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RESEARCH ARTICLE

Convergence to the planar interface for a nonlocal free-boundary evolution

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Abstract

We capture optimal decay for the Mullins–Sekerka evolution, a nonlocal, parabolic free boundary problem from materials science. Our main result establishes convergence of BV solutions to the planar profile in the physically relevant case of ambient space dimension three. Far from assuming small or well-prepared initial data, we allow for initial interfaces that do not have graph structure and are not connected, hence explicitly including the regime of Ostwald ripening. In terms only of initially finite (not small) excess mass and excess surface energy, we establish that the surface becomes a Lipschitz graph within a fixed timescale (quantitatively estimated) and remains trapped within this setting. To obtain the graph structure, we leverage regularity results from geometric measure theory. At the same time, we extend a duality method previously employed for one-dimensional PDE problems to higher dimensional, nonlocal geometric evolutions. Optimal algebraic decay rates of excess energy, dissipation, and graph height are obtained.

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1 | INTRODUCTION

The Mullins–Sekerka interfacial evolution (abbreviated MS in the following) is a nonlocal free boundary evolution: The normal velocity of the interface is a nonlocal function of the interface. Essential features of MS are the preservation of (signed) mass and the reduction of surface area. In addition, there is a scale invariance: The solution space is invariant under a rescaling of length by a factor of λ and time by a factor of λ^3. Hence MS is a geometric version of a third-order parabolic equation.

We are interested in the relaxation to a planar interface for fairly ill-prepared initial data: The initial interface is required neither to be a graph nor to consist of a single connected component. Singularities are not a matter of idle curiosity in MS but rather a fact of life: A well-known and naturally occurring configuration is that of *Ostwald ripening*, in which the positive phase is distributed in multiple, disconnected islands, and the smaller islands shrink and disappear (a singularity in the flow). We allow for such configurations.

We do not study existence but rather analyze the qualitative and quantitative properties of any BV solution (which are not known to be unique). We will obtain optimal relaxation rates to the flat configuration based on only monitoring the (absolute) excess mass or "volume" (between the positive phase $\Omega_+ \subseteq \mathbb{R}^{d+1}$ and $\{z > 0\}$):

$$\mathcal{V} := \int_{\mathbb{R}^{d+1}} |\chi| \, \mathrm{d}\mathbf{x}, \qquad \text{where} \qquad \chi := \mathbf{1}_{\Omega_+} - \mathbf{1}_{\mathbb{R}^d \times \{z>0\}} \tag{1.1}$$

and the excess surface area (compared to the flat configuration):

$$\mathcal{E} = \int_\Gamma 1 - e_z \cdot n \, \mathrm{d}S,$$

where, e_z is the unit vector in the z-direction and n is the **unit inward normal** to the boundary of the positive phase $\Gamma = \partial\Omega_+$.

In its strong form, the MS dynamics consists of the evolution of a d-dimensional hypersurface Γ, which is the boundary of a region Ω_+ and induces a potential $f : \mathbb{R}^{d+1} \to \mathbb{R}$ as the two-sided harmonic extension of the mean curvature (the sum of the principal curvatures) H of Γ according to

$$\begin{cases} \boldsymbol{\Delta} f = 0 & \text{in } \mathbb{R}^{d+1} \setminus \Gamma, \\ f = H & \text{on } \Gamma. \end{cases} \tag{1.2}$$

The surface Γ itself evolves with normal velocity determined by the jump in normal derivative of f across Γ via:

$$V := -[\boldsymbol{\nabla} f \cdot n], \tag{1.3}$$

where here and throughout we use square brackets to indicate $[\boldsymbol{\nabla} f \cdot n] = \nabla f_+ \cdot n - \nabla f_- \cdot n$.

Definition 1.1. For given initial data $\mathbf{1}_{\Omega_+(0)} \in BV_{\mathrm{loc}}(\mathbb{R}^{d+1})$, we call an L^1-continuous family of characteristic functions $\mathbf{1}_{\Omega_+(t)} \in BV_{\mathrm{loc}}(\mathbb{R}^{d+1})$ together with $f \in L^2(0, \infty; \dot{H}^1(\mathbb{R}^{d+1}))$ a weak BV solution of the MS dynamics if the following conditions are satisfied.

(1) For any $T > 0$, all $\varphi \in C_c^\infty(\mathbb{R}^{d+1} \times [0,T])$, and χ as in (1.1) there holds

$$\left(\int_{\mathbb{R}^{d+1}} \chi\varphi\,\mathrm{d}\mathbf{x}\right)(T) - \left(\int_{\mathbb{R}^{d+1}} \chi\varphi\,\mathrm{d}\mathbf{x}\right)(0) = \int_0^T \int_{\mathbb{R}^{d+1}} \chi\partial_t\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t - \int_0^T \int_{\mathbb{R}^{d+1}} \boldsymbol{\nabla} f \cdot \boldsymbol{\nabla}\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t. \tag{1.4}$$

(2) Let $\Gamma(t) \subseteq \mathbb{R}^{d+1}$ be the support of the Radon measure $\boldsymbol{\nabla}\mathbf{1}_{\Omega_+}$ and let n be the Radon–Nikodym derivative of $\boldsymbol{\nabla}\mathbf{1}_{\Omega_+}$ with respect to its total variation measure $|\boldsymbol{\nabla}\mathbf{1}_{\Omega_+}|$. For almost all $t > 0$, and all $\xi \in C_c^\infty(\mathbb{R}^{d+1}; \mathbb{R}^{d+1})$, there holds

$$\int_\Gamma (\mathrm{Id} - n \otimes n) : \nabla\xi\,\mathrm{d}S = \int_{\mathbb{R}^{d+1}} \mathbf{1}_{\Omega_+}\,\mathrm{div}(f\xi)\,\mathrm{d}\mathbf{x}, \tag{1.5}$$

where the notation $\int_\Gamma \mathrm{d}S$ is to be understood as $\int_{\mathbb{R}^{d+1}} \mathrm{d}|\boldsymbol{\nabla}\mathbf{1}_{\Omega_+}|$.

We will see in Section 4 that there exists a measurable function H such that the integrals in (1.5) are equal to $-\int_\Gamma H\xi \cdot n\,\mathrm{d}S$, and H can be considered the generalized mean curvature. (Note that with our choice of orientation, the mean curvature H is positive where Ω_+ is convex.) We remark that (1.4) encodes $\partial_t\chi - \Delta f = 0$ in the sense of distributions; in particular, f is harmonic outside Γ. For a smooth solution, (1.3) follows from (1.4) via integration by parts, and the boundary condition (1.2) follows from (1.5), integration by parts, and the equality with $-\int_\Gamma H\xi \cdot n\,\mathrm{d}S$ observed above.

We will work throughout the paper with BV solutions that satisfy the hypothesis:

Hypothesis (H). The energy is absolutely continuous and nonincreasing with a.e. time derivative

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} \geq D := \int_{\mathbb{R}^{d+1}} |\boldsymbol{\nabla} f|^2\,\mathrm{d}\mathbf{x}. \tag{1.6}$$

Furthermore we assume that the solution is smooth once it reaches a state with graph structure and small enough Lipschitz constant, that is, that there exists $\varepsilon_0 > 0$ such that

$$\Omega_+(t) = \left\{(x,z) \in \mathbb{R}^{d+1} : z > h(x,t)\right\} \quad \text{and} \quad \|\nabla h(\cdot,t)\|_\infty \leq \varepsilon_0$$

implies that the solution is smooth in space-time on (t,∞).

In particular, Hypothesis **(H)** allows for singular events that occur when connected components of Ω_+ disappear (as in Ostwald ripening) or collide. We motivate our hypothesis in the context of the available literature in Subsection 1.1 below. The reader who prefers not to consider weak solutions may alternatively suppose that there is a strong solution outside of finitely many singular times.

The MS evolution has the (formal) structure of a gradient flow of the excess surface energy $\mathcal{E}$ with respect to the $\dot{H}^{-1}$ metric, where the configuration space is given by characteristic functions.

Naturally Hypothesis **(H)** implies that for all $s < t$ there holds

$$\mathcal{E}(t) - \mathcal{E}(s) \le -\int_s^t D(\tau)\,\mathrm{d}\tau, \tag{1.7}$$

and we will regularly make use of the energy dissipation in the form

$$\mathcal{E} \le \mathcal{E}_0 \qquad \text{for all } t > 0. \tag{1.8}$$

Remark 1.2. The De-Giorgi type inequality from [15] suggests replacing (1.6) by $-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} \ge \frac{1}{2}D$, in which case our results carry through unchanged.

We obtain optimal algebraic-in-time decay rates for the energy gap within the L^1 setting, that is, for initially finite L^1 distance to a half-space, $\mathcal{V}_0 < \infty$. The L^1 norm is naturally related to the fact that the dynamics is mass conserving, that is, the value $\int_{\mathbb{R}^{d+1}} \chi\,\mathrm{d}\mathbf{x}$ is a conserved quantity of the evolution. (Note, however, that the L^1 norm $\mathcal{V}$ *is not typically preserved*; an elementary example is the shrinking of two initially circular islands symmetrically located on either side of a flat interface.) Our most interesting result is for $d = 2$, where we show that even for initial data that do not have graph structure or even connected phases, the evolution *enters the graph setting* within finite time, after which it remains trapped close to the linear evolution.

Our main result for $d = 2$ (ambient space dimension three) is:

Theorem 1.3 (Relaxation to flat). *Consider an initial set $\Omega_{+,0} \subseteq \mathbb{R}^3$ of locally finite perimeter with reduced boundary Γ_0 having finite excess mass $\mathcal{V}_0$ and excess energy $\mathcal{E}_0$. Corresponding to this initial data, consider any solution of MS satisfying **(H)**. For all times $t > 0$, the excess mass is bounded by*

$$\mathcal{V} \lesssim \mathcal{V}_0 + \mathcal{E}_0^{\frac{3}{2}}, \tag{1.9}$$

and the energy decays according to

$$\mathcal{E} \lesssim \min\left\{ \mathcal{E}_0, \frac{\mathcal{V}_0^2 + \mathcal{E}_0^3}{t^{\frac{4}{3}}} \right\}. \tag{1.10}$$

Moreover, there exists a time $T_g \sim \mathcal{E}_0^{3/2}$ ("graph-time") such that for all $t \ge T_g$, the interface $\Gamma(t)$ is the graph of a Lipschitz function

$$\Gamma(t) = \left\{ (x, z) \in \mathbb{R}^3 \,:\, z = h(x, t) \right\}, \tag{1.11}$$

with

$$\|\nabla h(\cdot, t)\|_\infty \le 1. \tag{1.12}$$

Finally, for $t \ge T_g$, the dissipation and height function h satisfy

$$D(t) \lesssim \frac{\mathcal{E}\left(\frac{t}{2}\right)}{t} \tag{1.13}$$

and

$$\|h\|_\infty + t^{\frac{1}{3}}\|\nabla h\|_\infty \lesssim \frac{\mathcal{V}_0 + \mathcal{E}_0^{\frac{3}{2}}}{t^{\frac{2}{3}}}, \quad \textit{respectively}. \tag{1.14}$$

Remark 1.4 (Notation). In order to get straight to the discussion of the main result, we relegate a discussion of the (hopefully rather natural) notation—including the order notation $\sim$, $\lesssim$ etcetera—to Subsection 1.4. The meticulous or skeptical reader may want to start there.

Remark 1.5 (Uniqueness). We do not require uniqueness, and the result is true for *any* weak solution. Once the solution has become a graph with small Lipschitz constant, we expect uniqueness to hold (see the discussion in Subsection 1.1, below), but we do not require it for our result.

Remark 1.6 (Dimension-dependent rates in the full-space problem). Because we work on the full space, there is no spectral gap for the linearized evolution and the algebraic rates are optimal; see Subsection 1.2. Unlike the generic exponential rates that one obtains on compact domains, the algebraic decay rates in (1.10), (1.13), and (1.14) are dimension-dependent (see also Proposition 2.3) and in this sense reflect **the structure of the evolution**.

Remark 1.7 (Flat initial data). The L^1 bound (1.9) reflects the linear behavior when

$$\mathcal{E}_0 \lesssim \mathcal{V}_0^{\frac{2}{3}}, \tag{1.15}$$

in the sense that the nonlinear system obeys the same bound $\mathcal{V} \lesssim \mathcal{V}_0$ satisfied by the linearization (see the discussion in Subsection 1.3.2 below). The condition (1.15) is a properly nondimensional way of imposing approximate flatness of the initial configuration. It does so only on average; it does not impose a graph structure, let alone of uniformly small slope.

Remark 1.8 (The dimensionless quantity). It is critical for our result that there exists a nondimensional combination of $\mathcal{E}$ and D whose smallness suffices to control the Lipschitz norm of the function h. In dimension $d \geq 3$, the slope cannot be controlled in terms of $\mathcal{E}$ and D alone (the quantity D^3 *just fails* in $d = 3$), and one would need a different approach to obtain Lipschitz control. The exciting observation of our paper in ambient dimension three ($d = 2$) is that the elementary but critical fact that the dimensionless quantity $\mathcal{E}D^2$ becomes small (cf. Lemma 1.10) can be leveraged to establish **generation and propagation of Lipschitz control** (cf. Section 4). In $d = 1$, as already exploited in [9], smallness of the dimensionless quantity $\mathcal{E}^2D$ is **propagated** and serves to maintain Lipschitz control.

Remark 1.9 ($d = 1$). Proposition 2.3 establishes that also in $d = 1$, the solution for appropriate initial data within the graph setting remains in the graph setting and decays optimally. Additionally, Proposition 2.1 in $d = 1$ shows that the excess mass remains bounded in terms of the initial values as expected (at least for a time quantified in terms of the initial excess energy). Hence “all” that is missing for $d = 1$ is the analogue of Proposition 2.2.

However, we do not believe that the analogue of Theorem 1.3 holds for $d = 1$. Indeed a heuristic calculation for a flat configuration perturbed only by a round island with radius R at distance L

from the interface has dissipation of order $(R^2 \log(L/R))^{-1}$. In particular, the dimensionless quantity scales like $\mathcal{E}^2 D \sim (\log(L/R))^{-1}$, so that the bound $\mathcal{E}^2 D \ll 1$ does not exclude islands if $R \ll L$ and there can be no analogue of Proposition 2.2 assuring graph structure based on this smallness condition. In addition, one cannot hope for a relaxation rate of the energy as in (1.10), since the normal velocity at the island can be made arbitrarily small by increasing L (which changes neither $\mathcal{E}_0$ nor $\mathcal{V}_0$).

1.1 | Review of literature

In the literature, the terms MS and Hele–Shaw are sometimes used interchangeably, but we will use the convention from [10]. The MS and Hele–Shaw evolutions are related: Their one-phase formulations coincide, while in the two-phase model, the Hele–Shaw model has normal velocity given by the (continuous over the boundary Γ) pressure gradient

$$V = \nabla q \cdot n,$$

where the pressure is harmonic outside Γ and has jump given by the mean curvature on the surface

$$\begin{cases} \Delta q = 0 & \text{in } \mathbb{R}^{d+1} \setminus \Gamma, \\ [q] = H & \text{on } \Gamma. \end{cases}$$

We refer to [10] for details.

We concentrate here on literature about the two-phase MS evolution and do not mention results in neighboring areas like the one-phase MS evolution, the mean curvature flow, and the Stefan problem with Gibbs–Thomson boundary condition. We note but do not comment additionally on the rich connection to the Cahn–Hilliard equation as a diffuse interface approximation of MS and corresponding convergence results. For an excellent, broad overview, we refer to the introduction of [15].

1.1.1 | Existence

Most of the available existence results for MS are on bounded domains. We comment separately on short-time and long-time results.

Short-time existence results for MS on bounded domains go back to [7] for weak solutions in $d + 1 = 2$ dimensions, and to [8, 12] for strong solutions in arbitrary dimension. As mentioned above, starting from non-connected initial data, components can merge or vanish. Even worse: starting from connected initial data, the evolution can produce non-connected phases. Thus the evolution is expected to be *non-smooth* at least at finitely many points in time, which poses a natural restriction on the existence time of strong solutions. Very recently, [11] proves short-time existence of strong solutions in the whole space in the graph setting in $d + 1 = 2$. This is the only existence result on unbounded domains of which we are aware.

Even though non-smooth events are expected during the evolution and need to be incorporated in a weak solution concept, the important properties of surface reduction and (signed) mass conservation are maintained across these events. Long-time existence of weak solutions via approximation by minimizing movements is proved in the BV setting in [19] with an additional assumption on the convergence of the energy. Building on results by Schätzle [26], the existence of varifold solutions that "contain" BV solutions is proved in [25] without the additional assumption. The recent work [15] provides long-time existence of weak gradient flow solutions that satisfy a De Giorgi type inequality for energy and dissipation; in particular an inequality corresponding to (1.7) (with factor $\frac{1}{2}$) holds true. These solutions satisfy a weak-strong uniqueness principle due to [14].

A combination of [11, 15], and [14] is our justification for Hypothesis **(H)**: Due to [15] it seems reasonable that weak solutions satisfying the correct dissipation behavior also exist on the whole space. An extension of [11] to $d + 1 = 3$ dimensions seems possible albeit challenging; in this case the long-time existence of a strong solution would be guaranteed by our uniform control of the dissipation, and weak-strong uniqueness would guarantee that a solution remains smooth once it has become regular.

1.1.2 | Ostwald ripening

Ostwald ripening consists of the coarsening of initial configurations in which many small, nearly spherical islands of one of the phases are present. The larger islands grow at the expense of the smaller ones, which eventually disappear. During this process the number of connected components decreases and the average size of islands grows until only the largest component (in our case unbounded) remains. We refer to [2, 21] and the references mentioned therein for a detailed analysis of this stage of the evolution.

Our result includes the regime of Ostwald ripening: One can consider initial configurations in which the unbounded components of positive and/or negative phase are complemented by spherical or nearly spherical islands of the opposite phase—or initial data that flows into such a regime.

1.1.3 | Relaxation rates

We are interested in capturing and quantifying convergence to the longtime limit. On bounded domains, convergence to the groundstate is exponential (see Remark 1.6) since the linearization exhibits a spectral gap. This is made rigorous in [7, 13] where exponential convergence is proved (in $d + 1 = 2$ and $d + 1 \geq 2$, respectively) for initial data that are close to a ball in a strong sense. In [18] exponential convergence to a collection of equally sized discs is proved for a so-called flat flow solution of MS on the torus of dimension $d + 1 = 2$. They consider a large class of initial data and their method relies on a quantitative Alexandrov Soap Bubble Theorem. An existence and stability analysis for an interface with $\frac{\pi}{2}$ contact angle at the boundary of the domain is conducted in [1, 17] (in $d + 1 = 2,\ 3$ and $d + 1 = 2$, respectively), including exponential stability for minimizers in appropriate situations.

