n,k}$ are the Stirling numbers of the second kind ([1], “The Berry–Esseen theorem” applications and [13]). The case $g(n) = 1$ leads to the Central Limit Theorem by de Moivre–Laplace (cf. [10]), since

$$a(n, k) = \binom{n-1}{k-1}.$$

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In this paper, we study the case where $g(n) = n$ in detail and show that the underlying expected values and variances involve Fibonacci numbers. We denote the associated polynomials and their coefficients by

$$Q_n(x) = \sum_{k=1}^{n} A_{n,k}\, x^k.$$

The Morgan-Voyce polynomials ([11], Chapter 41) are provided by $Q_{n+1}(x)/x$. In 1959, Morgan-Voyce [12] discovered the importance of these polynomials in the study of electric ladder networks of resistors. Further, in 1967 and 1968, several papers appeared by Swamy, Basin, Hoggatt, Jr., and Bicknell. These polynomials are closely related to Fibonacci polynomials. From [10], we know that

$$A_{n,k} := \binom{n+k-1}{2k-1}, \qquad n \in \mathbb{N},\ 0 \le k \le n.$$

The definition via Eq. 1.1 is equivalent to $Q_n(x) = x \sum_{k=1}^{n} k Q_{n-k}(x)$ with initial value $Q_0(x) = 1$. This hereditary recurrence relation can be reduced to a three-term recurrence relation

$$Q_{n+2}(x) - (2+x)Q_{n+1}(x) + Q_n(x) = 0, \qquad n \ge 1, \tag{1.2}$$

with initial values $Q_1(x) = x$ and $Q_2 = (x+2)x$. The double sequence $A_{n,k}$ is recorded in Sloane's database as A078812. For a glimpse on the asymptotic behavior of the coefficients, we refer to Table 1.

Asymptotic normality

Our first result, using techniques from singularity analysis (see Canfield [4], Section 3.6), states that the double sequence $A_{n,k}$ is asymptotically normal with asymptotic mean $a_n = \frac{1}{\sqrt{5}}\, n$ and asymptotic variance $b_n^2 = \frac{2}{5}\frac{1}{\sqrt{5}}\, n$.

Theorem 1.1 *There exist real sequences $(a_n)_n$ and $(b_n)_n$ with $b_n > 0$, such that*

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| \frac{1}{\sum_{k=0}^{n} A_{n,k}} \sum_{k \le a_n + x b_n} A_{n,k} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\, dt \right| = 0. \tag{1.3}$$

The sequences given by $a_n := n\, a$ and $b_n^2 := n\, b^2$ satisfy Eq. 1.3 with

$$a := \frac{1}{\sqrt{5}} \text{ and } b^2 := \frac{2}{5}\frac{1}{\sqrt{5}}.$$

Obviously, the sequences are not unique. In singularity analysis they are frequently called mean and variance (see Bender [1]) to indicate the link to probability theory and the classical Central Limit Theorem (cf. [7], Section 1.1).
Although we have here the Gaussian measure there is no direct connection to the Wiener chaos expansion, but it fits into the broader context of generating functions and recursive definitions of polynomials. The Hermite polynomials of the Wiener chaos expansion occur in another setting. For this, we refer to ([9], Section 3.4).

Table 1 Values of $\frac{1}{F_{2n}}\binom{n+\left\lfloor \frac{n}{\sqrt{5}} \right\rfloor - 1}{2\left\lfloor \frac{n}{\sqrt{5}} \right\rfloor - 1}$ (second columns) and $\left| \frac{2\sqrt{\pi}\sqrt{n}}{5^{\frac{3}{4}} F_{2n}} \binom{n+\left\lfloor \frac{n}{\sqrt{5}} \right\rfloor - 1}{2\left\lfloor \frac{n}{\sqrt{5}} \right\rfloor - 1} - 1 \right| \sqrt{n}$ (third columns)

n			n			n		
4	$1.9 \cdot 10^{-1}$	$1.1 \cdot 10^{0}$	40	$1.3 \cdot 10^{-1}$	$5.8 \cdot 10^{-1}$	400	$4.6 \cdot 10^{-2}$	$1.9 \cdot 10^{-1}$
6	$2.4 \cdot 10^{-1}$	$9.0 \cdot 10^{-1}$	60	$1.1 \cdot 10^{-1}$	$4.4 \cdot 10^{-1}$	600	$3.8 \cdot 10^{-2}$	$4.2 \cdot 10^{-2}$
8	$2.5 \cdot 10^{-1}$	$6.6 \cdot 10^{-1}$	80	$1.0 \cdot 10^{-1}$	$3.4 \cdot 10^{-1}$	800	$3.3 \cdot 10^{-2}$	$1.1 \cdot 10^{-1}$
10	$2.5 \cdot 10^{-1}$	$4.7 \cdot 10^{-1}$	100	$9.1 \cdot 10^{-2}$	$2.8 \cdot 10^{-1}$	1000	$2.9 \cdot 10^{-2}$	$2.1 \cdot 10^{-2}$