Regarding convergence rates for the whole space problem, where no spectral gap is available, a relaxation result in $d + 1 = 2$ is established in [9] in terms of the energy gap, the dissipation, and the squared $\dot{H}^{-1}$ distance

$$\mathcal{H} := \|\chi(0)\|^2_{\dot{H}^{-1}}$$

associated to the gradient flow. The assumptions on the initial data are $\mathcal{E}_0^2 D_0 \ll 1$, that Γ_0 is the graph of a function $h(0)$ with $\|h_x(0)\|_\infty \leq 1$, and

$$\mathcal{H}_0 := \|\chi(0)\|^2_{\dot{H}^{-1}} < \infty$$

(rather than $\mathcal{V}_0 < \infty$). Exploiting a method introduced in [24], the method makes use of

$$\mathcal{E} \lesssim \mathcal{H}^{\frac{1}{2}} D^{\frac{1}{2}}$$

to obtain the decay rates

$$\mathcal{E} \lesssim \frac{\mathcal{H}_0}{t}, \quad D \lesssim \frac{\mathcal{H}_0}{t^2}, \tag{1.16}$$

which are the same algebraic decay rates as pointed out by Brezis [6] for a gradient flow with respect to a *convex energy*. A corollary is the control

$$\|h\|_\infty + t^{\frac{1}{3}} \|\nabla h\|_\infty \lesssim \frac{\mathcal{H}_0}{t^{\frac{1}{3}}}.$$

Although the t^{-1} decay rate of the energy from (1.16) is the same algebraic rate that we will obtain for $d = 1$ in Proposition 2.3 below, the L^1 approach yields a stronger result for compactly supported perturbations, since it allows non-neutral perturbations of the halfspace, whereas $\mathcal{H}_0 < \infty$ implies $\int_{\mathbb{R}^d} h_0 \, \mathrm{d}x = 0$.

In $d = 2$, our L^1 result picks up the faster dimension-dependent decay, while the method of [9] is blind to the dimension.

The most important distinction to be drawn between [9] and the present work is that here, for $d = 2$, we show that the graph structure and Lipschitz bound on the height are *generated by the dynamics*, whereas, in [9] this is an assumption on the initial data. We remark that convergence rate results for MS for non-perturbative, non-graph initial data are extremely rare; to the best of our knowledge, there is only [18] for flat flow solutions in two dimensions (without resolving dependence of the rate on the initial data) and the present work.

The L^1 method used here was previously introduced to establish relaxation results for the one-dimensional Cahn–Hilliard equation [5, 23], and is shown here to be a useful tool for obtaining sharp relaxation rates for geometric evolutions.

1.2 | Discussion of optimality

The rates in (1.14) are optimal: It is exactly these rates that hold for the geometrically linearized problem. In order to see this, consider

$$-\Delta f = 0 \qquad \text{in } \mathbb{R}^d \times \{z > 0\}, \tag{1.17}$$

$$f = \Delta h \qquad \text{on } \mathbb{R}^d \times \{z = 0\}, \tag{1.18}$$

$$h_t = -2 f_z \qquad \text{on } \mathbb{R}^d \times \{z = 0\}, \tag{1.19}$$

$$h(0) = h_0 \qquad \text{on } \mathbb{R}^d \times \{z = 0\}, \tag{1.20}$$

where the flat geometry is fixed and decoupled from h. The boundary values for f are imposed by h in form of the linearized expression for the mean curvature (cf. 5.2) and the jump of the normal derivative of f at the flat interface determines the linearization of the normal velocity, which is just h_t (cf. 5.3).

In order to solve problem (1.17)–(1.20), we Fourier transform (1.17) and (1.18) in the x variable and deduce:

$$-\partial_{zz}^2 \hat{f}(k,z) = -|k|^2 \hat{f}(k,z),$$

$$\hat{f}(k,0) = -|k|^2 \hat{h}(k),$$

where k is the tangential wavenumber and $\hat{f}$ and $\hat{h}$ are the Fourier-transform of f and h in the tangential direction, respectively. Combining this with the growth condition on $\hat{f}(k,z)$ from $\|\nabla f\|_{L^2(\mathbb{R}^{d+1}\setminus\{z=0\})} < \infty$, we obtain

$$\widehat{f}(k,z) = -\exp(-|k|\,z)|k|^2 \widehat{h}(k).$$

Finally, using (1.19), we arrive at the closed equation

$$\partial_t \hat{h} + 2|k|^3 \hat{h} = 0. \tag{1.21}$$

By the above computation we see that the Dirichlet-to-Neumann map $A : f \mapsto -f_z$ on $\mathbb{R}^d$ is represented by the multiplier $|k|$ in Fourier-space which is why we denote $|\nabla| := A$. With this notation (1.21) can be written in physical space as

$$\partial_t h - 2|\nabla|\Delta h = 0, \tag{1.22}$$

and the Fourier symbol of $-2|\nabla|\Delta$ is exactly given by $2|k|^3$ in the tangential wave number k. From the Fourier representation one obtains

$$\hat{h}(t) = \hat{G}(t)\hat{h}_0,$$

with $\hat{G}(t,k) = \exp(-2\,|k|^3\, t)$. Thus, h itself is given by

$$h(t) = G(t) * h_0,$$

where

$$G(t,x) = \frac{1}{t^{\frac{d}{3}}} \tilde{G}\left(\frac{x}{t^{\frac{1}{3}}}\right) \tag{1.23}$$

for some profile $\tilde{G} = G(1)$. In particular

$$\|\hat{G}(t)\|_1 \lesssim t^{-\frac{d}{3}}, \qquad \|\,|k|\hat{G}(t)\|_1 \lesssim t^{-\frac{d+1}{3}},$$

which together with the estimates

$$\|h\|_\infty \le \|G\|_\infty \|h_0\|_1 \lesssim \|\hat{G}\|_1 \|h_0\|_1,$$

$$\|\nabla h\|_\infty \le \|\nabla G\|_\infty \|h_0\|_1 \lesssim \|\,|k|\hat{G}\|_1 \|h_0\|_1,$$

yields

$$\|h\|_\infty + t^{\frac{1}{3}}\|\nabla h\|_\infty \lesssim t^{-\frac{d}{3}} \int_{\mathbb{R}^d} |h_0|\, \mathrm{d}x.$$

Hence, initial data in L^1 lead to an L^∞ decay of $t^{-\frac{d}{3}}$, which is precisely the decay captured in (1.14).

1.3 | Method

There are two things to explain: Where does the graph structure come from and how are the relaxation rates obtained?

1.3.1 | Graph structure

We begin with the former. It is an elementary observation that

Lemma 1.10. *For every ε there exists $T_g \le \frac{2}{3\sqrt{\varepsilon}}\mathcal{E}_0^{\frac{3}{2}}$ such that*

$$\big(\mathcal{E}D^2\big)(T_g) \le \varepsilon. \tag{1.24}$$

Proof. The proof follows using $T = \frac{2}{3\sqrt{\varepsilon}}\mathcal{E}_0^{\frac{3}{2}}$ and the inequality almost everywhere in time:

$$(\mathcal{E}D^2)^{\frac{1}{2}} = \mathcal{E}^{\frac{1}{2}}D \overset{(1.6)}{\le} -\frac{2}{3}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^{\frac{3}{2}}. \tag{1.25}$$

Then since $\mathcal{E}$ and hence $\mathcal{E}^{\frac{3}{2}}$ is nonincreasing, we deduce

$$\inf_{t\le T}\big(\mathcal{E}D^2\big)^{\frac{1}{2}} \le \frac{1}{T}\int_0^T \big(\mathcal{E}D^2\big)^{1/2}\,\mathrm{d}t \overset{(1.25)}{\le} \frac{2}{3T}\mathcal{E}_0^{\frac{3}{2}} = \varepsilon^{\frac{1}{2}}. \qquad \square$$

For ε sufficiently small, $\mathcal{E}D^2 \le \varepsilon$ implies graph structure of Γ and the Lipschitz control

$$\|\nabla h\|_\infty \lesssim (\mathcal{E}D^2)^{\frac{1}{6}}, \tag{1.26}$$

which we show using on the one hand results of Meyers and Ziemer [20, 29] and Schätzle [26] to establish local control on the L^p norm of the mean curvature H, and on the other hand Allard's regularity theory [3] to convert L^p control of the curvature into graph structure and Lipschitz continuity with respect to the plane $\{z = 0\}$. An interpolation between small scale control of H via D and large scale control of the comparison plane via $\mathcal{E}$ yields exactly (1.26).

Based on (1.26), we will distinguish between an initial layer in time, where boundedness of the energy will suffice, and the small slope regime

$$\|\nabla h\|_\infty \ll 1, \tag{1.27}$$

where decay of energy and dissipation will be established and exploited. In the regime (1.27), we leverage the control of D, $\mathcal{E}D^2$ to deduce

$$\|\nabla^2 h\|_p \lesssim \mathcal{E}^{\frac{4-p}{6p}} D^{\frac{2(p-1)}{3p}} \tag{1.28}$$

for a suitable $p \in (2,4)$. This we derive by appealing to a trace-Sobolev estimate and a Meyers-type perturbative argument.

We then consider the dissipation of the dissipation; based on (1.28), we establish the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} D \lesssim D^4.$$

Together with (1.6), this immediately implies

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{E}D^2) \le 0 \qquad \text{for } t \ge T_g, \tag{1.29}$$

so that $\mathcal{E}D^2 \ll 1$ and hence also graph structure and the small slope regime are preserved.

1.3.2 | Relaxation rates via Nash and L^1 control

As far as the relaxation in time, the core idea is to use a Nash-type estimate controlling $\mathcal{E}$ in terms of the L^1-distance $\mathcal{V}$ and the dissipation D:

$$\mathcal{E} \lesssim \mathcal{V}^{\frac{6}{7}} D^{\frac{4}{7}},$$

which is not hard to show. If $\mathcal{V}$ remains bounded, then this algebraic relation combined with the differential equation (1.6) yields a differential inequality for the excess energy, which implies (1.10). Hence our main mathematical contribution is a duality argument that shows that $\mathcal{V}$ does not increase too much:

$$\mathcal{V} \lesssim \mathcal{V}_0 + \mathcal{E}_0^{\frac{3}{2}}. \tag{1.30}$$

One can view this result in the following way: Although the evolution is *not initially in a perturbative regime* and thus behaves genuinely nonlinearly, the gradient flow structure imbues the problem with an inherent relaxation towards the linear regime. We show with the duality argument that one can leverage this relaxation to capitalize on the linearization *even in the initial nonlinear stage*—essentially because the evolution *does not move away from the linear regime that much*.

In both the initial layer and the small slope regime, we employ a duality argument inspired by [22] and used previously in the L^1-method developed and employed in [5, 23]. The earliest use of this duality method may be the adjoint method of Evans used by Evans, Tran, and others in the context of Hamilton–Jacobi equations; see [16] and the citing references.

The duality argument is a nonlinear generalization of a duality argument for the linearization (1.22), which we will explain in the remainder of this subsection. The main task for the duality arguments of our paper will hence be to estimate the linearization errors in the right way.

But for now, let us look at the linear case. To see that (1.30) is true for the linear problem in any dimension, that is, that the L^1-norm of h is controlled in terms of the initial data, it is enough to use

$$\|h\|_1 \leq \|G\|_1 \|h_0\|_1$$

in combination with the fact that G is bounded in L^1 uniformly in t. This in turn follows from (1.23) and the L^1 control of the mask $\tilde{G}$, which is smooth and decays with rate

$$|x|^{-d-1} \quad \text{for large } x. \tag{1.31}$$

The decay can be derived via integration by parts as in

$$\begin{aligned}(2\pi)^{\frac{d}{2}} |x|^{d+1} \tilde{G}(x) &= \int_{\mathbb{R}^d} |x|^{d+1} e^{ix\cdot k} e^{-2|k|^3}\,\mathrm{d}k \\ &= -6i \int_{\mathbb{R}^d} |x|^{d-1} e^{ix\cdot k} x\cdot k|k| e^{-2|k|^3}\,\mathrm{d}k,\end{aligned}$$

repeating d more times and observing that $|k|^3$ is $(d+1)$-times weakly differentiable.

This approach is not well-suited for generalization to nonlinear equations. Hence, we turn instead to the dual characterization of the L^1 norm via

$$\|h(T)\|_1 = \int_{\mathbb{R}^d} |h(T)|\,\mathrm{d}x = \sup_{\psi\in L^\infty(\mathbb{R}^d), \|\psi\|_\infty \leq 1} \int_{\mathbb{R}^d} h(T)\psi\,\mathrm{d}x$$

and the dual or adjoint equation with terminal data ψ:

$$\begin{aligned}\partial_t u + 2|\nabla|\Delta u = 0 \quad &\text{on } [0,T)\times\mathbb{R}^d, \\ u(T) = \psi \quad &\text{on } \mathbb{R}^d,\end{aligned}$$

for which solutions exist, which can be seen by applying the kernel G backwards in time. For reference below, the property $\|G\|_1 < \infty$ yields

$$\|u(t=0)\|_\infty \lesssim \|\psi\|_\infty. \tag{1.32}$$

The idea is now to introduce the harmonic extension $\bar{u}$ of u to $\mathbb{R}^d \times \{z > 0\}$, which satisfies

$$-\partial_t \bar{u} + 2\partial_z \Delta \bar{u} = 0 \quad \text{on } (0,T)\times\mathbb{R}^d, \tag{1.33}$$

$$-\boldsymbol{\Delta} \bar{u} = 0 \quad \text{on } (0,T)\times\mathbb{R}^d \times \{z>0\}, \tag{1.34}$$

$$\bar{u}(T) = \psi \quad \text{on } \mathbb{R}^d, \tag{1.35}$$

and analogously for the harmonic extension $\bar{h}$ of the function h satisfying (1.22).

Using (1.33) and the corresponding equation for $\bar{h}$, we calculate

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} uh\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} \bar{u}\bar{h}\,\mathrm{d}x &= 2\int_{\mathbb{R}^d} (\partial_z \Delta \bar{u})\bar{h} - \bar{u}(\partial_z \Delta \bar{h})\,\mathrm{d}x \\ &= 2\int_{\mathbb{R}^d} \partial_z \bar{u} \Delta \bar{h} - \bar{u}\partial_z \Delta \bar{h}\,\mathrm{d}x\end{aligned}$$

$$\overset{(1.34)}{=} -2\int_{(z>0)} \boldsymbol{\nabla}\bar{u}\cdot\boldsymbol{\nabla}\Delta\bar{h} - \boldsymbol{\nabla}\bar{u}\cdot\boldsymbol{\nabla}\Delta\bar{h}\,\mathrm{d}x$$

$$= 0, \tag{1.36}$$

and deduce

$$\int_{\mathbb{R}^d} h(T)\psi\,\mathrm{d}x = \int_{\mathbb{R}^d} h_0 u(0)\,\mathrm{d}x \overset{(1.32)}{\lesssim} \|h_0\|_1\|\psi\|_\infty.$$

Taking the supremum over all ψ with $\|\psi\|_\infty \le 1$ gives $\|h\|_1 \lesssim \|h_0\|_1$. Applying this argument to a nonlinear perturbation of (1.22) leads to a nontrivial right-hand side in (1.36), for which suitable estimates are required. The advantage of the method is that it isolates and quantifies the nonlinear behavior, since the linear contributions cancel. This is the approach that we will use to establish L^1 control.

1.4 | Notation and organization

Notation 1.11 (Time-dependent functionals). We explicitly denote the t-dependence of the quantities Γ, $\mathcal{E}$, D, $\mathcal{V}$ only when we occasionally want to draw attention to it. We denote the initial values with index 0:

$$\Gamma_0, \quad \mathcal{E}_0, \quad D_0, \quad \mathcal{V}_0.$$

Notation 1.12 (Gradients and such). We use regular and boldface font to distinguish between d-dimensional and $(d+1)$-dimensional objects. We use $\mathrm{d}x$ to note integration with respect to the d-dimensional Lebesgue measure over $x \in \mathbb{R}^d$ and ∇, Δ to denote the gradient and Laplacian with respect to x; for the $(d+1)$-dimensional measure, gradient, and Laplacian, we use $\mathrm{d}\mathbf{x}$, $\boldsymbol{\nabla}$, and $\boldsymbol{\Delta}$:

$$\mathrm{d}\mathbf{x} = \mathrm{d}x\,dz, \qquad \boldsymbol{\nabla} f = \begin{pmatrix} \nabla f \\ \partial_z f \end{pmatrix}, \qquad \boldsymbol{\Delta} := \Delta + \partial_{zz}.$$

We refer to x as the tangential variable.

Notation 1.13 (Norms). Our convention is to denote norms on the interface explicitly and suppress the domain on $\mathbb{R}^d$, for example,

$$\|g\|_{L^p(\Gamma)}, \qquad \|g\|_p = \|g\|_{L^p(\mathbb{R}^d)},$$

except when $\mathbb{R}^d$ is made explicit for emphasis.

Notation 1.14 (Jump). For a quantity $q : \mathbb{R}^d \to \mathbb{R}$ that is defined on both Ω_+ and Ω_- and has a well-defined trace q_+ and q_- on Γ from each side, respectively, we denote the jump by

$$[q] := q_+ - q_-.$$

Notation 1.15 (Minimum). We occasionally use the notation

$$A \wedge B := \min\{A, B\}.$$

Notation 1.16 (Order). For $B \geq 0$, we use the notation

$$A \lesssim B$$

if there exists a universal constant $C \in (0, \infty)$ depending at most on the dimension d, such that $A \leq CB$. If for $A, B \geq 0$ there holds $A \lesssim B$ and $B \lesssim A$, we write $A \sim B$.

We say

$$A \lesssim B \quad \text{implies} \quad E \lesssim F$$

if for every $C_1 < \infty$ there exists $C_2 < \infty$ such that

$$A \leq C_1 B \quad \text{implies} \quad E \leq C_2 F,$$

and analogously for statements involving $A \sim B$ or $E \sim F$.

Notation 1.17. When we say let $\mathcal{V}$ and $\mathcal{E}$ be finite, we mean that there exists a set Ω_+ of locally finite perimeter with reduced boundary Γ and that the associated excess mass and excess energy are finite.

1.4.1 | Organization

The rest of the paper is organized as follows. Section 2 announces the three central propositions (which will be proved in Sections 3, 4, and 5) and establishes the main theorem based on these results. Section 3 establishes control of the excess mass in the initial layer. Section 4 proves that the graph regime and Lipschitz control are reached by time T_g. Finally, Section 5 establishes control and decay for later times $t > T_g$. A few auxiliary results are gathered in the appendix.

2 | CENTRAL PROPOSITIONS AND PROOF OF MAIN THEOREM

As described in Subsection 1.3, in the initial layer we will use merely $\mathcal{E} \leq \mathcal{E}_0$ but need an argument to control $\mathcal{V}$; the result is recorded in Proposition 2.1.

Proposition 2.1. *Let $d = 2$ or $d = 1$. Within the setting of Theorem 1.3 and for any $T \lesssim \mathcal{E}_0^{\frac{3}{d}}$, there holds*

$$\sup_{0 \leq t \leq T} \mathcal{V}(t) \lesssim \mathcal{V}_0 + \mathcal{E}_0^{\frac{d+1}{d}}. \tag{2.1}$$

The next main ingredient in $d = 2$ is to show that a graph structure is achieved once $\mathcal{E}D^2$ is small enough.

Proposition 2.2. *Let $d = 2$. Let Γ be the reduced boundary of a set Ω_+ of locally finite perimeter and suppose finite excess energy $\mathcal{E}$ and dissipation D. There exists $\varepsilon_1 > 0$ such that if $\mathcal{E}D^2 \leq \varepsilon_1$, then there exists a Lipschitz function $h : \mathbb{R}^2 \to \mathbb{R}$ such that*

$$\Gamma = \{(x,z) : x \in \mathbb{R}^2,\ z = h(x)\} \qquad \textit{and} \qquad \|\nabla h\|_\infty \lesssim (\mathcal{E}D^2)^{\frac{1}{6}}. \tag{2.2}$$

We will use Lemma 1.10 with an ε small enough so that Proposition 2.2 identifies a time $T_g \lesssim \mathcal{E}_0^{3/2}$ such that graph structure is achieved. In addition, as described in Subsection 1.3, we will choose ε sufficiently small so that (1.29) and the control from (2.2) locks the dynamics within the small slope regime. The next and final proposition encapsulates relaxation to flat for graphs with small Lipschitz norm. We use this in Theorem 1.3 in $d = 2$ to establish relaxation once the evolution has entered the graph setting; in $d = 1$, the proposition says that for initial data as given, the relaxation rates hold. We state it as a "stand-alone result" since it already gives a stronger relaxation result for the graph setting in $d = 1$ than has previously been established (see Subsection 1.1).

Proposition 2.3. *Let $d = 2$ or $d = 1$. There exists $\varepsilon_2 > 0$ with the following property. Consider the the MS dynamics under the hypothesis **(H)** in the graph setting, that is, where Γ_0 is the graph of a function $h_0 : \mathbb{R}^d \to \mathbb{R}$. If $\mathcal{E}_0^{3-d}D_0^d \leq \varepsilon_2$ and $\|\nabla h_0\|_\infty \leq 1$, then Γ remains a graph satisfying $\|\nabla h\|_\infty \leq 1$ and*

$$\mathcal{V} \lesssim \mathcal{V}_0 + \mathcal{E}_0^{\frac{d+1}{d}}, \tag{2.3}$$

$$\mathcal{E} \lesssim \min\left\{\mathcal{E}_0, \frac{\mathcal{V}_0^2 + \mathcal{E}_0^{\frac{2(d+1)}{d}}}{t^{\frac{d+2}{3}}}\right\} \tag{2.4}$$

hold for all future times. In addition there exists $T_{diss} \sim \mathcal{E}_0^{\frac{3}{d}}$ such that

$$D(t) \lesssim \frac{\mathcal{E}\left(\frac{t}{2}\right)}{t}, \tag{2.5}$$

$$\|h\|_\infty + t^{\frac{1}{3}}\|\nabla h\|_\infty \lesssim \frac{\mathcal{V}_0 + \mathcal{E}_0^{\frac{d+1}{d}}}{t^{\frac{d}{3}}} \tag{2.6}$$

holds for all $t \geq T_{diss}$.

Proof of Theorem 1.3. We fix $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$ for the constants ε_1 and ε_2 from Propositions 2.2 and 2.3. If necessary we reduce ε additionally so that for the implicit constant on the right-hand side of (2.2), there holds

$$\|\nabla h\|_\infty \leq C(\mathcal{E}D^2)^{\frac{1}{6}} \leq 1.$$

For this $\varepsilon > 0$, we apply Lemma 1.10 to define $T_g \lesssim \mathcal{E}_0^{3/2}$ so that

$$\mathcal{E}D^2(T_g) \leq \varepsilon.$$

On $[0, T_g]$, we use Proposition 2.1 to control

$$\sup_{0\le t\le T_g} \mathcal{V}(t) \lesssim \mathcal{V}_0 + \mathcal{E}_0^{\frac{3}{2}}. \tag{2.7}$$

By choice of ε, Proposition 2.2 yields that the interface at time T_g is a graph, and for $t \ge T_g$ Proposition 2.3 yields that the excess mass is bounded by

$$\mathcal{V} \lesssim \mathcal{V}(T_g) + \mathcal{E}(T_g)^{\frac{3}{2}} \overset{(1.8)(2.7)}{\lesssim} \mathcal{V}_0 + \mathcal{E}_0^{\frac{3}{2}} \qquad \text{for } t \ge T_g,$$

and the energy decays according to

$$\mathcal{E} \lesssim \min\left\{\mathcal{E}_0, \frac{\mathcal{V}(T_g)^2 + \mathcal{E}(T_g)^3}{(t-T_g)^{\frac{4}{3}}}\right\} \overset{(1.8)(2.7)}{\lesssim} \min\left\{\mathcal{E}_0, \frac{\mathcal{V}_0^2 + \mathcal{E}_0^3}{(t-T_g)^{\frac{4}{3}}}\right\} \quad \text{for } t \ge T_g. \tag{2.8}$$

In view of

$$\frac{\mathcal{E}_0^3}{t^{\frac{4}{3}}} \gtrsim \mathcal{E}_0 \qquad \text{for } t \lesssim \mathcal{E}_0^{\frac{3}{2}},$$

(1.8) and (2.8) combine to give (1.10).

Increasing from T_g to $2(T_g + T_{\text{diss}})$ for T_{diss} from Proposition 2.3, we again argue as for (2.7) to control $\mathcal{V}$ up to this point and then use (2.5)–(2.6) to obtain (1.13)–(1.14). □

3 | CONTROL OF THE INITIAL LAYER

This section is devoted to the proof of Proposition 2.1. Throughout the section we will let $d \in \{1, 2\}$ and χ be as defined in (1.1).

We represent $\mathcal{V}$ via duality in the form

$$\mathcal{V} = \sup_{\psi\in L^\infty(\mathbb{R}^{d+1}), \|\psi\|_\infty\le 1} \int_{\mathbb{R}^{d+1}} \psi\chi \, d\mathbf{x} = \sup_{\psi\in C_c^\infty(\mathbb{R}^{d+1}), \|\psi\|_\infty\le 1} \int_{\mathbb{R}^{d+1}} \psi\chi \, d\mathbf{x}, \tag{3.1}$$

because it will be convenient below to work with smooth and compactly supported test functions. (The restriction can be justified by passing to the limit in a standard cut-off and mollification argument.) Rather than work with test functions on $\mathbb{R}^{d+1}$, it will be convenient to argue for (2.1) by way of the following modified version of the excess mass

$$\overline{\mathcal{V}} := \sup_{\psi\in L^\infty(\mathbb{R}^{d}), \|\psi\|_\infty\le 1} \int_{\mathbb{R}^{d+1}} \bar{\psi}\chi \, d\mathbf{x} = \sup_{\psi\in C_c^\infty(\mathbb{R}^{d}), \|\psi\|_\infty\le 1} \int_{\mathbb{R}^{d+1}} \bar{\psi}\chi \, d\mathbf{x}, \tag{3.2}$$

where $\bar{\psi}$ is the harmonic extension of ψ (and the second equality is justified as in (3.1)). Control of $\overline{\mathcal{V}}$ will deliver control of $\mathcal{V}$ using

Lemma 3.1. *If $\mathcal{V}, \mathcal{E} < \infty$, then*

$$\mathcal{V} \lesssim \overline{\mathcal{V}} + \mathcal{E}^{\frac{d+1}{d}}. \tag{3.3}$$

As explained in Section 1.3, the idea to bound $\overline{\mathcal{V}}$ is to introduce the adjoint harmonic extension $\bar{u}$ of ψ using (1.33)–(1.35) and to estimate $\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^{d+1}} \chi\bar{u}\,\mathrm{d}\mathbf{x}$. In the following auxiliary lemma we split the error and announce the corresponding estimates. We denote by $\tilde{u}$ and $\tilde{v}$ the *constant extensions in the z-direction* of u and $v = -|\nabla|\,u$ to $\mathbb{R}^{d+1}$.

Lemma 3.2 (Splitting the error and preprocessing). *Let $T > 0$, $\psi \in C_c^\infty(\mathbb{R}^d)$ with $\|\psi\|_\infty \le 1$, and let $\bar{u}$ satisfy (1.33)–(1.35), extended by even reflection to $\mathbb{R}^d \times \{z < 0\}$. There holds*

$$\Big(\int_{\mathbb{R}^{d+1}} \chi\bar{u}\,\mathrm{d}\mathbf{x}\Big)(T) - \Big(\int_{\mathbb{R}^{d+1}} \chi\bar{u}\,\mathrm{d}\mathbf{x}\Big)(0)$$
$$= \int_0^T \Big(2\int_{\mathbb{R}^{d+1}} \chi\tilde{v}\partial_z f\ \mathrm{d}\mathbf{x} - 2\int_\Gamma (1 - e_z\cdot n)(\boldsymbol{\nabla}\tilde{v}\cdot n)\,\mathrm{d}S + \int_{\mathbb{R}^{d+1}} \chi(\partial_t\bar{u} - \partial_t\tilde{u})\,\mathrm{d}\mathbf{x}\Big)\,\mathrm{d}t. \tag{3.4}$$

Moreover, the error terms can be estimated for almost every t as

$$A_1 := \Big|2\int_{\mathbb{R}^{d+1}} \chi\ \tilde{v}\,\partial_z f\ \mathrm{d}\mathbf{x}\Big| \lesssim \frac{\mathcal{V}^{\frac12} D^{\frac12}}{(T-t)^{\frac13}}, \tag{3.5}$$

$$A_2 := \Big|2\int_\Gamma (1 - e_z\cdot n)(\boldsymbol{\nabla}\tilde{v}\cdot n)\,\mathrm{d}S\Big| \lesssim \frac{\mathcal{E}}{(T-t)^{\frac23}}, \tag{3.6}$$

$$A_3 := \Big|\int_{\mathbb{R}^{d+1}} \chi\ (\partial_t\bar{u} - \partial_t\tilde{u})\ \mathrm{d}\mathbf{x}\Big| \lesssim \frac{\mathcal{V}^{\frac12}\mathcal{E}^{\frac12}}{(T-t)^{\frac56}} + \frac{\mathcal{E}}{(T-t)^{\frac23}}. \tag{3.7}$$

For the proof of Proposition 2.1 it will be convenient to use the notation:

$$\mathcal{V}_T := \sup_{t\in[0,T]} \mathcal{V}(t), \qquad \overline{\mathcal{V}}_T := \sup_{t\in[0,T]} \overline{\mathcal{V}}(t). \tag{3.8}$$

Proof of Proposition 2.1. For given $\psi \in C_c^\infty(\mathbb{R}^d)$ with $\|\psi\|_\infty \le 1$ and $T > 0$, let $\bar{u}$ be the solution to (1.33)–(1.35). Using Lemma 3.2 we write

$$\Big(\int_{\mathbb{R}^{d+1}} \bar{\psi}\chi\,\mathrm{d}\mathbf{x}\Big)(T) \le \Big(\int_{\mathbb{R}^{d+1}} \bar{u}\chi\,\mathrm{d}\mathbf{x}\Big)(0) + \int_0^T A_1 + A_2 + A_3\,\mathrm{d}t.$$

Employing the estimates (3.5)–(3.7), taking the supremum over ψ, and using (A.3) in the form $\|\bar{u}\|_\infty \le \|\psi\|_\infty \le 1$, we obtain

$$\overline{\mathcal{V}}(T) \le \mathcal{V}_0 + C\int_0^T (T-t)^{-\frac13}\mathcal{V}^{\frac12}D^{\frac12} + (T-t)^{-\frac23}\mathcal{E} + (T-t)^{-\frac56}\mathcal{V}^{\frac12}\mathcal{E}^{\frac12}\,\mathrm{d}t. \tag{3.9}$$

It remains to estimate the time integral, which we do term by term:

$$\int_0^T (T-t)^{-\frac{1}{3}} \mathcal{V}^{\frac{1}{2}} D^{\frac{1}{2}}\,\mathrm{d}t \lesssim \mathcal{V}_T^{\frac{1}{2}} \left(\int_0^T (T-t)^{-\frac{2}{3}}\,\mathrm{d}t \int_0^T D\,\mathrm{d}t \right)^{\frac{1}{2}} \lesssim \mathcal{V}_T^{\frac{1}{2}} \left(T^{\frac{1}{3}} \mathcal{E}_0 \right)^{\frac{1}{2}},$$
$$\int_0^T (T-t)^{-\frac{2}{3}} \mathcal{E}\,\mathrm{d}t \lesssim \mathcal{E}_0 \int_0^T (T-t)^{-\frac{2}{3}}\,\mathrm{d}t \lesssim T^{\frac{1}{3}} \mathcal{E}_0,$$
$$\int_0^T (T-t)^{-\frac{5}{6}} \mathcal{V}^{\frac{1}{2}} \mathcal{E}^{\frac{1}{2}}\,\mathrm{d}t \lesssim \mathcal{V}_T^{\frac{1}{2}} \mathcal{E}_0^{\frac{1}{2}} \int_0^T (T-t)^{-\frac{5}{6}}\,\mathrm{d}t \lesssim \mathcal{V}_T^{\frac{1}{2}} \mathcal{E}_0^{\frac{1}{2}} T^{\frac{1}{6}}.$$

Inserting the right-hand sides with $T \lesssim \mathcal{E}_0^{\frac{3}{d}}$ into (3.9), using Lemma 3.1, and applying Young's inequality yields

$$\overline{\mathcal{V}}(T) \le \mathcal{V}_0 + \frac{1}{2}\overline{\mathcal{V}}_T + C\mathcal{E}_0^{\frac{d+1}{d}}.$$

Taking the supremum in T, absorbing the second term from the right-hand side in the left-hand side, and again applying Lemma 3.1 yields the result. □

3.1 | Proofs of auxiliary statements

For the proof of Lemma 3.1, we will need one technical lemma, which we state here and prove after establishing Lemma 3.1.

Lemma 3.3. *If $\mathcal{V}, \mathcal{E} < \infty$, then*

$$\int_\Gamma |n'|^2\,\mathrm{d}S \lesssim \mathcal{E}, \tag{3.10}$$

where $n' = n - (n \cdot e_z)e_z$. Furthermore, for $R > 0$, there holds

$$\int_\Gamma (|z| \wedge R)^2\,\mathrm{d}S \lesssim R\mathcal{V} + R^2\mathcal{E}, \tag{3.11}$$

where we recall the notation $|z| \wedge R = \min\{|z|, R\}$.

Proof of Lemma 3.1. It will be useful to introduce the intermediary functional

$$\tilde{\mathcal{V}} := \sup_{\psi \in L^\infty(\mathbb{R}^d), \|\psi\|_\infty \le 1} \int_{\mathbb{R}^{d+1}} \tilde{\psi}\chi\,\mathrm{d}\mathbf{x},$$

where as usual $\tilde{\psi}$ denotes the constant extension of ψ and we again recall the definition of χ from (1.1).

The proof consists of several steps and makes use of a cut-off lengthscale $R > 0$, corresponding to which we define

$$\mathcal{V}_R := \int_{\mathbb{R}^d \times \{|z| \le R\}} |\chi| \, \mathrm{d}\mathbf{x}, \qquad \widetilde{\mathcal{V}}_R = \sup_{\psi \in L^\infty(\mathbb{R}^d), \|\psi\|_\infty \le 1} \int_{\mathbb{R}^d \times \{|z| \le R\}} \tilde{\psi} \chi \, \mathrm{d}\mathbf{x},$$

and analogously for $\overline{\mathcal{V}}_R$ (cf. 3.2).

It suffices to show:

Step 1: There exists $C_0 < \infty$ such that for any $R \ge C_0 \mathcal{V}^{\frac{1}{d+1}}$, there holds

$$\mathcal{V} - \mathcal{V}_R \lesssim \mathcal{E}^{\frac{d+1}{d}}. \tag{3.12}$$

Step 2: For any $R \ge 0$ there holds

$$\mathcal{V}_R - \widetilde{\mathcal{V}}_R \lesssim \mathcal{E}^{\frac{d+1}{d}}. \tag{3.13}$$

Step 3: There exists $C < \infty$ such that for any $R \sim \mathcal{V}^{\frac{1}{d+1}}$, there holds

$$\widetilde{\mathcal{V}}_R - \overline{\mathcal{V}}_R \le \frac{1}{2}\mathcal{V} + C\mathcal{E}^{\frac{d+1}{d}}. \tag{3.14}$$

Indeed, choosing C_0 from Step 1 and $R = C_0 \mathcal{V}^{\frac{1}{d+1}}$, it follows that

$$\mathcal{V} \overset{(3.12)}{\le} \mathcal{V}_R + C\mathcal{E}^{\frac{d+1}{d}} \overset{(3.13)}{\le} \widetilde{\mathcal{V}}_R + C\mathcal{E}^{\frac{d+1}{d}} \overset{(3.14)}{\le} \frac{1}{2}\mathcal{V} + \overline{\mathcal{V}}_R + C\mathcal{E}^{\frac{d+1}{d}} \le \frac{1}{2}\mathcal{V} + \overline{\mathcal{V}} + C\mathcal{E}^{\frac{d+1}{d}},$$

from which we deduce (3.3).

Step 1: Let

$$\chi_R := \chi \, \mathbf{1}_{\mathbb{R}^d \times \{|z| \le R\}}, \tag{3.15}$$

so that $\mathcal{V}_R = \|\chi_R\|_1$. We denote $\Delta\chi = \chi - \chi_R$ and by $\Delta V = \Delta V(R) = \|\Delta\chi\|_1 = \mathcal{V} - \mathcal{V}_R$ the portion of the mass in $\mathbb{R}^d \times \{|z| > R\}$ bounded by Γ and the hyperplanes $\mathbb{R}^d \times \{z = \pm R\}$. Note that ΔV is monotone and by Fubini's theorem absolutely continuous with

$$\frac{\mathrm{d}\Delta V}{\mathrm{d}R} = \int_{\mathbb{R}^d} \chi(x, R) - \chi(x, -R) \, \mathrm{d}x \le 0 \quad \text{for almost every } R > 0.$$

Without loss of generality, we may assume $\mathcal{V} \gg \mathcal{E}^{\frac{d+1}{d}}$ (since otherwise (3.12) holds trivially for any $R \ge 0$).