In this paper, we provide sequences $(\mu_n)_n$ and $(\sigma_n^2)_n$, the expected value and variance of a suitable sequence of a random variable X_n. We obtain a refined version of Theorem 1.1. We apply a method of Harper [8] (see also [4] section 3.4: the method of negative roots). The values μ_n and σ_n^2 involve Fibonacci numbers F_n and the golden ratio φ. They characterize the peaks and modes of the coefficients of the generating series. These are real-rooted polynomials, which are related to the Jonquière function (also called polylogarithm $\mathrm{Li}_{-1}(t)$) given by $G(t) := \sum_{n=1}^{\infty} n\, t^n$. The radius of convergence is 1.

Probabilistic approach

Let X be a random variable. Then, X is called normally distributed, if the probability $P(X \leq x)$ is equal to the normal distribution

$$\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\, dt,$$

for all real x. The distribution Φ is called normal or Gaussian distribution. Further, let $\phi(x)$ be the underlying density function.

Definition A sequence $(S_n)_n$ of random variables satisfies a Central Limit Theorem, if there exist sequences $(a_n)_n$ and $(b_n)_n$ with $a_n, b_n \in \mathbb{R}$ and $b_n > 0$, such that the normalized random variables

$$S_n^* := \frac{S_n - a_n}{b_n},$$

converges in distribution to the normal distribution

$$P(S_n^* \leq x) \stackrel{D}{\longrightarrow} \Phi(x).$$

Consider the sequence of random variables $X_n \in \{0, 1, \dots, n\}$ defined by:

$$P(X_n = k) := \frac{A_{n,k}}{\sum_{m=0}^{n} A_{n,m}}, \text{ where } Q_n(x) = \sum_{k=0}^{n} A_{n,k}\, x^k. \tag{1.4}$$

Let $S_n := \sum_{m=1}^{n} X_m$. Then,

$$P(S_n^* \leq x) = \frac{1}{\sum_{m=0}^{n} A_{n,m}} \sum_{k \leq a_n + x\, b_n} A_{n,k}, \tag{1.5}$$

leads to a probabilistic interpretation of Theorem 1.1.

Theorem 1.2 *Let S_n^* be the normalized random variables as defined in Eq. 1.5 with a_n and b_n^2 provided by the expected value $\mathbb{E}(X_n)$ and variance $\mathbb{V}(X_n)$ of the random variable X_n, associated with the double sequence $A_{n,k}$. Then, $P(S_n^* \leq x) \stackrel{D}{\longrightarrow} \Phi(x)$. Therefore,*

$$\lim_{n \to \infty} \left\| \frac{1}{\sum_{m=0}^{n} A_{n,m}} \sum_{k \leq \mathbb{E}(S_n) + \sqrt{\mathbb{V}(S_n)}\, x} A_{n,k} - \Phi(x) \right\|_{\mathbb{R}} = 0. \tag{1.6}$$

Here, $\|f(x)\|_M$ denotes the supremum norm of f on $M \subset \mathbb{R}$. This theorem is obtained by a method introduced by Harper [4, 8] and a result by Pólya for continuous distribution functions. We determine the explicit values of the mean and variance of X_n. These are expressed in terms of Fibonacci numbers F_n and are related to the peaks and plateaus of the unimodal sequence $A_{n,0} \leq A_{n,1} \leq \ldots \leq A_{n,m} \geq \ldots \geq A_{n,n}$ by Darroch's Theorem [5]. The sequence is log-concave and therefore, unimodal, since the polynomials $Q_n(x)$ are orthogonal polynomials. Indeed, $Q_n(x) = x\, U_{n-1}(x/2 + 1)$, where $U_n(x)$ are the Chebyshev polynomials of the second kind [10]. The next lemma shows that the normalizing factor that appears in Eq. 1.6 is a Fibonacci number.