Letting $\mathcal{E}_R = \int_{\Gamma \cap (\mathbb{R}^d \times \{|z| > R\})} 1 - e_z \cdot n \, \mathrm{d}S$ denote the excess energy of Γ in $\mathbb{R}^d \times \{|z| > R\}$, we apply the divergence theorem on the domain $\left\{(x, z) \in \mathbb{R}^{d+1} : \chi(x, z) \ne 0, |z| > R\right\}$ for the constant function e_z to derive

$$\mathcal{E}_R = \int_{\mathbb{R}^d \times \{|z| > R\}} |\nabla \chi| \, \mathrm{d}\mathbf{x} + \frac{\mathrm{d}\Delta V}{\mathrm{d}R}, \quad \text{for almost every } R > 0. \tag{3.16}$$

On the other hand for almost every $R > 0$ there holds

$$\int_{\mathbb{R}^{d+1}} |\nabla \Delta\chi| \,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^d \times \{|z|>R\}} |\nabla \chi| \,\mathrm{d}\mathbf{x} - \frac{\mathrm{d}\Delta V}{\mathrm{d}R} \overset{(3.16)}{=} \mathcal{E}_R - 2\frac{\mathrm{d}\Delta V}{\mathrm{d}R}.$$

Clearly

$$\mathcal{E}_R - 2\frac{d\Delta V}{dR} \leq \mathcal{E} - 2\frac{d\Delta V}{dR},$$

so that by the isoperimetric inequality for $\Delta\chi$, we have

$$\Delta V \leq C\left(\mathcal{E} - 2\frac{\mathrm{d}\Delta V}{\mathrm{d}R}\right)^{\frac{d+1}{d}}.$$

Rewriting this as

$$-\frac{\mathrm{d}\Delta V}{\mathrm{d}R} \geq \frac{1}{C}\Delta V^{\frac{d}{d+1}} - \frac{1}{2}\mathcal{E},$$

and observing that $\Delta V(0) = \mathcal{V}$, we obtain that as long as $\Delta V \geq (C\mathcal{E})^{\frac{d+1}{d}}$, there holds

$$-\frac{\mathrm{d}\Delta V}{\mathrm{d}R} \geq \frac{1}{2C}\Delta V^{\frac{d}{d+1}}.$$

Integration from 0 to R yields $\mathcal{V}^{\frac{1}{d+1}} - \Delta V^{\frac{1}{d+1}} \geq \frac{1}{2C(d+1)}R$ and hence $R \leq 2C(d+1)\mathcal{V}^{\frac{1}{d+1}}$. By contraposition this implies that for any $R \geq 6C\mathcal{V}^{\frac{1}{d+1}}$, there holds $\Delta V \leq (C\mathcal{E})^{\frac{d+1}{d}} \lesssim \mathcal{E}^{\frac{d+1}{d}}$.

Step 2: The cut-off plays no role in this step (i.e., the argument is the same for any $R \geq 0$), so without loss of generality, we will establish (3.13) for $R = 0$. Integrating out the z-component, we define

$$g(x) := \int_{\mathbb{R}} |\chi|(x,z)\,\mathrm{d}z, \qquad h(x) := \int_{\mathbb{R}} \chi(x,z)\,\mathrm{d}z,$$

so that $\mathcal{V}$ and $\widetilde{\mathcal{V}}$ can be represented as

$$\mathcal{V} = \int_{\mathbb{R}^d} g\,\mathrm{d}x, \qquad \widetilde{\mathcal{V}} = \int_{\mathbb{R}^d} |h|\,\mathrm{d}x$$

and compared by studying the set $F_R := \left\{x \in \mathbb{R}^d \,:\, |x| < R, |h(x)| < g(x)\right\}$.

Naturally $|h| \leq g$. Also because of the structure of χ, there holds

$$\int_{\mathbb{R}} \left|\partial_z \mathbf{1}_{\Omega_+}(x,z)\right| \mathrm{d}z \geq 3 \qquad \text{on the set } F_R.$$

Thus, we have

$$\int_{F_R \times \mathbb{R}} \left|\nabla \mathbf{1}_{\Omega_+}\right| \,\mathrm{d}\mathbf{x} \geq \int_{F_R}\int_{\mathbb{R}} \left|\partial_z \mathbf{1}_{\Omega_+}(x,z)\right| \mathrm{d}z\,\mathrm{d}x \geq 3|F_R|. \tag{3.17}$$

As usual, by using the divergence theorem on a cylindrical domain over F_R, we have

$$\int_{\Gamma\cap(F_R\times\mathbb{R})} e_z \cdot n \, \mathrm{d}S = |F_R|. \tag{3.18}$$

Combining these facts, we deduce

$$\begin{aligned}\mathcal{E} &\geq \int_{\Gamma\cap(F_R\times\mathbb{R})} 1 - e_z \cdot n \, \mathrm{d}S \overset{(3.18)}{=} \int_{F_R\times\mathbb{R}} \left|\nabla \mathbf{1}_{\Omega_+}\right| \mathrm{d}\mathbf{x} - |F_R| \\ &\overset{(3.17)}{\geq} \frac{2}{3}\int_{F_R\times\mathbb{R}} \left|\nabla \mathbf{1}_{\Omega_+}\right| \mathrm{d}\mathbf{x} \overset{(3.17)}{\geq} 2|F_R|. \end{aligned}\tag{3.19}$$

As (3.19) is true independently of R, we conclude that the set $F := \{|h| < g\}$ has finite measure and

$$\mathcal{E} \geq \frac{2}{3}\int_{F\times\mathbb{R}} \left|\nabla \mathbf{1}_{\Omega_+}\right| \mathrm{d}\mathbf{x}. \tag{3.20}$$

For $w = g, h$ we decompose $\nabla w = \nabla^r w + \nabla^s w$, where $\nabla^r w$ is regular with respect to the Lebesgue measure and $\nabla^s w$ denotes the singular part. We will show that

$$\int_F \sqrt{|\nabla^r g|^2 + 1}\, \mathrm{d}x + \int_F |\nabla^s g| \, \mathrm{d}x \lesssim \int_{F\times\mathbb{R}} \left|\nabla \mathbf{1}_{\Omega_+}\right| \mathrm{d}\mathbf{x}, \tag{3.21}$$

$$\int_F \sqrt{|\nabla^r h|^2 + 1}\, \mathrm{d}x + \int_F |\nabla^s h| \, \mathrm{d}x \lesssim \int_{F\times\mathbb{R}} \left|\nabla \mathbf{1}_{\Omega_+}\right| \mathrm{d}\mathbf{x}, \tag{3.22}$$

which together with the isoperimetric inequality on $\{(x,z) : |h|(x) < z < g(x)\}$ implies

$$\begin{aligned}\mathcal{V} - \widetilde{\mathcal{V}} &= \int_{\mathbb{R}^d} (g - |h|)\, \mathrm{d}x \\ &\lesssim \left(\int_F \sqrt{|\nabla^r g|^2 + 1}\, \mathrm{d}x + \int_F |\nabla^s g| + \int_F \sqrt{|\nabla^r h|^2 + 1}\, \mathrm{d}x + \int_F |\nabla^s h| \, \mathrm{d}x\right)^{\frac{d+1}{d}} \\ &\overset{(3.21)(3.22)}{\lesssim} \left(\int_{F\times\mathbb{R}} \left|\nabla \mathbf{1}_{\Omega_+}\right| \mathrm{d}\mathbf{x}\right)^{\frac{d+1}{d}} \\ &\overset{(3.20)}{\lesssim} \mathcal{E}^{\frac{d+1}{d}}. \end{aligned}$$

The proof of (3.21) and (3.22) are the same, and we will show (3.21). Notice that we can write distributionally

$$\nabla g(x) = \int_{\mathbb{R}} \sigma \nabla \mathbf{1}_{\Omega_+} \, \mathrm{d}z,$$

where $\sigma = \pm 1$ on $(-\infty, 0)$ and $(0, \infty)$, respectively and

$$1 = \int_{\mathbb{R}} \partial_z \mathbf{1}_{\Omega_+} \, \mathrm{d}z.$$

Thus, for any $\xi \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^d)$, $\zeta \in C_c^\infty(\mathbb{R}^d;\mathbb{R})$, there holds

$$\int_{\mathbb{R}^d} \xi\cdot\nabla g + \zeta\,\mathrm{d}x = \int_{\mathbb{R}^{d+1}} \left(\sigma\xi\cdot\nabla\mathbf{1}_{\Omega_+} + \zeta\partial_z\mathbf{1}_{\Omega_+}\right)\mathrm{d}\mathbf{x} \le \int_{\mathbb{R}^{d+1}} \sqrt{|\xi|^2+\zeta^2}\left|\boldsymbol{\nabla}\mathbf{1}_{\Omega_+}\right|\mathrm{d}\mathbf{x}.$$

Taking the supremum over ξ,ζ with $|\xi|^2+\zeta^2 \le 1$ yields the bound in (3.21).

Step 3: We recall the definition of χ_R from (3.15) and denote $\Gamma_R = \Gamma\cap\left(\mathbb{R}^d\times\{|z|\le R\}\right)$. We claim that to establish (3.14), it suffices to show

$$\left|\int_{\mathbb{R}^{d+1}} \chi_R(\tilde\psi-\bar\psi)\,\mathrm{d}\mathbf{x}\right| \le \left(\mathcal{V}_R\int_{\Gamma_R}|z||n'|\,\mathrm{d}S\right)^{\frac{1}{2}}. \tag{3.23}$$

(Recall our notation $n' = n-(n\cdot e_z)e_z$.) Indeed, observe that

$$\begin{aligned}\int_{\Gamma_R}|z||n'|\,\mathrm{d}S &= \int_{\Gamma_R}(|z|\wedge R)|n'|\,\mathrm{d}S\\ &\le \left(\int_\Gamma(|z|\wedge R)^2\,\mathrm{d}S\int_\Gamma|n'|^2\,\mathrm{d}S\right)^{\frac{1}{2}}\\ &\overset{(3.10)(3.11)}{\le} R^{\frac{1}{2}}\mathcal{V}^{\frac{1}{2}}\mathcal{E}^{\frac{1}{2}} + R\mathcal{E},\end{aligned}$$

where we used Lemma 3.3. Using $R\sim\mathcal{V}^{\frac{1}{d+1}}$, we arrive at

$$\int_{\Gamma_R}|z||n'|\,\mathrm{d}S \lesssim \mathcal{V}^{\frac{d+2}{2(d+1)}}\mathcal{E}^{\frac{1}{2}} + \mathcal{V}^{\frac{1}{d+1}}\mathcal{E}. \tag{3.24}$$

Inserting (3.24) into (3.23) (and using $\mathcal{V}_R\le\mathcal{V}$) gives

$$\left|\int_{\mathbb{R}^{d+1}} \chi_R(\tilde\psi-\bar\psi)\,\mathrm{d}\mathbf{x}\right| \lesssim \mathcal{V}^{\frac{3d+4}{4(d+1)}}\mathcal{E}^{\frac{1}{4}} + \mathcal{V}^{\frac{d+2}{2(d+1)}}\mathcal{E}^{\frac{1}{2}},$$

so that Young's inequality and considering the supremum over ψ leads to (3.14).

It hence remains only to establish (3.23). The idea is to write $\tilde\psi-\bar\psi = (\mathrm{Id}-\mathcal{P})\psi$, where $\mathcal{P}$ is the convolution with the Poisson kernel

$$P(x,z) = C\frac{|z|}{\left(|x|^2+z^2\right)^{\frac{d+1}{2}}}, \tag{3.25}$$

and then shift $(\mathrm{Id}-\mathcal{P})$ onto χ_R:

$$
\begin{aligned}
\int_{\mathbb{R}^{d+1}} \chi_R(\tilde{\psi}-\bar{\psi})\mathrm{d}\mathbf{x} &= \int_{\mathbb{R}^{d+1}} \chi_R(y,z)\psi(y)\mathrm{d}y\ \mathrm{d}z - \int_{\mathbb{R}^{d+1}} \left(\int_{\mathbb{R}^d} \chi_R(x,z)P(x-y,z)\psi(y)\mathrm{d}y \right) \mathrm{d}x\ \mathrm{d}z \\
&= \int_{\mathbb{R}^{d+1}} \chi_R(y,z)\psi(y)\mathrm{d}y\ \mathrm{d}z - \int_{\mathbb{R}^d} \psi(y) \left(\int_{\mathbb{R}} \underbrace{\int_{\mathbb{R}^d} P(y-x,z)\chi_R(x,z)\,\mathrm{d}x\,\mathrm{d}z}_{=:\chi_{R,z}(y)} \right) \mathrm{d}y \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}^d} \psi(y)\big(\chi_R(y,z)-\chi_{R,z}(y)\big)\mathrm{d}y\,\mathrm{d}z \\
&\le \int_{\mathbb{R}} \int_{\mathbb{R}^d} \big|\chi_R(y,z)-\chi_{R,z}(y)\big|\mathrm{d}y\,\mathrm{d}z.
\end{aligned}
$$

For fixed $M > 0$, using that

$$
\int_{\mathbb{R}^d} P(\eta,z)\mathrm{d}\eta = 1 \text{ for all } z, \tag{3.26}
$$

we estimate

$$
\begin{aligned}
\int_{\mathbb{R}^d} \big|\chi_R(y,z)-\chi_{R,z}(y)\big|\mathrm{d}y &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} P(\eta,z)\chi_R(y,z) - \chi_R(y+\eta,z)\mathrm{d}\eta \right| \mathrm{d}y \\
&\le \int_{\mathbb{R}^d} \int_{\{|\eta|\le M|z|\}} P(\eta,z)|\chi_R(y,z)-\chi_R(y+\eta,z)|\mathrm{d}\eta\mathrm{d}y \\
&\quad + \int_{\mathbb{R}^d} \int_{\{|\eta|> M|z|\}} P(\eta,z)|\chi_R(y,z)-\chi_R(y+\eta,z)|\mathrm{d}\eta\mathrm{d}y \\
&\le \int_{\{|\eta|\le M|z|\}} \left(P(\eta,z) \int_{\mathbb{R}^d} \int_0^1 |\nabla\chi_R(y+s\eta,z)\cdot\eta|\mathrm{d}s\,\mathrm{d}y \right)\mathrm{d}\eta \\
&\quad + 2\int_{\mathbb{R}^d} |\chi_R|(x,z)\,\mathrm{d}x \int_{\{|\eta|> M|z|\}} P(\eta,z)\mathrm{d}\eta \\
&\overset{(3.25)}{\lesssim} \int_{\{|\eta|\le M|z|\}} \int_0^1 \int_{\mathbb{R}^d} |\nabla\chi_R(y+s\eta,z)|\mathrm{d}y\mathrm{d}s|\eta|P(\eta,z)\mathrm{d}\eta \\
&\quad + \int_{\mathbb{R}^d} |\chi_R|(x,z)\,\mathrm{d}x \int_{\{|\eta|> M|z|\}} \frac{|z|}{\left(|\eta|^2+z^2\right)^{\frac{d+1}{2}}}\mathrm{d}\eta \\
&\overset{(3.26)}{\lesssim} M|z| \int_{\mathbb{R}^d} |\nabla\chi_R|(x,z)\,\mathrm{d}x + \frac{1}{M}\int_{\mathbb{R}^d} |\chi_R|(x,z)\,\mathrm{d}x.
\end{aligned}
$$

Integration over z and optimization in M yields (3.23), where we have used

$$
\nabla\chi_R = n'\,|\boldsymbol{\nabla}\chi| \llcorner \left(\mathbb{R}^d \cap \{|z|\le R\}\right). \qquad \square
$$

Proof of Lemma 3.3. Estimate (3.10) follows from

$$|n'|^2 = 1-(e_z\cdot n)^2 = (1+e_z\cdot n)(1-e_z\cdot n) \le 2(1-e_z\cdot n), \tag{3.27}$$

where we used the fact that $|e_z\cdot n|\le 1$. For the proof of (3.11) we compute

$$\begin{aligned}\int_\Gamma |z|e_z\cdot n\,\mathrm{d}S &= \int_\Gamma |z|e_z\cdot n\,\mathrm{d}S - \int_{\mathbb{R}^d\times\{0\}} |z|e_z\cdot e_z\,\mathrm{d}S = -\int_{\mathbb{R}^{d+1}} \chi\,\mathrm{div}\,(|z|e_z)\,\mathrm{d}\mathbf{x}\\ &= \int_{\mathbb{R}^{d+1}} |\chi|\,\mathrm{d}\mathbf{x} = \mathcal{V},\end{aligned}$$

and thus

$$\begin{aligned}\int_\Gamma (|z|\wedge R)^2\,\mathrm{d}S &\le \int_\Gamma (|z|\wedge R)^2 e_z\cdot n\,\mathrm{d}S + \int_\Gamma (|z|\wedge R)^2(1-e_z\cdot n)\,\mathrm{d}S\\ &\le R\int_\Gamma |z|e_z\cdot n\,\mathrm{d}S + R^2\int_\Gamma 1-e_z\cdot n\,\mathrm{d}S\\ &= R\mathcal{V} + R^2\mathcal{E}.\end{aligned}$$

□

Proof of Lemma 3.2. The idea is to use $\bar u$ as a test function in the weak equation (1.4):

$$\begin{aligned}\left(\int_{\mathbb{R}^{d+1}} \chi\bar u\,\mathrm{d}\mathbf{x}\right)(T) - \left(\int_{\mathbb{R}^{d+1}} \chi\bar u\,\mathrm{d}\mathbf{x}\right)(0) &= \int_0^T \left(\int_{\mathbb{R}^{d+1}} \chi\partial_t\bar u\,\mathrm{d}\mathbf{x} - \int_{\mathbb{R}^{d+1}} \nabla f\cdot\nabla\bar u\,\mathrm{d}\mathbf{x}\right)\mathrm{d}t\\ &= \int_0^T \left(\int_{\mathbb{R}^{d+1}} \chi\partial_t\bar u\,\mathrm{d}\mathbf{x} + 2\int_{\mathbb{R}^d} f\partial_z\bar u\,\mathrm{d}x\right)\mathrm{d}t. \end{aligned}\tag{3.28}$$

Because $\bar u$ is not an admissible test function (it is merely continuous across $\mathbb{R}^d\times\{z=0\}$ and does not have compact support in space), this requires an approximation argument. Extending (1.4) to test with functions that are smooth on $\mathbb{R}^d\times\{z<0\}$ and $\mathbb{R}^d\times\{z>0\}$ and continuous across $\mathbb{R}^d\times\{z=0\}$ is straightforward. For the integrability/support, we take a smooth cut-off function $\eta_R\in C_c^\infty(B_{R+1}(0))$ with $\eta_R\equiv 1$ on $B_R(0)$ and $\|\eta_R\|_\infty + \|\nabla\eta_R\|_\infty \lesssim 1$ and test (1.4) by $\varphi_R = \eta_R\bar u$ to obtain

$$\begin{aligned}&\left(\int_{\mathbb{R}^{d+1}} \chi\varphi_R\,\mathrm{d}\mathbf{x}\right)(T) - \left(\int_{\mathbb{R}^{d+1}} \chi\varphi_R\,\mathrm{d}\mathbf{x}\right)(0)\\ &= \int_0^T \left(\int_{\mathbb{R}^{d+1}} \chi\partial_t\varphi_R\,\mathrm{d}\mathbf{x} - \int_{\mathbb{R}^{d+1}} \nabla f\cdot\nabla\varphi_R\,\mathrm{d}\mathbf{x}\right)\mathrm{d}t\\ &= \int_0^T \left(\int_{\mathbb{R}^{d+1}} \chi\eta_R\partial_t\bar u\,\mathrm{d}\mathbf{x} - \int_{\mathbb{R}^{d+1}} \eta_R\nabla f\cdot\nabla\bar u\,\mathrm{d}\mathbf{x} - \int_{\mathbb{R}^{d+1}} \bar u\nabla f\cdot\nabla\eta_R\,\mathrm{d}\mathbf{x}\right)\mathrm{d}t. \end{aligned}\tag{3.29}$$

To pass to the limit in the various terms, we will argue using regularity and decay.

First note that by L^1-continuity (cf. Definition 1.1), χ is in $L^\infty(0,T;L^1(\mathbb{R}^{d+1}))$. We turn now to a closer examination of $u = G*\psi$ and $\bar u(t) = P*G(T-t)*\psi$, where G is the kernel from (1.23) and P is the Poisson kernel (3.25). For the second term on the left-hand side of (3.29), we observe that $\psi\in C_c^\infty(\mathbb{R}^d)$ and $G\in L^1(\mathbb{R}^d)$ for $t<T$ implies $u(0)\in L^\infty(\mathbb{R}^d)$ and hence, by

$\bar{u} = P * u$, also $\bar{u}(0) \in L^\infty(\mathbb{R}^{d+1})$. Similarly for the first term on the right-hand side of (3.29), notice that $|\nabla|\,\Delta\psi \in L^\infty(\mathbb{R}^d)$ and G is uniformly bounded in L^1 whence $|\nabla|\,\Delta u = G(T-t) * |\nabla|\,\Delta\psi$, is in $L^\infty(0,T;L^\infty(\mathbb{R}^d))$. Thus, we argue for and exploit that $\partial_t\bar{u}$, as the harmonic extension of $|\nabla|\,\Delta u$, is in $L^1(0,T;L^\infty(\mathbb{R}^{d+1})) \subseteq L^\infty(0,T;L^\infty(\mathbb{R}^{d+1}))$.

To pass to the limit in the second term on the right-hand side of (3.29), we deduce from $u \in L^2(0,T;H^1(\mathbb{R}^d)) \subseteq L^2(0,T;\dot{H}^{\frac{1}{2}}(\mathbb{R}^d))$ that $\boldsymbol{\nabla}\bar{u} \in L^2(0,T;L^2(\mathbb{R}^{d+1}))$. (For our definition of $\dot{H}^{\frac{1}{2}}$, see (5) below.)

Finally for the last term on the right-hand side of (3.29), we will use the decay of $\bar{u}$:

$$|\bar{u}(\mathbf{x})| \lesssim \frac{1}{|\mathbf{x}|^d} \quad \text{for } |\mathbf{x}| \gg 1, \tag{3.30}$$

cf. Lemma A.1. The decay (3.30) together with the L^2 control of ∇f and L^∞ control on $\nabla\eta_R$ allows one to conclude that the last term on the right-hand side of (3.29) vanishes in the limit and we may pass from (3.29) to (3.28).

As explained above the statement of the lemma, we define $v = -|\nabla|\,u$ on the flat interface $\mathbb{R}^d \times \{z=0\}$, that is, $v = \partial_z\bar{u}$, and introduce $\tilde{v}$ as the constant continuation of v in the z-direction. To establish (3.4), it suffices to argue that for almost every time there holds

$$\int_{\mathbb{R}^d} f\partial_z\bar{u}\,\mathrm{d}x = \int_{\mathbb{R}^{d+1}} \chi\tilde{v}\partial_z f\;\,\mathrm{d}\mathbf{x} - \int_\Gamma (1-e_z\cdot n)(\boldsymbol{\nabla}\tilde{v}\cdot n)\,\mathrm{d}S - \frac{1}{2}\int_{\mathbb{R}^{d+1}} \chi\,\partial_t\tilde{u}\,\mathrm{d}\mathbf{x}. \tag{3.31}$$

We record for reference below that $\boldsymbol{\nabla}(\tilde{v}e_z) = e_z \otimes \boldsymbol{\nabla}\tilde{v}$ and

$$\operatorname{div}(\tilde{v}e_z) = 0 \tag{3.32}$$

and hence also

$$\operatorname{div}_{\tan}(\tilde{v}e_z) := (\mathrm{Id} - n\otimes n) : \boldsymbol{\nabla}(\tilde{v}e_z) = -n\cdot\boldsymbol{\nabla}(\tilde{v}e_z)n = -(e_z\cdot n)(\boldsymbol{\nabla}\tilde{v}\cdot n). \tag{3.33}$$

By the divergence theorem and the distributional definition (1.5) of the mean curvature on Γ, there holds

$$\begin{aligned}\int_{\mathbb{R}^d} f\partial_z\bar{u}\,\mathrm{d}x &= \int_{\mathbb{R}^d} fve_z\cdot e_z\,\mathrm{d}x = -\int_{\mathbb{R}^{d+1}} \mathbf{1}_{\mathbb{R}^d\times\{z>0\}}\operatorname{div}(f\tilde{v}e_z)\,\mathrm{d}\mathbf{x}\\ &= \int_{\mathbb{R}^{d+1}} \chi\operatorname{div}(f\tilde{v}e_z)\,\mathrm{d}\mathbf{x} - \int_\Gamma \operatorname{div}_{\tan}(\tilde{v}e_z)\,\mathrm{d}S.\end{aligned} \tag{3.34}$$

Using (3.32), the first integral on the right-hand side simplifies to

$$\int_{\mathbb{R}^{d+1}} \chi\operatorname{div}(f\tilde{v}e_z)\,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^{d+1}} \chi\tilde{v}\partial_z f\;\,\mathrm{d}\mathbf{x}. \tag{3.35}$$

Substituting (3.35) into (3.34), it suffices for (3.31) to show

$$\int_\Gamma \operatorname{div}_{\tan}(\tilde{v}e_z)\,\mathrm{d}S = \int_\Gamma (1-e_z\cdot n)(\boldsymbol{\nabla}\tilde{v}\cdot n)\,\mathrm{d}S + \frac{1}{2}\int_{\mathbb{R}^{d+1}} \chi\,\partial_t\tilde{u}\,\mathrm{d}\mathbf{x}. \tag{3.36}$$

To this end, we compute

$$\int_\Gamma \operatorname{div}_{\tan}(\tilde{v}e_z)\,\mathrm{d}S \overset{(3.33)}{=} -\int_\Gamma (e_z\cdot n)(\nabla\tilde{v}\cdot n)\,\mathrm{d}S$$
$$= \int_\Gamma (1-e_z\cdot n)(\nabla\tilde{v}\cdot n)\,\mathrm{d}S - \int_\Gamma \nabla\tilde{v}\cdot n\,\mathrm{d}S. \tag{3.37}$$

Finally we transform the second integral on the right-hand side as

$$\int_\Gamma \nabla\tilde{v}\cdot n\,\mathrm{d}S = \int_\Gamma \nabla\tilde{v}\cdot n\,\mathrm{d}S - \int_{\mathbb{R}^d}\nabla\tilde{v}\cdot e_z\,\mathrm{d}S = -\int_{\mathbb{R}^{d+1}}\chi\,\mathrm{div}(\nabla\tilde{v})\,\mathrm{d}\mathbf{x}$$
$$= \int_{\mathbb{R}^{d+1}}\chi|\nabla|\Delta\tilde{u}\,\mathrm{d}\mathbf{x}$$
$$= -\frac{1}{2}\int_{\mathbb{R}^{d+1}}\chi\partial_t\tilde{u}\,\mathrm{d}\mathbf{x}. \tag{3.38}$$

Substituting (3.38) into (3.37) demonstrates (3.36). This concludes the proof of (3.4).

We start by estimating A_1. We estimate

$$A_1 \le \|v\|_\infty\int_{\mathbb{R}^{d+1}}|\chi||\nabla f|\,\mathrm{d}\mathbf{x} \le \|v\|_\infty\left(\int_{\mathbb{R}^{d+1}}|\chi|^2\,\mathrm{d}\mathbf{x}\int_{\mathbb{R}^{d+1}}|\nabla f|^2\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{2}}$$
$$\le \|v\|_\infty\mathcal{V}^{\frac{1}{2}}D^{\frac{1}{2}}.$$

An application of (A.4) yields (3.5). Inserting (A.5) in

$$A_2 \lesssim \|\nabla v\|_\infty\int_\Gamma 1-e_z\cdot n\,\mathrm{d}S \lesssim \|\nabla v\|_\infty\mathcal{E}$$

delivers (3.6).

Finally, we turn to A_3, which we rewrite as

$$A_3 = \left|\int_{\mathbb{R}^{d+1}}\chi(\partial_t\bar{u}-\partial_t\tilde{u})\,\mathrm{d}\mathbf{x}\right| = 2\left|\int_{\mathbb{R}^{d+1}}\chi(\Delta\tilde{v}-\Delta\bar{v})\,\mathrm{d}\mathbf{x}\right|.$$

Using the divergence theorem and $\nabla\tilde{v}(x,0)=\nabla\bar{v}(x,0)=\nabla v(x)$, we reformulate the right-hand integral as

$$\left|\int_{\mathbb{R}^{d+1}}\chi(\Delta\tilde{v}-\Delta\bar{v})\,\mathrm{d}\mathbf{x}\right| = \left|\int_\Gamma n'\cdot\nabla(\tilde{v}-\bar{v})\,\mathrm{d}S\right|, \tag{3.39}$$

and estimate for $R>0$ via

$$\left|\int_\Gamma n'\cdot\nabla(\tilde{v}-\bar{v})\,\mathrm{d}S\right|$$
$$\lesssim \left(R^{-1}\|\nabla v\|_{L^\infty(\mathbb{R}^d)} + \||z|^{-1}\nabla(\tilde{v}-\bar{v})\|_{L^\infty(\mathbb{R}^{d+1})}\right)\int_\Gamma |n'|(|z|\wedge R)\,\mathrm{d}S$$

$$\lesssim \left(R^{-1}\|\nabla v\|_{L^\infty(\mathbb{R}^d)} + \|\partial_z \nabla v\|_{L^\infty(\mathbb{R}^d)}\right) \int_\Gamma |n'|(|z| \wedge R)\,\mathrm{d}S, \tag{3.40}$$

where we have used $\|\,|z|^{-1}\,\nabla(\tilde{v} - \bar{v})\|_\infty \leq \|\partial_z \nabla \bar{v}\|_\infty$ and the maximum principle (cf. Lemma A.2).

Next, using Lemma 3.3, we estimate

$$\int_\Gamma |n'|(|z| \wedge R)\,\mathrm{d}S \leq \left(\int_\Gamma |n'|^2 \int_\Gamma (|z| \wedge R)^2\right)^{\frac{1}{2}} \leq \mathcal{E}^{\frac{1}{2}}\left(R\mathcal{V} + R^2\mathcal{E}\right)^{\frac{1}{2}}. \tag{3.41}$$

We insert (3.41) in (3.40) to obtain

$$\left|\int_\Gamma n' \cdot \nabla'(\tilde{v} - \bar{v})\,\mathrm{d}S\right| \lesssim \left(R^{-\frac{1}{2}}\|\nabla v\|_\infty + R^{\frac{1}{2}}\|\partial_z \nabla v\|_\infty\right)\mathcal{E}^{\frac{1}{2}}(\mathcal{V} + R\mathcal{E})^{\frac{1}{2}},$$

which we in turn substitute into (3.39) to deduce

$$|A_3| \lesssim \left(R^{-\frac{1}{2}}\|\nabla v\|_\infty + R^{\frac{1}{2}}\|\partial_z \nabla v\|_\infty\right)\mathcal{E}^{\frac{1}{2}}(\mathcal{V} + R\mathcal{E})^{\frac{1}{2}}.$$

The choice $R = (T - t)^{\frac{1}{3}}$ and the use of (A.5) and (A.6) finishes the proof of (3.7). □

4 | ENTERING THE GRAPH SETTING

In this section we prove Proposition 2.2; in particular, $d = 2$. We will, in the spirit of Notation 1.12, write balls in dimension 2 in regular font and in dimension 3 boldface, that is, for $x_0 \in \mathbb{R}^2$ and $\mathbf{x}_0 \in \mathbb{R}^3$, we denote

$$B_\rho(x_0) = \left\{x \in \mathbb{R}^2 : |x - x_0| < \rho\right\}, \qquad \mathbf{B}_\rho(\mathbf{x}_0) = \left\{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x} - \mathbf{x}_0| < \rho\right\}.$$

For the proof of Proposition 2.2, we need two ingredients. First, we need a version of Allard's regularity theorem. We define the tilt excess with respect to $\mathbb{R}^2$ as

$$E(\mathbf{x}, \rho) = \rho^{-2}\int_{\mathbf{B}_\rho(\mathbf{x})\cap\Gamma} 1 - (n \cdot e_z)^2\,\mathrm{d}S.$$

We rely on Allard's Regularity Theorem [3, Section 8] in the form given by Simon [27, Theorem 23.1], which can be stated in our BV setting as:

Theorem 4.1 (Allard's Regularity Theorem). *For any $p > 2$ and $\alpha \in (0, 1)$ there are $\varepsilon_\alpha > 0$, $\gamma \in (0, 1)$, and $C < \infty$ with the following property. Let Ω_+ be a set with locally finite perimeter, $\Gamma = \operatorname{supp}\boldsymbol{\nabla}\mathbf{1}_{\Omega_+}$, and the generalized scalar mean curvature H of Γ given by a measurable function on Γ. If for $\mathbf{x} = (x, z) \in \Gamma$ the three conditions*

(i) $\left|\boldsymbol{\nabla}\mathbf{1}_{\Omega_+}\right|(\mathbf{B}_\rho(\mathbf{x})) \leq 2(1 - \alpha)\pi\rho^2$,
(ii) $E(\mathbf{x}, \rho) \leq \varepsilon_\alpha$,
(iii) $\rho^{2\left(1-\frac{2}{p}\right)}\|H\|^2_{L^p(\mathbf{B}_\rho(\mathbf{x})\cap\Gamma)} \leq \varepsilon_\alpha^2$,

hold, then $\mathbf{B}_{\gamma\rho}(\mathbf{x}) \cap \Gamma$ *is the graph of a Lipschitz function* h *over* $B_{\gamma\rho}(x)$ *satisfying the estimate*

$$\|\nabla h\|_{L^\infty(B_{\gamma\rho}(x))} \leq C\left(E(\mathbf{x},\rho)^{\frac{1}{2}} + \rho^{1-\frac{2}{p}}\|H\|_{L^p(\mathbf{B}_\rho(\mathbf{x})\cap\Gamma)}\right). \tag{4.1}$$

Equivalently, one can work with the newer version [28, Theorem 5.5.2] and straightforward modifications of our proof.

To establish (iii) and control the second term on the right-hand side of (4.1), we need L^p control of the mean curvature for $p > 2 = d$, which we obtain in the following form.

Lemma 4.2. *Assume that* $\mathcal{E}D^2 < \infty$. *Then the distributional mean curvature is given by a locally integrable function* $H \in L^4(\Gamma)$ *and*

$$\|H\|_{L^4(\Gamma)} \lesssim \left(1 + (\mathcal{E}D^2)^{\frac{1}{20}}\right)D^{\frac{1}{2}}. \tag{4.2}$$

We will deduce this estimate by combining a trace estimate of Meyers and Ziemer with a monotonicity formula of Schätzle; see Subsection 4.1 below.

Proof of Proposition 2.2. Fix $p = 4$, $\alpha \in (0, \frac{1}{2})$, and $\mathbf{x}_0 = (x_0, z_0) \in \Gamma$. We will check that there exists $\rho > 0$ such that the conditions of Theorem 4.1 hold. Note for reference below that the tilt excess is controlled by the energy as in the proof of Lemma 3.3 via

$$E(\mathbf{x}_0,\rho) = \rho^{-2}\int_{\mathbf{B}_\rho(\mathbf{x}_0)\cap\Gamma} 1 - (e_z\cdot n)^2\,\mathrm{d}S \overset{(3.27)}{\leq} 2\rho^{-2}\mathcal{E}. \tag{4.3}$$

For condition (i) we calculate as for (3.16), using the divergence theorem on the intersection of an infinite cylinder of radius $\rho > 0$ with Γ and $\{z = 0\}$, that

$$\int_{\Gamma\cap\{(x,z)\,:\,|x-x_0|<\rho\}} e_z\cdot n\,\mathrm{d}S = \pi\rho^2. \tag{4.4}$$

Consequently for any $\rho > 0$, by expanding the ball to a cylinder, there holds

$$\begin{aligned}\left|\nabla\mathbf{1}_{\Omega_+}\right|(\mathbf{B}_\rho(\mathbf{x}_0)) &\leq \int_{\Gamma\cap\{(x,z)\,:\,|x-x_0|<\rho\}} 1\,\mathrm{d}S = \int_{\Gamma\cap\{(x,z)\,:\,|x|<\rho\}} e_z\cdot n\,\mathrm{d}S + \mathcal{E}\\ &\overset{(4.4)}{=} \pi\rho^2 + \mathcal{E}.\end{aligned} \tag{4.5}$$

From here we read off that condition (i) holds as long as ρ is large enough so that

$$\rho^{-2}\mathcal{E} \leq (1-2\alpha)\pi, \quad \text{that is,} \quad \rho \geq \sqrt{\frac{\mathcal{E}}{(1-2\alpha)\pi}}. \tag{4.6}$$

At the same time, the estimate (4.3) guarantees condition (ii) as long as

$$\rho^{-2}\mathcal{E} \leq \frac{1}{2}\varepsilon_\alpha, \quad \text{that is,} \quad \rho \geq \sqrt{\frac{2\mathcal{E}}{\varepsilon_\alpha}}. \tag{4.7}$$

Finally, restricting ε_1 from Proposition 2.2 to $\varepsilon_1 \le 1$, Lemma 4.2 yields (iii) for all ρ small enough so that

$$2C\rho D \le \varepsilon_\alpha^2, \tag{4.8}$$

where C is the implicit constant in (4.2). Choosing

$$\rho = \left(\frac{\mathcal{E}}{D}\right)^{\frac{1}{3}} \tag{4.9}$$

and $\mathcal{E}D^2 \le \varepsilon_1$ for $\varepsilon_1 > 0$ small enough delivers (4.6), (4.7), and (4.8).