Lemma 1.3 *We have*

$$\sum_{k=0}^{n} A_{n,k} = Q_n(1) = F_{2n}.$$

Proposition 1.4 *The random variable X_n as defined in Eq. 1.4 has the expected value $\mu_n := \mathbb{E}(X_n)$ and variance $\sigma_n^2 := \mathbb{V}(X_n)$ given by*

$$\mu_n = \frac{2}{5}\left(\frac{F_{2n+1}}{F_{2n}} - \frac{1}{2} + \frac{1}{n}\right)n,$$
$$\sigma_n^2 = \frac{4}{25}\left(\frac{F_{2n+1}}{F_{2n}} - \frac{1}{2} - \frac{n}{F_{2n}^2} - \frac{1}{2n}\right)n.$$

Remark

a) Note that μ_n is never an integer for $n \geq 2$. If μ_n is an integer, then $2\frac{F_{2n+1}}{F_{2n}}n$ is also an integer. Since F_{2n+1} and F_{2n} are coprime, $\frac{2n}{F_{2n}}$ is also an integer. Since $F_{2n} > 2n$ for $n \geq 3$ and the only remaining case is $n = 1$.
b) The reciprocal polynomial of $Q_n(x)$ has the coefficients $\binom{N-k-1}{k}$ with $N = 2n$. These sequences have been investigated by Tanny and Zucker [14, 15] and Benoumhani [3]. It has been shown that the sequence is strictly log-concave, the smallest mode has been determined as well as the indices, at which a double maximum occurs.

Darroch's theorem [5] (see also Benoumhani [2]) leads to:

Corollary 1.5 *The modes of $Q_n(x)$ are located around μ_n. Let m_n be a mode, then*

$$0 < \left|\frac{2}{5}\left(\frac{F_{2n+1}}{F_{2n}} - \frac{1}{2} + \frac{1}{n}\right)n - m_n\right| < 1.$$

Elementary calculations and explicit solutions of the Pell–Fermat equation lead to:

Theorem 1.6 *Let $n \in \mathbb{N}$. The polynomials $Q_n(x)$ are unimodal and have at most two modes. The smallest mode m is uniquely determined by*

$$\frac{\sqrt{5n^2+1}-1}{5} \leq m < \frac{\sqrt{5n^2+1}+4}{5}.$$

The mode is unique if $5m^2 + 2m \neq n^2$. We have two modes m_k and $m_k + 1$, if and only if

$$\begin{pmatrix} 5m_k + 1 \\ n_k \end{pmatrix} = \begin{pmatrix} 161 & 360 \\ 72 & 161 \end{pmatrix}^k \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{for } k \geq 1.$$

Remark If $5m^2 + 2m = n^2$ has positive integer solutions, there are two modes of $A_{n,m}$ at m and $m + 1$. For example, for $n = n_1 = 72$ and $m = m_1 = 32$, we have $\binom{103}{63} = 612181827433047018914314 82520 = \binom{104}{65}$. The sequence of these n_k starts with

$$72, 23184, 7465176, 2403763488, 774004377960, 249227005939632, \ldots$$

and the corresponding m_k are

$$32, 10368, 3338528, 1074995712, 346145280800, 111457705421952, \ldots$$

The expected values μ_n and variances σ_n^2 converge against the sequences a_n and b_n, obtained in Theorem 1.1 by methods of singularity analysis.

Corollary 1.7 *Let X_n be the random variable defined by Eq. 1.4 with expected value μ_n and variance σ_n^2. Let a and b be as in Theorem 1.1. Let φ denote the golden ratio. Then,*

$$\lim_{n\to\infty} \frac{\mu_n}{n} = \frac{2}{5}\left(\varphi - \frac{1}{2}\right) = \frac{1}{\sqrt{5}} = a,$$
$$\lim_{n\to\infty} \frac{\sigma_n^2}{n} = \frac{4}{25}\left(\varphi - \frac{1}{2}\right) = \frac{2}{5}\,\frac{1}{\sqrt{5}} = b^2.$$

This follows from Proposition 1.4, since we know that

$$\lim_{n\to\infty} \frac{F_{2n+1}}{F_{2n}} = \varphi \qquad \text{(Kepler)}.$$

It is possible to obtain a rate of convergence of Eq. 1.6 by utilizing the Berry–Esseen theorem (see, for example, [4], section 3.2), which implies Theorem 1.2, since the variance satisfies $\lim_{n\to\infty} \sigma_n = \infty$.