For this radius ρ, the surface $\Gamma \cap \mathbf{B}_{\gamma\rho}(\mathbf{x}_0)$ is a graph over the disk $B_{\gamma\rho}(x_0)$. Since $\mathbf{x}_0 \in \Gamma$ was arbitrary, Γ is locally a graph over $\mathbb{R}^2$. We consider the set

$$M = \left\{x \in \mathbb{R}^2 \,:\, \#(\Gamma \cap (\{x\} \times \mathbb{R})) = 1\right\}.$$

Note that $\#(\Gamma \cap (\{x\} \times \mathbb{R})) \ge 1$ everywhere. Since $\mathcal{E} < \infty$, the set M is not empty. Because of the local graph property, M and its complement M^c are open, and hence $M = \mathbb{R}^2$ and Γ is given by a global Lipschitz graph function h. Finally, the bound from Theorem 4.1 in combination with (4.3) and Lemma 4.2 yields

$$\|\nabla h\|_\infty \lesssim \rho^{-1}\mathcal{E}^{\frac{1}{2}} + \rho^{\frac{1}{2}}D^{\frac{1}{2}} \overset{(4.9)}{=} 2(\mathcal{E}D^2)^{\frac{1}{6}}. \qquad \square$$

4.1 | Proof of Lemma 4.2

We capitalize on the fact that H is the trace of f and employ the following Meyers–Ziemer trace estimate, contained in [20, Theorem 4.7]; see also [29, Theorem 5.12.4] and [26, p. 386]. (We again formulate the results in the BV setting.)

Lemma 4.3 (Meyers–Ziemer trace estimate). *Let Ω_+ be a set with locally finite perimeter and* $\Gamma = \operatorname{supp}\boldsymbol{\nabla}\mathbf{1}_{\Omega_+}$. *Let*

$$M(\Omega_+) := \sup_{\mathbf{x}\in\mathbb{R}^3, \rho>0} \rho^{-2}\left|\boldsymbol{\nabla}\mathbf{1}_{\Omega_+}\right|(\mathbf{B}_\rho(\mathbf{x})).$$

If $M(\Omega_+) < \infty$, then for all $\varphi \in C_0^1(\mathbb{R}^3)$ there holds

$$\left|\int_\Gamma \varphi \,\mathrm{d}S\right| \lesssim M(\Omega_+)\|\nabla\varphi\|_1.$$

To obtain a uniform bound on $M(\Omega_+)$, we will use the following slight adaption of a monotonicity formula of Schätzle, cf. [26, Lemma 2.1], whose proof follows [27, Section 17].

Lemma 4.4 (Schätzle's monotonicity formula). *There is a constant $C < \infty$ with the following property. Let Ω_+ be a set with locally finite perimeter and* $\Gamma = \operatorname{supp}\boldsymbol{\nabla}\mathbf{1}_{\Omega_+}$, *and let $f \in \dot{H}^1(\mathbb{R}^3)$ satisfy (1.5).*

Then, for any $\mathbf{x} \in \mathbb{R}^3$, *the function*

$$\rho \mapsto \rho^{-2}\left|\nabla \mathbf{1}_{\Omega_+}\right|(\mathbf{B}_\rho(\mathbf{x})) + C\rho^{\frac{1}{2}}\|\nabla f\|_2$$

is nondecreasing.

Proof of Lemma 4.2. Using density we apply the Meyers–Ziemer estimate from Lemma 4.3 with $\varphi = f^4$ to obtain

$$\begin{aligned}\|f\|_{L^4(\Gamma)}^4 &\lesssim M(\Omega_+)\|\nabla(f^4)\|_1 \lesssim M(\Omega_+)\|f^3\nabla f\|_1 \lesssim M(\Omega_+)\|\nabla f\|_2\|f\|_6^3 \lesssim M(\Omega_+)\|\nabla f\|_2^4 \\ &\lesssim M(\Omega_+)D^2,\end{aligned}$$

where we used Sobolev embedding in the penultimate step. Since f has a trace in L^4 on Γ, an application of the divergence theorem to the right-hand side of (1.5) reveals that $f = H$ on Γ in $L^4(\Gamma)$. To deduce (4.2) it hence suffices to show

$$M(\Omega_+) \lesssim 1 + (\mathcal{E}D^2)^{\frac{1}{5}}. \tag{4.10}$$

To this end, we observe that for any fixed $R > 0$, Schätzle's monotonicity formula (Lemma 4.4) implies for $\rho \leq R$ that

$$\begin{aligned}\rho^{-2}\left|\nabla \mathbf{1}_{\Omega_+}\right|(\mathbf{B}_\rho(\mathbf{x})) &\leq R^{-2}\left|\nabla \mathbf{1}_{\Omega_+}\right|(\mathbf{B}_R(\mathbf{x})) + CR^{\frac{1}{2}}D^{\frac{1}{2}} \\ &\overset{(4.5)}{\leq} \pi + R^{-2}\mathcal{E} + CR^{\frac{1}{2}}D^{\frac{1}{2}} \lesssim 1 + (\mathcal{E}D^2)^{\frac{1}{5}},\end{aligned} \tag{4.11}$$

where in the last step we have optimized with

$$R = \left(\frac{\mathcal{E}}{D^{\frac{1}{2}}}\right)^{\frac{2}{5}}.$$

It remains to control the supremum over larger radii, which is achieved by observing that (4.5) for $\rho \geq R$ gives

$$\rho^{-2}\left|\nabla \mathbf{1}_{\Omega_+}\right|(\mathbf{B}_\rho(\mathbf{x})) \leq \pi + R^{-2}\mathcal{E} \lesssim 1 + (\mathcal{E}D^2)^{\frac{1}{5}}. \tag{4.12}$$

The combination of (4.11) and (4.12) establishes (4.10). □

5 | RELAXATION RATES IN THE GRAPH SETTING

In this section, we work in the graph setting with small Lipschitz constant and hence, by the last part of Hypothesis **(H)**, assume that all quantities are smooth. In particular, Equations (1.2) and (1.3) hold pointwise.

Because Γ is given as the graph of a function as in (1.11), the excess mass $\mathcal{V}$ reduces to the L^1-norm of the height function

$$\mathcal{V} = \int_{\mathbb{R}^d} |h| \, \mathrm{d}x.$$

Moreover, the following geometric quantities can also be expressed in terms of h as

$$n = \frac{(-\nabla h, 1)}{\sqrt{1+|\nabla h|^2}}, \tag{5.1}$$

$$H = \operatorname{div} \frac{\nabla h}{\sqrt{1+|\nabla h|^2}} = \frac{1}{\sqrt{1+|\nabla h|^2}}\left(\Delta h - \frac{\nabla^2 h : (\nabla h \otimes \nabla h)}{1+|\nabla h|^2}\right), \tag{5.2}$$

$$V = \frac{h_t}{\sqrt{1+|\nabla h|^2}}. \tag{5.3}$$

Here the left-hand side quantities are only defined on Γ and are hence evaluated at $(x, h(x))$ while the right-hand side quantities are defined on $\mathbb{R}^d$ and are evaluated at x. We will mildly abuse notation by allowing context to make clear whether H, V take arguments from $\mathbb{R}^d$ or Γ. Because of the Lipschitz bound (1.12) on h, within the Lipschitz regime we have

$$\mathrm{d}S = \sqrt{1+|\nabla h|^2}\, \mathrm{d}x \sim \mathrm{d}x$$

and hence we can compare quantities on Γ to quantities on $\mathbb{R}^d$. A direct consequence of this and (5.1) is that the energy can be expressed as

$$\mathcal{E} = \int_{\mathbb{R}^d} \sqrt{1+|\nabla h|^2} - 1 \, \mathrm{d}x.$$

A second advantage of the Lipschitz setting is that we can deduce nonlinear estimates from the linearized ones and then take advantage of the (nonlinear) gradient flow structure of the system.

Within the Lipschitz regime, the rough plan is as follows: A simple differential inequality (Lemma 5.8) implies that the solution remains trapped in the Lipschitz regime. The main interpolation estimate (Proposition 5.6) and the gradient flow structure imply that as long as the L^1 norm remains controlled, the natural algebraic decay estimates hold. A duality argument verifies that the L^1 control is indeed guaranteed; see Subsection 5.3. The pieces are assembled via a buckling argument in Subsection 5.4.

Fractional and negative spaces, notation. We will work with the spaces $\dot{H}^s$ for $s = \pm\frac{1}{2}, \pm 1$. For $s = \frac{1}{2}$, we define $\dot{H}^{\frac{1}{2}}(\mathbb{R}^d)$ and $\dot{H}^{\frac{1}{2}}(\Gamma)$ via trace, that is, the space consists of traces of $\dot{H}^1(\mathbb{R}^{d+1})$ functions and the norm is given by the Dirichlet norm of the corresponding harmonic extension (or equivalently the infimum of the Dirichlet norm for all extensions). The negative spaces are defined by duality.

As in the introduction, we use $|\nabla|^\alpha$, $\alpha \in \mathbb{R}$ in the Fourier multiplier sense, that is, if $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier and the inverse Fourier transform, respectively, then

$$|\nabla|^\alpha g = \mathcal{F}^{-1}\big(|k|^\alpha \mathcal{F} g\big).$$

Within our small Lipschitz setting, the trace and Fourier definitions are equivalent in the sense that $\|g\|_{\dot{H}^{\pm1/2}(\mathbb{R}^d)} \sim \| |\nabla|^{\pm 1/2} g\|_2$ (see e.g., [9, Proposition A.1]). Also the Sobolev norms on $\mathbb{R}^d$ and Γ are equivalent; that is, for a function $w_\Gamma : \Gamma \to \mathbb{R}$ and $w : \mathbb{R}^d \to \mathbb{R}$ defined by $w(x) = w_\Gamma(x, h(x))$, we have

$$\|w_\Gamma\|_{\dot{H}^s(\Gamma)} \sim \|w\|_{H^s(\mathbb{R}^d)}$$

for $s = \pm\frac{1}{2}, \pm 1$. For $s = \frac{1}{2}$ this follows from the fact that the transformation $z' = z - h(x)$ maps Ω_+ to the half-space while the the Dirichlet energy of the corresponding extensions remains comparable (since h is Lipschitz). For $s = 1$ it is an easy computation. For the negative spaces it follows by duality. We will make regular use of this fact.

We continue to denote norms on the interface explicitly and suppress the domain on $\mathbb{R}^d$, for example,

$$\|g\|_{\dot{H}^{-\frac{1}{2}}(\Gamma)}, \qquad \|g\|_{\dot{H}^{-\frac{1}{2}}},$$

except when $\mathbb{R}^d$ is made explicit for emphasis.

5.1 | Algebraic relationships

The aim of this subsection is to relate the nonlinear quantities tied to the MS evolution to linear ones (for which we can take advantage of standard interpolation estimates and Fourier techniques). All statements are for general dimension $d \geq 1$ unless otherwise indicated.

As a first easy consequence of the Lipschitz bound, we obtain that $\mathcal{E}$ scales like the Dirichlet energy of h, and D controls a negative norm of the normal velocity V. A more elementary proof of (5.5) in the case $d = 1$ is contained in [9, Lemma 3.1 and proof of Lemma 4.1].

Lemma 5.1. *Under the condition $\|\nabla h\|_\infty \leq 1$, there holds*

$$\mathcal{E} \sim \|\nabla h\|_2^2, \tag{5.4}$$

and

$$\|V\|^2_{\dot{H}^{-\frac{1}{2}}(\Gamma)} \lesssim \|\nabla f_+ \cdot n\|^2_{\dot{H}^{-\frac{1}{2}}(\Gamma)} + \|\nabla f_- \cdot n\|^2_{\dot{H}^{-\frac{1}{2}}(\Gamma)} \lesssim D. \tag{5.5}$$

Proof. We deduce (5.4) directly from the identity

$$|\nabla h|^2 = \left(\sqrt{1 + |\nabla h|^2} - 1\right)\left(\sqrt{1 + |\nabla h|^2} + 1\right).$$

The first estimate in (5.5) follows from (1.3). For the second estimate in (5.5), we test with $\varphi \in \dot{H}^{\frac{1}{2}}(\Gamma)$ (with harmonic extension $\bar{\varphi}$) and use the divergence theorem (and $\boldsymbol{\Delta} f_\pm = 0$) to estimate

$$\left|\int_\Gamma \varphi \nabla f_\pm \cdot n \mathrm{d}S\right| = \left|\int_{\Omega_\pm} \nabla\bar{\varphi} \cdot \boldsymbol{\nabla} f_\pm \mathrm{d}S\right| \leq \|\nabla f\|_2 \|\nabla\bar{\varphi}\|_2.$$

Taking the supremum over φ yields the claim. □

Lemma 5.2. *Assume $\|\nabla h\|_\infty \le 1$. In any dimension, there holds*

$$\|H\|^2_{\dot H^{-1}(\Gamma)} \lesssim \mathcal{E}, \tag{5.6}$$

$$\|H\|^2_{\dot H^{\frac12}(\Gamma)} = D, \tag{5.7}$$

$$\|H\|^2_{L^2(\Gamma)} \lesssim \left(\mathcal{E}D^2\right)^{\frac13}. \tag{5.8}$$

Corollary 5.3. *Let $d = 2$. Assume $\|\nabla h\|_\infty \le 1$. Then for any $2 \le p \le 4$ there holds*

$$\|H\|_{L^p(\Gamma)} \lesssim \mathcal{E}^{\frac{4-p}{6p}} D^{\frac{2(p-1)}{3p}}. \tag{5.9}$$

Proof. By embedding (comparing to norms on $\mathbb{R}^d$) we have $\|H\|_{L^4(\Gamma)} \lesssim \|H\|_{\dot H^{\frac12}(\Gamma)}$. Then (5.9) follows by interpolation between this inequality and (5.8), and by using (5.7), where we have again used the equivalence of the norms on Γ and $\mathbb{R}^d$ and the Fourier representation, for which the interpolation is readily available. □

Proof of Lemma 5.2. A proof in $d = 1$ appeared already in [9, Lemma 3.1].

Estimate (5.8) follows by interpolation between (5.6) and (5.7). Equality (5.7) holds by definition of the norm. Inequality (5.6) follows from testing equation (5.2) with an $\dot H^1(\mathbb{R}^d)$ function, integrating by parts, applying the Cauchy–Schwarz inequality, and using the equivalence of the norms on Γ and $\mathbb{R}^d$. □

We next turn to p-norm control of the full Hessian of h. Our first observation is that in any dimension, one can directly deduce L^2 control of the Hessian from (5.2) (see (5.10) below). In $d = 1$, this yields control of $\|\nabla h\|_\infty$ in terms of $\mathcal{E}$ and D (cf. Corollary 5.5 below and [9, Lemma 3.2]). In $d = 2$ the norm $\|\nabla^2 h\|_2$ *just fails* to control $\|\nabla h\|_\infty$. Although we establish control of $\|\nabla h\|_\infty$ in $d = 2$ by heavier machinery in Proposition 2.2, we can also deduce it in the graph case in an elementary way from higher p-integrability of the Hessian (see Corollary 5.5). We also use the $p > 2$ integrability in Lemma 5.9 below to control the time change of the dissipation, which we need for our duality argument for large times.

Lemma 5.4 (Control of the Hessian by mean curvature). *Assume $\|\nabla h\|_\infty \le 1$. In any dimension, there holds*

$$\|\nabla^2 h\|_2 \lesssim \|H\|_2, \tag{5.10}$$

and there exists $\delta > 0$ such that for any $p \in [2, 2+\delta)$, there holds

$$\|\nabla^2 h\|_p \lesssim \|H\|_p. \tag{5.11}$$

Proof. It suffices to show (5.11). We begin with the representation (5.2) in the form

$$\Delta h = \sqrt{1 + |\nabla h|^2} H + \frac{\nabla h \otimes \nabla h}{1 + |\nabla h|^2} : \nabla^2 h. \tag{5.12}$$

According to the Calderon–Zygmund theory, there exists $M_p < \infty$ such that solutions of $\Delta w = g$ in $\mathbb{R}^d$ satisfy

$$\|\nabla^2 w\|_p \le M_p \|g\|_p \qquad \text{and} \qquad M_2 = 1.$$

Applying this to (5.12) under the assumption that $\|\nabla^2 h\|_p < \infty$ yields the estimate

$$\|\nabla^2 h\|_p \le M_p \left(\left\| \sqrt{1+|\nabla h|^2} H \right\|_p + \left\| \frac{\nabla h \otimes \nabla h}{1+|\nabla h|^2} \right\|_\infty \|\nabla^2 h\|_p \right).$$

Since

$$\|\nabla h\|_\infty \le 1, \qquad \text{and hence also} \qquad \left\| \frac{\nabla h \otimes \nabla h}{1+|\nabla h|^2} \right\|_\infty \le \frac{1}{2},$$

we obtain

$$\|\nabla^2 h\|_p \le \sqrt{2} M_p \|H\|_p + \frac{1}{2} M_p \|\nabla^2 h\|_p. \tag{5.13}$$

We now use that $\lim_{p\to 2} M_p = 1$, which is a consequence of the Riesz–Thorin interpolation theorem and $M_2 = 1$. Choosing $p > 2$ close enough to 2, we have $\frac{1}{2} M_p < 1$ and we can absorb the second term on the right-hand side of (5.13) into the left-hand side to obtain

$$\|\nabla^2 h\|_p \lesssim \|H\|_p.$$

Finally we justify the assumption $\|\nabla^2 h\|_p < \infty$ with a fixed point argument that confirms $\nabla^2 h \in L^p$ as long as $\frac{1}{2} M_p < 1$.

We remark that the proof simplifies in $d = 1$, since (5.2) takes the form

$$H = \frac{d}{dx} \frac{h_x}{\sqrt{1+h_x^2}} = \frac{h_{xx}}{(\sqrt{1+h_x^2})^3}.$$

□

We now confirm that the integral estimates on the Hessian and previous algebraic relationships can be converted into a Lipschitz bound in $d = 2$ and $d = 1$.