Theorem 1.8 *Let S_n^* be the normalized random variables as defined in Eq. 1.5, associated with $A_{n,k}$. Then,*

$$|P(S_n^* \le x) - \Phi(x)| \;\le\; C\,\sigma_n^{-1}.$$

Here, $C > 0$ can be chosen as $C = 0.7975$.

The Central Limit Theorem, to quote Bender [1], provides a certain qualitative feel for the numbers $A_{n,k}$. Further information is provided by local limit theorems (cf. [4], section 3.7). Let $A_{n,k}^* := A_{n,k}/Q_n(1)$. We consider subsets $K \subset \{0, 1, \ldots, n\}$ and $k \in K$ with the asymptotic behavior

$$A_{n,k}^* \;\sim\; \left.\frac{\phi(x)}{\sigma_n}\right|_{x=(k-\mu_n)/\sigma_n}.$$

More generally, we say the doubly indexed sequence $a(n, k)$ satisfies a local limit theorem on a set S of real numbers provided

$$\sup_{x\in S} \left| \frac{\sigma_n\, a(n, \lfloor \mu_n + x\,\sigma_n \rfloor)}{\sum_k a(n,k)} - \phi(x) \right| \longrightarrow 0,$$

where μ_n and σ_n are the expected values and variances ([4], definition 3.7.1). Further, we quote the following result.

Theorem 1.9 (Bender, [4], Theorem 3.7.2) *Suppose that $a(n, k)$ are asymptotically normal and $\sigma_n^2 \to \infty$. If for each n the sequence $a(n, k)$ is unimodal in k, then $a(n, k)$ satisfies a local limit theorem on the set $\{x \;:\; |x| \ge \varepsilon\}$, for any $\varepsilon > 0$. If for each n the sequence $a(n, k)$ is log-concave in k, the $a(n, k)$ satisfies a local limit theorem on $\mathbb{R}$.*

We have that $A(n, k)$ are asymptotically normal by Theorem 1.2 and $\sigma_n^2 \to \infty$. Further, that $Q_n(x)$ has real roots. By Theorem 1.9, this implies:

Corollary 1.10

$$\lim_{n\to\infty} \left\| \sigma_n A^*_{n,\lfloor \mu_n + x\sigma_n \rfloor} - \phi(x) \right\|_{\mathbb{R}} = 0.$$

Corollary 1.11 *Further, we have the asymptotic formula*

$$A_{n,k} \sim F_{2n} \frac{\phi(x)}{\sigma_n} \text{ for } n \to \infty,$$

where $k = \mu_n + x\sigma_n$ and x is bounded.

Singularity analysis

Bender [1] provided a criterion for a double sequence $a(n, k)$ to be asymptotically normal. Typically the criterion applies if the generating function has only one singularity on the circle of convergence (see also [4], section 3.6 Method 4).

Theorem 2.1 (Bender) *Let $f(x, t) = \sum_{k,n} a(n, k)\, x^k\, t^n$, with $a(n, k) \geq 0$. Suppose there exist*

(i) *a function $A(s)$ continuous and non-zero near 0,*
(ii) *a function $r(s)$ with bounded third derivative near 0,*
(iii) *a non-negative integer m, and*
(iv) *positive numbers ε and δ such that*

$$\left(1 - \frac{t}{r(s)}\right)^m f(e^s, t) - \frac{A(s)}{1 - \frac{t}{r(s)}}$$

is analytic and bounded for $|s| < \varepsilon, |t| < r(0) + \delta$.

Put $a = -r'(0)/r(0)$ and $b^2 = a^2 - r''(0)/r(0)$. If $b^2 \neq 0$, then the numbers $a(n, k)$ are asymptotically normal with $a_n = n\, a$ and $b_n^2 = b^2\, n$.