Corollary 5.5. *Let $d = 2$ or $d = 1$ and assume $\|\nabla h\|_\infty \le 1$. Then*

$$\|\nabla h\|_\infty \lesssim \left(\mathcal{E}^{3-d} D^d\right)^{\frac{1}{6}}. \tag{5.14}$$

Proof. For $d = 1$ this is contained in [9, Lemma 3.2] and can be deduced from the interpolation estimate

$$\|h_x\|_\infty \lesssim \|h_x\|_2^{\frac{1}{2}} \|h_{xx}\|_2^{\frac{1}{2}} \overset{(5.4)(5.10)}{\lesssim} \mathcal{E}^{\frac{1}{4}} \|H\|_2^{\frac{1}{2}} \overset{(5.8)}{\lesssim} \mathcal{E}^{\frac{1}{3}} D^{\frac{1}{6}}.$$

For $d = 2$, let $p > 2$ be an admissible exponent from Lemma 5.4. Then

$$\|\nabla h\|_\infty \lesssim \|\nabla h\|_2^{\frac{p-2}{2(p-1)}} \|\nabla^2 h\|_p^{\frac{p}{2(p-1)}}$$

(cf. Lemma A.5(i)) together with (5.4), (5.11) and (5.9) yields (5.14) for $d = 2$. □

Of central importance is the following interpolation estimate controlling the energy in terms of the excess mass and dissipation. We present it here for arbitrary dimension; for the proof of Theorem 1.3, we use it for $d = 2$ and $d = 1$.

Proposition 5.6 (Main interpolation estimate). *Under the condition* $\|\nabla h\|_\infty \leq 1$*, there holds*

$$\mathcal{E} \lesssim \mathcal{V}^{\frac{6}{d+5}} D^{\frac{d+2}{d+5}}.$$

Proof. This is a consequence of Lemma 5.1, the interpolation inequality

$$\|\nabla h\|_2^2 \lesssim \|h\|_1^{\frac{4}{d+4}} \|\nabla^2 h\|_2^{\frac{2d+4}{d+4}}$$

(cf. Lemma A.5(iii)), (5.10), and (5.8). ☐

5.2 | Differential relationships and decay estimates

We begin with a few elementary results that are true for general nonnegative quantities $\mathcal{V}, \mathcal{E}$, and D satisfying given algebraic and differential relationships. Then starting with Lemma 5.10 below, we show that the assumed relationships hold true for the MS evolution.

Our first observation is an ODE lemma: The gradient flow structure and the interpolation estimate from Proposition 5.6 immediately yield decay of $\mathcal{E}$ on any interval $[0, T]$ in terms of the supremum $\mathcal{V}_T$ defined in (3.8).

Lemma 5.7. *Suppose that for some $T > 0$ the quantities $\mathcal{V}, \mathcal{E}, D : [0, T] \to [0, \infty)$ satisfy*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} \leq -D, \quad \mathcal{E} \lesssim \mathcal{V}^{\frac{6}{d+5}} D^{\frac{d+2}{d+5}}. \tag{5.15}$$

Then, for all $t \in [0, T]$:

$$\mathcal{E}(t) \lesssim \min\left\{ \mathcal{E}_0, \frac{\mathcal{V}_T^2}{t^{\frac{d+2}{3}}} \right\}.$$

Proof. Combining the relation $\dot{\mathcal{E}} \leq -D$ with the inequality (5.15) yields

$$-\dot{\mathcal{E}} \gtrsim \mathcal{V}_T^{-\frac{6}{d+2}} \mathcal{E}^{\frac{d+5}{d+2}}.$$

An integration in time completes the proof. ☐

Lemma 5.8. *Let $d = 2$ or $d = 1$. For any $C < \infty$, there exists $\varepsilon_1 > 0$ with the following property. For any $T > 0$, if $\mathcal{E}, D : [0, T] \to [0, \infty)$ satisfy*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} \leq -D,$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}D \leq CD^{\frac{6-d}{3-d}},$$

then $\mathcal{E}^{3-d}D^d \leq \varepsilon_1$ *implies*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{E}^{3-d}D^d\right) \leq 0.$$

Proof. This follows from the straightforward computation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{E}^{3-d}D^d\right) &= (3-d)\,\dot{\mathcal{E}}\,\mathcal{E}^{2-d}\,D^d + d\,\dot{D}\,\mathcal{E}^{3-d}\,D^{d-1}\\ &\leq -(3-d)\,\mathcal{E}^{2-d}\,D^{d+1} + Cd\,\mathcal{E}^{3-d}\,D^{d-1}\,D^{\frac{6-d}{3-d}}\\ &\leq \mathcal{E}^{2-d}\,D^{d+1}\left(dC\left(\mathcal{E}^{3-d}D^d\right)^{\frac{1}{3-d}} - (3-d)\right),\end{aligned}$$

which has the right sign if $dC\varepsilon_1^{\frac{1}{3-d}} < 3-d$. This calculation for $d=1$ appeared already in [9, Lemma 5.1]. □

Lemma 5.9. *Let $d=2$ or $d=1$ and $T>0$. Suppose $\mathcal{E}, D : [0,T] \to [0,\infty)$ satisfy the relations*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} \leq -D, \quad \textit{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}D \lesssim D^{\frac{6-d}{3-d}} \quad \textit{on } [0,T].$$

Then, there exists $t_* \sim \mathcal{E}_0^{\frac{3}{d}}$ such that for all $t \in [t_*, T]$ there holds

$$D(t) \lesssim \frac{\mathcal{E}\left(\frac{t}{2}\right)}{t}.$$

Proof. Let $d=2$. By integrating the inequality

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(D^{-3}\right) \leq C$$

for $C \geq 1$ on the interval $[s,t]$ and multiplying with $D(s)^3D(t)^3$, we obtain the inequality

$$D(s)^3 \geq \frac{D(t)^3}{1+C(t-s)D(t)^3} \geq \frac{D(t)^3}{C+C(t-s)D(t)^3}$$

and deduce

$$D(s) \gtrsim \frac{D(t)}{(1+(t-s)D(t)^3)^{\frac{1}{3}}}.$$

We insert this in

$$\int_\tau^t D(s)\mathrm{d}s \leq \mathcal{E}(\tau) - \mathcal{E}(t) \leq \mathcal{E}(\tau)$$

to obtain

$$\mathcal{E}(\tau) \gtrsim D(t)\int_\tau^t \frac{1}{(1+(t-s)D(t)^3)^{\frac{1}{3}}}\mathrm{d}s = D(t)^{-2}\int_0^{(t-\tau)D(t)^3}\frac{1}{(1+\sigma)^{\frac{1}{3}}}\mathrm{d}\sigma$$
$$\gtrsim \min\left\{(t-\tau)D(t)(t-\tau),(t-\tau)^{\frac{2}{3}}\right\}.$$

We finish by choosing $\tau = \frac{t}{2}$.

The proof for $d = 1$ is similar and is contained in [9, Proof of (1.13)]. □

We now check that the assumed differential estimates from Lemmas 5.8 and 5.9 hold true for the MS evolution. The differential inequality for the energy gap is contained in **(H)**. It remains to derive a differential inequality for the dissipation.

Lemma 5.10. *Consider a smooth solution of the MS evolution with graph structure on* $[0,T]$ *in* $d = 2$ *or* $d = 1$ *and assume* $\|\nabla h\|_\infty \leq 1$. *In* $d = 1$, *there holds*

$$\frac{\mathrm{d}}{\mathrm{d}t}D \lesssim D^{\frac{5}{2}}\left(1+\left(\mathcal{E}^2 D\right)^{\frac{1}{2}}\right).$$

In $d = 2$, *let* $p > 2$ *be such that the assertion of Lemma 5.4 is satisfied. Then there holds*

$$\frac{\mathrm{d}}{\mathrm{d}t}D \lesssim D^4\left(1+\left(\mathcal{E}D^2\right)^{\frac{4-p}{2(p-2)}}\right). \tag{5.16}$$

Remark 5.11. In view of Corollary 5.5 and Lemmas 5.8 and 5.10, the condition $\|\nabla h\|_\infty \leq 1$ will be preserved for the MS evolution in $d = 2$ and $d = 1$ if $\mathcal{E}^{3-d}D^d$ is small enough initially.

Proof of Lemma 5.10. In $d = 1$, the result is contained in [9, Lemma 4.1 and (4.3)].

For $d = 2$, we recall the well-known evolution equation for the mean curvature (which can be derived for instance from [4, (3.8) and (3.10)]):

$$\dot{H} = -\operatorname{div}_{\tan}\nabla_{\tan}V - |\nabla_{\tan}n|^2 V, \tag{5.17}$$

where, $\nabla_{\tan}$ and $\operatorname{div}_{\tan}$ are the surface gradient and the surface divergence, respectively, and $\dot{H}$ represents the change of the curvature in the normal direction. We compute

$$\frac{\mathrm{d}}{\mathrm{d}t}D = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|\nabla f|^2\,\mathrm{d}\mathbf{x} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_+(t)}|\nabla f_+|^2\,\mathrm{d}\mathbf{x} + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_-(t)}|\nabla f_-|^2\,\mathrm{d}\mathbf{x}$$
$$= \int_{\Omega_+(t)}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla f_+|^2\,\mathrm{d}\mathbf{x} + \int_{\Omega_-}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla f_-|^2\,\mathrm{d}\mathbf{x} - \int_\Gamma V\left(|\nabla f_+|^2 - |\nabla f_-|^2\right)\mathrm{d}S$$
$$\overset{(1.2)}{=} \int_\Gamma -2\dot{f}_+\nabla f_+\cdot n + 2\dot{f}_-\nabla f_-\cdot n - V\left(|\nabla f_+|^2 - |\nabla f_-|^2\right)\mathrm{d}S$$
$$\overset{(1.2)}{=} \int_\Gamma -2\dot{H}[\nabla f\cdot n] - V\left(|\nabla f_+|^2 - |\nabla f_-|^2\right)\mathrm{d}S$$
$$\overset{(5.17)}{=} \int_\Gamma 2\left(\operatorname{div}_\Gamma\nabla_\Gamma V + |\nabla_\Gamma n|^2 V\right)V - V\left(|\nabla f_+|^2 - |\nabla f_-|^2\right)\mathrm{d}S, \tag{5.18}$$

where in the third line we have applied the divergence theorem and $\dot{f}_\pm = \frac{d}{dt}H(h(x,t),t)$ is the (total) time derivative of the curvature of a boundary point. Because ∇f is continuous in the tangential direction, the difference on the right-hand side of (5.18) satisfies

$$|\nabla f_+|^2 - |\nabla f_-|^2 = (\nabla f_+ \cdot n)^2 - (\nabla f_- \cdot n)^2 = [\nabla f \cdot n](\nabla f_+ \cdot n + \nabla f_- \cdot n)$$
$$\overset{(1.2)}{=} -V(\nabla f_+ \cdot n + \nabla f_- \cdot n). \tag{5.19}$$

Inserting (5.19) into the right-hand side of (5.18) and integrating by parts, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}D + 2\int_\Gamma |\nabla_\Gamma V|^2 \,\mathrm{d}S$$
$$\leq \underbrace{2\int_\Gamma |\nabla_\Gamma n|^2 V^2 \,\mathrm{d}S}_{=:A} + \underbrace{\int_\Gamma V^2(\nabla f_+ \cdot n + \nabla f_- \cdot n)\,\mathrm{d}S}_{=:B}. \tag{5.20}$$

We will deduce (5.16) from (5.20), the error estimates

$$A \lesssim \mathcal{E}^{\frac{4-p}{3p}} D^{\frac{2(3p-4)}{3p}} \|\nabla V\|_2^{\frac{2(p+4)}{3p}}, \tag{5.21}$$

$$|B| \lesssim D^{\frac{2}{3}} \|\nabla V\|_2^{\frac{5}{3}}, \tag{5.22}$$

and Young's inequality, absorbing the $\|\nabla V\|_2^2$ term in the left-hand side, where we use that $\|\nabla V\|_2^2 \lesssim \int_\Gamma |\nabla_\Gamma V|^2 \,\mathrm{d}S$. (Notice that because of $p > 2$, one has $2(p+4)/(3p) < 2$.)

We start by estimating

$$A \lesssim \|\nabla_\Gamma n\|_{L^p(\Gamma)}^2 \|V\|^2_{L^{\frac{2p}{p-2}}(\Gamma)}. \tag{5.23}$$

On the one hand, a simple computation based on (5.1) and using $\|\nabla h\|_\infty \leq 1$ reveals

$$\|\nabla_\Gamma n\|_{L^p(\Gamma)}^2 \lesssim \|\nabla_\Gamma n\|_p^2 \lesssim \|\nabla^2 h\|_p^2 \overset{(5.11)(5.9)}{\lesssim} \mathcal{E}^{\frac{4-p}{3p}} D^{\frac{4(p-1)}{3p}}. \tag{5.24}$$

On the other hand, we estimate

$$\|V\|^2_{L^{\frac{2p}{p-2}}(\Gamma)} \lesssim \|V\|^2_{\frac{2p}{p-2}} \lesssim \|V\|_{\dot H^{-1/2}}^{\frac{4(p-2)}{3p}} \|\nabla V\|_2^{\frac{2(p+4)}{3p}}. \tag{5.25}$$

The last inequality follows from the interpolation inequalities

$$\|V\|_2 \lesssim \|V\|_{\dot H^{-1/2}}^{\frac{2}{3}} \|\nabla V\|_2^{\frac{1}{3}}, \qquad \text{and} \qquad \|V\|_q \lesssim \|V\|_2^\theta \|\nabla V\|_2^{1-\theta},$$

for $\theta = 2/q$, see Lemma A.5(v). Inserting (5.24) and (5.25) into (5.23) and using $\|V\|_{\dot H^{-\frac{1}{2}}} \lesssim \|V\|_{\dot H^{-\frac{1}{2}}(\Gamma)}$ in combination with Lemma 5.1, we arrive at (5.21).

We now turn to establishing (5.22) for B. The terms with f_+ and f_- are handled in the same way, so without loss of generality, we consider f_+ and begin with

$$\left|\int_\Gamma \nabla f_+ \cdot nV^2 \,\mathrm{d}S\right| \lesssim \|\nabla f_+ \cdot n\|_{\dot H^{-\frac12}(\Gamma)} \|V^2\|_{\dot H^{\frac12}(\Gamma)} \lesssim D^{\frac12}\|V^2\|_{\dot H^{\frac12}(\Gamma)}, \tag{5.26}$$

where for the second estimate we have applied Lemma 5.1. For the second term on the right-hand side, we use

$$\|V^2\|_{\dot H^{\frac12}(\Gamma)} \lesssim \|V\|_6^{\frac32}\|\nabla V\|_2^{\frac12}$$

(cf. Lemma A.4) and insert (5.25) with $p = 3$ to obtain

$$\|V^2\|_{\dot H^{\frac12}(\Gamma)} \lesssim \|V\|_{\dot H^{-\frac12}}^{\frac13}\|\nabla V\|_2^{\frac53}.$$

Again applying Lemma 5.1 and inserting the result into (5.26) yields (5.22). □

5.3 | Duality argument

In this subsection we address the main mathematical challenge of this section, which is to prove within the graph regime that $\mathcal{V}$ remains bounded in terms of its initial data for all time. As in Section 3, the starting point is a dual representation of $\mathcal{V}$. In the graph setting, we use:

$$\mathcal{V} = \sup_{\psi\in L^\infty(\mathbb{R}^d),\|\psi\|_\infty\le 1}\int \psi h \,\mathrm{d}x,$$

and again we use the solution $\bar u$ of (1.33)–(1.35). To obtain uniform-in-time error estimates for large times, the decay of $\mathcal{E}$ and D will play a central role. Before stating and proving the duality result, we introduce a splitting of the linearization error into kinetic and geometric nonlinearity.

Lemma 5.12 (Splitting the error and preprocessing). *Let $\psi \in C_c^\infty(\mathbb{R}^d)$ with $\|\psi\|_\infty \le 1$. In $d = 2$ and $d = 1$ for a smooth solution of MS with graph structure on $[0, T]$, and u satisfying (1.33)–(1.35), there holds*

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d} hu\,\mathrm{d}x &= \frac{d}{dt}\int_{\mathbb{R}^d}\bar h\bar u\\ &= \left(-\int_\Gamma V\,\bar u(x,0)\,\mathrm{d}S + 2\int_{\mathbb{R}^d} f(x,0)\,\partial_z\bar u\,\mathrm{d}x\right)\\ &\quad + 2\int_{\mathbb{R}^d}(f(x,h(x)) - f(x,0))\,\partial_z\bar u\,\mathrm{d}x - 2\int_{\mathbb{R}^d}(H(x,h(x)) - \Delta h)\partial_z\bar u\,\mathrm{d}x.\end{aligned}$$

Moreover, the terms

$$\begin{aligned}A_4 &:= -\textstyle\int_\Gamma V\bar u(x,0)\mathrm{d}S + 2\int_{\mathbb{R}^d} f(x,0)\partial_z\bar u\mathrm{d}x,\\ A_5 &:= 2\textstyle\int_{\mathbb{R}^d}(f(x,h(x)) - f(x,0))\,\partial_z\bar u\mathrm{d}x,\\ A_6 &:= 2\textstyle\int_{\mathbb{R}^d}(H(x,h(x)) - \Delta h)\,\partial_z\bar u\mathrm{d}x,\end{aligned}$$

obey the estimates:

$$|A_4| \lesssim D^{\frac{1}{2}} \left(\frac{\mathcal{V}_T^{\frac{1}{d+2}} \mathcal{E}^{\frac{d+1}{2(d+2)}}}{(T-t)^{\frac{1}{3}}} + \frac{\mathcal{V}_T^{\frac{2}{d+2}} \mathcal{E}^{\frac{d}{2(d+2)}}}{(T-t)^{\frac{1}{2}}} \right), \tag{5.27}$$

$$|A_5| \lesssim \frac{\mathcal{V}_T^{\frac{1}{2}} D^{\frac{1}{2}}}{(T-t)^{\frac{1}{3}}}, \tag{5.28}$$

$$|A_6| \lesssim \frac{1}{(T-t)^{\frac{2}{3}}} \min\left\{ \|\nabla h\|_2^2, \|\nabla h\|_3^3 \right\}. \tag{5.29}$$

Proof. A direct calculation yields

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}^d} \bar{h}\bar{u}\,\mathrm{d}x &= \int_{\mathbb{R}^d} \bar{u}\partial_t \bar{h}\,\mathrm{d}x + \int_{\mathbb{R}^d} \bar{h}\partial_t \bar{u}\,\mathrm{d}x \\
&\overset{(5.3)(1.33)}{=} -\int_{\mathbb{R}^d} \sqrt{1+|\nabla h|^2}\, V(x,h(x))\,\bar{u}\,\mathrm{d}x + 2\int_{\mathbb{R}^d} h\,\partial_z \Delta\bar{u}\,\mathrm{d}x \\
&= -\int_\Gamma V\,\bar{u}(x,0)\,\mathrm{d}S + 2\int_{\mathbb{R}^d} \Delta h\,\partial_z \bar{u}\,\mathrm{d}x \\
&= -\int_\Gamma V\,\bar{u}(x,0)\,\mathrm{d}S + 2\int_{\mathbb{R}^d} H(x,h(x))\,\partial_z \bar{u}\,\mathrm{d}x \\
&\quad -2\int_{\mathbb{R}^d} (H(x,h(x)) - \Delta h)\partial_z \bar{u}\,\mathrm{d}x \\
&= -\int_\Gamma V\,\bar{u}(x,0)\,\mathrm{d}S + 2\int_{\mathbb{R}^d} f(x,0)\,\partial_z \bar{u}\,\mathrm{d}x \\
&\quad +2\int_{\mathbb{R}^d} (f(x,h(x)) - f(x,0))\,\partial_z \bar{u}\,\mathrm{d}x \\
&\quad -2\int_{\mathbb{R}^d} (H(x,h(x)) - \Delta h)\partial_z \bar{u}\,\mathrm{d}x,
\end{aligned}$$

as desired.