Proof of Theorem 1.1

Let $a(n, k) = A_{n,k}$. Let $G(t) = \sum_{n=1}^{\infty} n\, t^n$. This power series has a radius of convergence 1 and has a pole at $t = 1$. The generating series of $A_{n,k}$ (we refer to [10]) is provided by

$$f(x, t) = \frac{1}{1 - x\, G(t)} = \sum_{n=0}^{\infty} Q_n(x)\, t^n.$$

We have $G(t) = \frac{t}{(t-1)^2}$, thus $G(t) = 1$ has two real solutions. The smallest in absolute value is given by $t_1 = \frac{3-\sqrt{5}}{2}$. Suppose $x \neq 0$. If $G(t) = \frac{1}{x}$, then $xt = (1 - t)^2$ i. e. $t^2 - (2 + x)t + 1 = 0$. Therefore, $t = 1 + \frac{x}{2} - \sqrt{\frac{x^2}{4} + x}$ with the principle branch of the square root for $|x - 1| < 1$. Let $r(s) = 1 + \frac{e^s}{2} - \sqrt{\frac{e^{2s}}{4} + e^s}$ for $|s| < \ln 2$. Then, the pole of $f(x, t)$ closest to 0 is located at $r(s)$. The other pole is located at $1/r(s) = 1 + \frac{e^s}{2} + \sqrt{\frac{e^{2s}}{4} + e^s}$. Since

$$f(e^s, t) = 1 + \frac{e^s}{2\sqrt{\frac{e^{2s}}{4} + e^s}} \left(\frac{1}{1 - t/r(s)} - \frac{1}{1 - r(s)t} \right),$$

we obtain

$$f(e^s,t) - \frac{e^s}{2\sqrt{\frac{e^{2s}}{4}+e^s}}\frac{1}{1-t/r(s)} = 1 - \frac{e^s}{2\sqrt{\frac{e^{2s}}{4}+e^s}}\frac{1}{1-r(s)t}. \tag{2.1}$$

For $|s| < \ln 2$, we have $|r(s)| < \frac{1}{2}$. Therefore, $|1/r(s)| > 2$ and Eq. 2.1 is analytic and bounded for $|t| < \frac{3}{2}$. We obtain $r'(s) = \frac{e^s}{2} - \frac{1}{2}\left(\frac{e^{2s}}{4}+e^s\right)^{-1/2}\left(\frac{e^{2s}}{2}+e^s\right)$ and

$$r''(s) = \frac{e^s}{2} + \frac{1}{4}\left(\frac{e^{2s}}{4}+e^s\right)^{-3/2}\left(\frac{e^{2s}}{2}+e^s\right)^2 - \frac{1}{2}\left(\frac{e^{2s}}{4}+e^s\right)^{-1/2}\left(e^{2s}+e^s\right).$$

Further,

$$\begin{aligned} r'''(s) =& \frac{e^s}{2} - \frac{3}{8}\left(\frac{e^{2s}}{4}+e^s\right)^{-5/2}\left(\frac{e^{2s}}{2}+e^s\right)^3 \\ &+ \frac{1}{2}\left(\frac{e^{2s}}{4}+e^s\right)^{-3/2}\left(\frac{e^{2s}}{2}+e^s\right)\left(e^{2s}+e^s\right) \\ &+ \frac{1}{4}\left(\frac{e^{2s}}{4}+e^s\right)^{-3/2}\left(e^{2s}+e^s\right)^2 - \frac{1}{2}\left(\frac{e^{2s}}{4}+e^s\right)^{-1/2}\left(2e^{2s}+e^s\right), \end{aligned}$$

which shows that the third derivative is bounded near 0,

$$r'(0) = \frac{1}{2} - \frac{1}{2}\frac{2}{\sqrt{5}}\frac{3}{2} = \frac{1}{2} - \frac{3}{2\sqrt{5}}$$

and

$$r''(0) = \frac{1}{2} + \frac{1}{4}\frac{8}{5\sqrt{5}}\frac{9}{4} - \frac{2}{\sqrt{5}} = \frac{1}{2} - \frac{11}{10\sqrt{5}}.$$

With $r(0) = \frac{3}{2} - \frac{\sqrt{5}}{2} = \left(\frac{3}{2\sqrt{5}} - \frac{1}{2}\right)\sqrt{5}$, this yields $a = \frac{1}{\sqrt{5}}$. As $1/r(0) = \frac{3}{2} + \frac{\sqrt{5}}{2}$, we obtain

$$b = \frac{1}{5} - \left(\frac{3}{2} + \frac{\sqrt{5}}{2}\right)\left(\frac{1}{2} - \frac{11}{10\sqrt{5}}\right) = \frac{1}{5} - \frac{3}{4} + \frac{11}{20} - \frac{\sqrt{5}}{4} + \frac{33}{20\sqrt{5}} = \frac{2}{5\sqrt{5}}.$$

The Berry–Esseen theorem

Let $X \in \{0, 1, \dots, n\}$ be a random variable with expected value $\mathbb{E}(X)$ and variance $\mathbb{V}(X)$ for $n \in \mathbb{N}$. Let $P_X(x) = \sum_{k=0}^{n} P(X=k)\,x^k$ be the probability generating function. A straightforward calculation leads to

$$\begin{aligned} \mathbb{E}(X) =& P'(1), \\ \mathbb{V}(X) =& P''(1) - \left(P'(1)\right)^2 + P'(1). \end{aligned}$$

There is a Central Limit Theorem for a sequence of independent, but not necessarily identically distributed random variables. We work this out in the setting of a triangular array of Bernoulli random variables. To control the rate of convergence, we utilize a Berry–Esseen theorem (we refer to [4], section 3).

Theorem 3.1 *Let $X_{n,k}$ for $1 \leq k \leq n$ be independent random variables with expected values $\mu_{n,k}$, variances $\sigma_{n,k}^2$ and absolute third central moment*

$$\rho_{n,k} = \mathbb{E}(|X_{n,k} - \mu_{n,k}|^3) < \infty.$$

Let $\mu_n = \sum_{k=1}^n \mu_{n,k}$, $\sigma_n^2 := \sum_{k=1}^n \sigma_{n,k}^2$ and $S_n := \sum_{k=1}^n X_{n,k}$. Let $S_n^* = (S_n - \mu_n)/\sigma_n$. Then,

$$\left\| P(S_n^* < x) - \Phi(x) \right\|_{\mathbb{R}} \leq C \, \frac{\sum_{k=1}^n \rho_{n,k}}{\sigma_n^3},$$

where $C > 0$ is a universal constant. This constant can be chosen as $C = 0.7975$ [16].

Harper's method

Let $P_n(x) = \sum_{k=0}^n a_{n,k} x^k$ be a monic polynomial of degree $n \geq 1$ and $a_{n,k} \geq 0$. Suppose the roots are real and $P_n(x) = \prod_{k=1}^n (x + r_k)$. Harper [8] introduced a triangular array of Bernoulli random variables $X_{n,j}$ with distribution

$$P(X_{n,j} = 0) := \frac{r_j}{1 + r_j} \text{ and } P(X_{n,j} = 1) := \frac{1}{1 + r_j}.$$

Let $S_n := \sum_{j=1}^n X_{n,j}$. Then, $P(S_n = k) = \frac{a_{n,k}}{P_n(1)}$.

Lemma 3.2 *Let $X_{n,j}$ be given. Then,*

$$\mathbb{E}(X_{n,j}) = \frac{1}{1 + r_j}, \quad \mathbb{V}(X_{n,j}) = \frac{r_j}{(1 + r_j)^2}, \quad \mathbb{E}(|X_{n,j} - \mathbb{E}(X_{n,j})|^3) = \frac{r_j(1 + r_j^2)}{(1 + r_j)^4}.$$

This implies that $\mathbb{E}(|X_{n,j} - \mathbb{E}(X_{n,j})|^3) < \mathbb{V}(X_{n,j})$.

Proof of Theorem 1.8 Using Lemma 3.2, we obtain

$$\frac{\sum_{k=1}^n \rho_{n,k}}{\sigma_n^3} \leq \frac{\sum_{k=1}^n \sigma_{n,k}^2}{\sigma_n^3} = \frac{1}{\sigma_n}.$$

Invoking Theorem 3.1 then gives the result.

Expected values μ_n and variances σ_n^2

The polynomials of the family $\{Q_n(x)\}_n$ satisfy the three term recurrence relation in Eq. 1.2. Then, $u_n = Q_n(1)$ satisfies the recurrence

$$u_n - 3u_{n-1} + u_{n-2} = 0. \tag{3.1}$$

Proof of Lemma 1.3 The Fibonacci numbers F_n satisfy the recurrence relation $F_n - F_{n-1} - F_{n-2} = 0$. Therefore, $F_{n-1} = F_n - F_{n-2} = F_{n+1} - 2F_{n-1} + F_{n-3}$, which implies $F_{n+1} - 3F_{n-1} + F_{n-3} = 0$. Therefore, the sequences constituted by the F_{2n} or F_{2n+1} satisfy the same recurrence relation as u_n. Now, $u_1 = 1 = F_2$ and $u_2 = 3 = F_4$, so we obtain $u_n = F_{2n}$ for $n \geq 1$.

Proof of Proposition 1.4 To determine μ_n and σ_n, we follow the strategy offered by the standard method of solving a linear non-homogeneous difference equation (we refer to [6], Section 2.4). For $v_n = Q_n'(1)$ holds $v_n - 3v_{n-1} + v_{n-2} = u_{n-1} = F_{2n-2}$. The sequences constituted by F_{2n} or F_{2n+1}, resp., are linearly independent solutions of the homogeneous difference equation Eq. 3.1. Therefore, the solution of the inhomogeneous difference equation

$$v_n - 3v_{n-1} + v_{n-2} = u_{n-1}.$$

is a linear combination of $F_{2n}, nF_{2n}, F_{2n+1}, nF_{2n+1}$. From the initial conditions $v_1 = 1, v_2 = 4, v_3 = 14$, and $v_4 = 46$, we can determine the coefficients and obtain $v_n = \frac{2}{5}nF_{2n+1} + \frac{2}{5}F_{2n} - \frac{1}{5}nF_{2n}$ for $n \geq 1$ and observe that this also holds for $n = 0$.

Let $w_n = Q_n''(1)$. Then, $w_n - 3w_{n-1} + w_{n-2} = v_{n-1}$ and $N(E) = p(E)^2$ is an annihilator of the right-hand side. Since the sequences constituted by F_{2n} and F_{2n+1} are a fundamental system of solutions of Eq. 3.1, we obtain a fundamental system of solutions of the inhomogeneous one by $F_{2n}, F_{2n+1}, nF_{2n}, nF_{2n+1}, n^2F_{2n}$, and n^2F_{2n+1}. The coefficients can be determined from the values $w_1 = 0$, $w_2 = 2$, $w_3 = 14$, $w_4 = 68$, $w_5 = 282$, and $w_6 = 1068$ and we obtain $w_n = \left(\frac{1}{5}n^2 - \frac{1}{25}n - \frac{8}{25}\right)F_{2n} + \frac{2}{25}nF_{2n+1}$. Recall that

$$F_{2n}^2 + F_{2n}F_{2n+1} - F_{2n+1}^2 = -1 \text{ for } n \geq 0.$$

This leads to the explicit formula for σ_n^2.

Location of the modes: proof of Theorem 1.6

Proof Let $A_{n,m} = \binom{n+m-1}{2m-1}$. Therefore, $A_{n,m} - A_{n,m+1} = \binom{n+m-1}{2m-1}\left(1 - \frac{(n+m)(n-m)}{(2m+1)2m}\right)$. For the numerator of the expression in brackets, we obtain $5m^2 + 2m - n^2 = 5\left(m + \frac{1}{5}\right)^2 - \frac{1}{5} - n^2$. Therefore, $A_{n,m} > A_{n,m+1}$ for $m > \frac{\sqrt{5n^2+1}-1}{5}$ and $A_{n,m-1} < A_{n,m}$ for $m < \frac{\sqrt{5n^2+1}-1}{5} + 1$.

Obviously, for $n^2 = 5m^2 + 2m$, we have two modes. This equation is equivalent to $5n^2 = (5m+1)^2 - 1$. With $j = 5m + 1$, we obtain the Pell–Fermat equation $j^2 - 5n^2 = 1$. All its non-negative solutions are $\begin{pmatrix} j \\ n \end{pmatrix} = \begin{pmatrix} 9 & 20 \\ 4 & 9 \end{pmatrix}^{k'} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for $k' \geq 0$. To be a solution to the original problem, the integer j must satisfy $j \equiv 1 \mod 5$. Exactly even powers k' yield such a solution.

Proof of Theorem 1.2 To apply Theorem 1.8, we need to show that the variance σ_n^2 proceeds to infinity. But this follows from Corollary 1.7.

Local limit theorem: numerical data

We consider Corollary 1.11 for $k = a_n + b_n$. Here, $a_n = n\,a$ and $b_n = \sqrt{n}\;b$, where we approximated μ_n by a_n and σ_n by b_n. A local Berry–Esseen result suggests a rate of convergence by $1/\sqrt{n}$ (see Table 1). We have

$$\binom{n + \left\lfloor \frac{n}{\sqrt{5}} \right\rfloor - 1}{2\left\lfloor \frac{n}{\sqrt{5}} \right\rfloor - 1} \sim \frac{5^{\frac{3}{4}}}{2\sqrt{\pi}\,\sqrt{n}} F_{2n}.$$

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Declarations

Conflict of interest The authors declare no competing interests.

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