For A_4 we compute

$$\begin{aligned}
A_4 &= -\int_\Gamma V\bar{u}(x,0)\,\mathrm{d}S + 2\int_{\mathbb{R}^d} f(x,0)\,\partial_z \bar{u}\,\mathrm{d}x \\
&\overset{(1.34)}{=} -\int_\Gamma V\bar{u}(x,0)\,\mathrm{d}S + \int_{\mathbb{R}^{d+1}} \boldsymbol{\nabla} f \cdot \boldsymbol{\nabla}\bar{u}\,\mathrm{d}\mathbf{x} \\
&\overset{(1.2)}{=} -\int_\Gamma V\bar{u}(x,0) + [\boldsymbol{\nabla} f\cdot n]\bar{u}(x,h(x))\,\mathrm{d}S \\
&\overset{(1.3)}{=} \int_\Gamma V(\bar{u}(x,h(x)) - \bar{u}(x,0))\,\mathrm{d}S.
\end{aligned}$$

We use $\|\nabla h\|_\infty \leq 1$ and $\|\psi\|_\infty \leq 1$ to estimate

$$
\begin{aligned}
|A_4| &\lesssim \|V\|_{\dot{H}^{-\frac{1}{2}}(\Gamma)} \|(\bar{u}(\cdot, h(\cdot)) - \bar{u}(\cdot, 0))\|_{\dot{H}^{\frac{1}{2}}(\Gamma)} \\
&\overset{(5.5)}{\lesssim} D^{\frac{1}{2}} \|(\bar{u}(\cdot, h(\cdot)) - \bar{u}(\cdot, 0))\|_{L^2(\mathbb{R}^d)}^{\frac{1}{2}} \|\nabla(\bar{u}(\cdot, h(\cdot)) - \bar{u}(\cdot, 0))\|_{L^2(\mathbb{R}^d)}^{\frac{1}{2}} \\
&\lesssim D^{\frac{1}{2}} \left(\frac{\mathcal{V}^{\frac{1}{d+2}} \mathcal{E}^{\frac{d+1}{2(d+2)}}}{(T-t)^{\frac{1}{3}}} + \frac{\mathcal{V}^{\frac{2}{d+2}} \mathcal{E}^{\frac{d}{2(d+2)}}}{(T-t)^{\frac{1}{2}}} \right),
\end{aligned}
$$

where we have used

$$
\|(\bar{u}(\cdot, h(\cdot)) - \bar{u}(\cdot, 0))\|_2 = \| \int_0^{h(\cdot)} \partial_z \bar{u}(\cdot, z) \mathrm{d}z \|_2 \lesssim \|\partial_z \bar{u}\|_\infty \|h\|_2,
$$

and

$$
\begin{aligned}
\|\nabla(\bar{u}(\cdot, h(\cdot)) - \bar{u}(\cdot, 0))\|_2 &= \|\partial_z \bar{u}(\cdot, h(\cdot)) \nabla h(\cdot) + \int_0^{h(\cdot)} \partial_z \nabla \bar{u}(\cdot, z) \mathrm{d}z\|_2 \\
&\lesssim \|\partial_z \bar{u}\|_\infty \|\nabla h\|_2 + \|\partial_z \nabla \bar{u}\|_\infty \|h\|_2,
\end{aligned}
$$

together with (A.4), (A.5), Lemma A.5(ii), and (5.4).

We now turn to A_5. Starting with

$$
\begin{aligned}
A_5 &= 2 \int_{\mathbb{R}^d} (f(x, h(x)) - f(x, 0)) \partial_z \bar{u} \, \mathrm{d}x \\
&= 2 \int_{\mathbb{R}^d} \partial_z \bar{u} \int_0^{h(x)} \partial_z f(x, z) \, \mathrm{d}z \, \mathrm{d}x,
\end{aligned}
$$

we estimate

$$
\begin{aligned}
|A_5| &\lesssim \|\partial_z \bar{u}\|_\infty \int_{\mathbb{R}^d} |h(x)|^{\frac{1}{2}} \left(\int_0^{h(x)} |\boldsymbol{\nabla} f(x, z)|^2 \, \mathrm{d}z \right)^{\frac{1}{2}} \mathrm{d}x \\
&\lesssim \|\partial_z \bar{u}\|_\infty \left(\int_{\mathbb{R}^d} |h| \, \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^{d+1}} |\boldsymbol{\nabla} f|^2 \, \mathrm{d}\mathbf{x} \right)^{\frac{1}{2}} \\
&\overset{(A.4)}{\lesssim} \mathcal{V}_T^{\frac{1}{2}} \frac{D^{\frac{1}{2}}}{(T-t)^{\frac{1}{3}}}.
\end{aligned}
$$

Finally, we turn to A_6, expressing it in the form

$$
\begin{aligned}
A_6 &= -2 \int_{\mathbb{R}^d} (H(x, h(x)) - \Delta h) \partial_z \bar{u} \, \mathrm{d}x \\
&\overset{(5.2)}{=} -2 \int_{\mathbb{R}^d} \mathrm{div} \left(\left(\frac{1}{\sqrt{1 + |\nabla h|^2}} - 1 \right) \nabla h \right) \partial_z \bar{u} \, \mathrm{d}x
\end{aligned}
$$

$$= 2\int_{\mathbb{R}^d}\left(\frac{1}{\sqrt{1+|\nabla h|^2}}-1\right)\nabla h\nabla\partial_z\bar{u}\,\mathrm{d}x$$

$$= -2\int_{\mathbb{R}^d}\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\left(\sqrt{1+|\nabla h|^2}-1\right)\nabla\partial_z\bar{u}\,\mathrm{d}x,$$

which we estimate using

$$\frac{|\nabla h|}{\sqrt{1+|\nabla h|^2}} \le \min\{1, |\nabla h|\},$$

to deduce

$$|A_6| \lesssim \|\nabla\partial_z\bar{u}\|_\infty \min\left\{\|\nabla h\|_2^2, \|\nabla h\|_3^3\right\} \overset{(A.5)}{\lesssim} \frac{1}{(T-t)^{\frac{2}{3}}}\min\left\{\|\nabla h\|_2^2, \|\nabla h\|_3^3\right\}. \qquad \square$$

We are now ready for the duality argument.

Proposition 5.13. *Let $d = 2$ or $d = 1$, $T > 0$, and consider a smooth solution of the MS problem with graph structure on $[0, T]$. Suppose that $\mathcal{V}_T < \infty$, $\|\nabla h\|_\infty \le 1$ hold for all $t \in [0, T]$ and that D satisfies*

$$D(t) \lesssim \min\left\{\frac{\mathcal{E}_0}{t}, \frac{\mathcal{V}_T^2}{t^{\frac{d+5}{3}}}\right\}. \tag{5.30}$$

Then $\mathcal{V}$ obeys the bound

$$\mathcal{V}_T = \sup_{t\in[0,T]}\mathcal{V}(t) \lesssim \mathcal{V}_0 + \mathcal{E}_0^{\frac{d+1}{d}}. \tag{5.31}$$

Proof. Let u satisfy (1.33)–(1.35). We claim that it suffices to establish

$$\int_0^T \frac{d}{dt}\int_{\mathbb{R}^d} hu\,\mathrm{d}x\,\mathrm{d}t \lesssim \mathcal{E}_0^{\frac{3}{2}} + \left(\mathcal{V}_T^{\frac{1}{2}}\mathcal{E}_0^{\frac{3}{4}} + \mathcal{V}_T^{\frac{3}{4}}\mathcal{E}_0^{\frac{3}{8}}\right) \qquad \text{in } d = 2, \text{ and} \tag{5.32}$$

$$\int_0^T \frac{d}{dt}\int_{\mathbb{R}^d} hu\,\mathrm{d}x\,\mathrm{d}t \lesssim \mathcal{V}_T^{\frac{1}{3}}\mathcal{E}_0^{\frac{4}{3}} + \mathcal{V}_T^{\frac{2}{3}}\mathcal{E}_0^{\frac{2}{3}} + \mathcal{V}_T^{\frac{5}{6}}\mathcal{E}_0^{\frac{1}{3}} \qquad \text{in } d = 1. \tag{5.33}$$

Indeed, evaluating the integrals on the left-hand side, taking the supremum over ψ, recalling (A.3), and applying Young's inequality leads to (5.31) for all $T > 0$.

We will establish (5.32)–(5.33) based on the error estimates (5.27)–(5.29) in $d = 2$ and $d = 1$. The strategy is that we need enough decay for integrability at infinity, but no larger power than $\mathcal{V}_T^1$ so that we can, as described above, absorb powers of $\mathcal{V}_T$ from the right-hand side into the left-hand side. To this end, we use either the first or second estimate from (5.30), as needed. Note that the dissipation decay (5.30) is better than the energy decay (1.10) for this purpose, since we get "more time decay for the same power of $\mathcal{V}_T$." Note also that we will make repeated use of Lemma A.3 with $a + b = 1$.

For A_6, we use (5.29) together with

$$\|\nabla h\|_3^3 \le \|\nabla h\|_\infty \|\nabla h\|_2^2 \overset{(5.14)(5.4)}{\lesssim} \mathcal{E}^{\frac{9-d}{6}} D^{\frac{d}{6}}$$

to derive

$$\int_0^T |A_6|\,\mathrm{d}t \lesssim \int_0^T \frac{\mathcal{E}^{\frac{9-d}{6}} D^{\frac{d}{6}}}{(T-t)^{\frac{2}{3}}}\,\mathrm{d}t \overset{(5.30)}{\lesssim} \mathcal{E}_0^{\frac{9-d}{6}} \int_0^T \frac{1}{(T-t)^{\frac{2}{3}}} \left(\frac{\mathcal{V}_T^2}{t^{\frac{d+5}{3}}}\right)^{\frac{\alpha d}{6}} \left(\frac{\mathcal{E}_0}{t}\right)^{\frac{(1-\alpha)d}{6}} \mathrm{d}t.$$

We choose α to give $t^{\frac{1}{3}}$ decay and invoke (A.7), which leads to

$$\alpha = 0 \qquad \text{and an error} \quad \mathcal{E}_0^{\frac{3}{2}} \qquad \text{in } d = 2,$$

$$\alpha = 1 \qquad \text{and an error} \quad \mathcal{V}_T^{\frac{1}{3}} \mathcal{E}_0^{\frac{4}{3}} \qquad \text{in } d = 1.$$

To estimate A_5, we use (5.30) to compute

$$\int_0^T |A_5|\,\mathrm{d}t \overset{(5.28)}{\lesssim} \mathcal{V}_T^{\frac{1}{2}} \int_0^T \frac{1}{(T-t)^{\frac{1}{3}}} \left(\frac{\mathcal{V}_T^2}{t^{\frac{d+5}{3}}}\right)^{\frac{\alpha}{2}} \left(\frac{\mathcal{E}_0}{t}\right)^{\frac{1-\alpha}{2}} \qquad \alpha = \frac{1}{d+2}$$

$$\overset{(A.7)}{\lesssim} \mathcal{V}_T^{\frac{d+4}{2(d+2)}} \mathcal{E}_0^{\frac{d+1}{2(d+2)}}.$$

Finally we turn to A_4. Estimating (5.27) as in A_5 where for the first time integral we again take $\alpha = \frac{1}{d+2}$ and in the second time integral we instead take $\alpha = 0$, we obtain an error

$$\mathcal{V}_T^{\frac{2}{d+2}} \mathcal{E}_0^{\frac{d+1}{d+2}}. \qquad \square$$

5.4 | Proof of Proposition 2.3

Proof. **Step 1: Control of the Lipschitz constant**. We begin by establishing control on $\mathcal{E}^{3-d}D^d$ and $\|\nabla h\|_\infty$ for all times. Let ε_1 be the constant from Lemma 5.8 and $\hat{\varepsilon}$ be such that $C\hat{\varepsilon}^{\frac{1}{6}} < 1$, where $C < \infty$ is the implicit constant in (5.14). We set $\varepsilon_2 := \frac{1}{2}\min\{\varepsilon_1, \hat{\varepsilon}\}$. This implies

$$\|\nabla h_0\|_\infty < 1. \tag{5.34}$$

We set

$$T_1 := \sup\left\{T > 0 : \mathcal{E}^{3-d}D^d \le 2\varepsilon_2 \text{ for all } t \le T\right\}$$
$$\text{and} \quad T_2 := \sup\{T > 0 : \|\nabla h\|_\infty < 1 \text{ for all } t \le T\}$$

and note that by smoothness, (1.24), and (5.34), there holds $T_1 > 0$ and $T_2 > 0$. Now we define

$$T := \min\{T_1, T_2\}.$$

On the one hand, we can apply Corollary 5.5 so that $\|\nabla h\|_\infty < 1$ on $[0, T]$. On the other hand, $\|\nabla h\|_\infty \leq 1$ implies that we can apply Lemma 5.10. Using $\mathcal{E}^{3-d}D^d \leq \varepsilon_1$, we apply Lemma 5.8 to get $\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{E}^{3-d}D^d) \leq 0$ on $[0, T]$. We deduce $T = \infty$.

Step 2: Proof of (2.3), (2.4), (2.5). Next we define

$$T_3 := \sup\left\{ T > 0 : \mathcal{V} \leq \tilde{C}(\mathcal{V}_0 + \mathcal{E}_0^{\frac{d+1}{d}}) \text{ for all } t \leq T \right\},$$

where, $\tilde{C} > 1$ is a universal constant to be specified below and smoothness implies $T_3 > 0$.

Our goal is to prove $T_3 = \infty$ for $\tilde{C}$ large enough. By Proposition 5.6 and Lemma 5.7 we see that

$$\mathcal{E} \lesssim \frac{\mathcal{V}_{T_3}^2}{t^{\frac{d+2}{3}}} \qquad \text{for} \quad t \in [0, T_3].$$

Combining this with Lemma 5.10, $\mathcal{E}^{3-d}D^d < 2\varepsilon_2$, and Lemma 5.9, we obtain

$$D \lesssim \frac{\mathcal{E}\left(\frac{t}{2}\right)}{t} \lesssim \frac{\mathcal{V}_{T_3}^2}{t^{\frac{d+5}{3}}} \qquad \text{for all } t \in [T_{\text{diss}}, T_3],$$

where, $T_{\text{diss}} \sim \mathcal{E}_0^{\frac{3}{d}}$ is the time from Lemma 5.9. Combining Proposition 2.1 for $t \leq T_{\text{diss}}$ with Proposition 5.13 for $t \geq T_{\text{diss}}$, we deduce

$$\mathcal{V}_{T_3} \leq C\left(\mathcal{V}_0 + \mathcal{E}_0^{\frac{d+1}{d}}\right)$$

for a universal constant $C < \infty$. Choosing $\tilde{C} > C$ implies $T_3 = \infty$.

Step 3: Proof of (2.6). A combination of Corollary 5.5 and Step 2 delivers the bound on $\|\nabla h\|_\infty$ in (2.6) for $t \geq T_{\text{diss}}$. The bound on $\|h\|_\infty$ follows from item (iv) of Lemma A.5, Lemma 5.1, and the bound on $\|\nabla h\|_\infty$. □

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APPENDIX A: ELEMENTARY BOUNDS

For completeness, we collect here a few elementary results that we apply in the paper.

Lemma A.1. *Let $\psi \in C_c^\infty(\mathbb{R}^d)$. For the function $\bar{u}$ satisfying (1.33)–(1.35), there holds*

$$|\bar{u}(\mathbf{x})| \lesssim \frac{1}{|\mathbf{x}|^d} \quad \textit{for } |\mathbf{x}| \gg 1. \tag{A.1}$$

Proof. To establish (A.1), we observe that because of the decay rate of G from (1.31) and $\psi \in C_c^\infty(\mathbb{R}^d)$, there holds

$$\|u(t)\|_{L^1(\mathbb{R}^d)} \lesssim 1 \qquad \text{and} \qquad |u(x,t)| \lesssim T^{1/3}|x|^{-(d+1)} \text{ for large } x \text{ and all } t < T. \tag{A.2}$$

For $|z| > |x|$, (A.1) directly follows from the decay of the Poisson kernel $|P(x,z)| \lesssim |z|^{-d}$ and $\|u\|_{L^1(\mathbb{R}^d)} \lesssim 1$. For $|x| \geq |z|$ and $|x|$ large enough, we write

$$\begin{aligned}
|\bar{u}(x,z)| &= \left| \int_{|x-y|\geq \frac{1}{2}|x|} P(x-y,z)u(y)\mathrm{d}y + \int_{|x-y|\leq \frac{1}{2}|x|} P(x-y,z)u(y)\mathrm{d}y \right| \\
&\lesssim \sup_{|x-y|\geq\frac{1}{2}|x|} \frac{|z|}{|x-y|^{d+1}}\|u\|_1 + \|P(\cdot,z)\|_1 \sup_{|x-y|\leq \frac{1}{2}|x|} |u(y)| \\
&\overset{(A.2)}{\lesssim} \frac{|z|}{|x|^{(d+1)}} + \frac{1}{|x|^d} \lesssim \frac{1}{|\mathbf{x}|^d}.
\end{aligned}$$

□

Lemma A.2. *Let $\psi \in L^\infty(\mathbb{R}^d)$ and let $\bar{u}$ satisfy (1.33)–(1.35). Then we have the following estimates in terms of the terminal data:*

$$\begin{aligned}
\|\bar{u}\|_\infty &\lesssim \|u\|_\infty \lesssim \|\psi\|_\infty, \\
\|\nabla\bar{u}\|_\infty &\lesssim \|\nabla u\|_\infty \lesssim (T-t)^{-\frac{1}{3}}\|\psi\|_\infty, \\
\|\nabla^2\bar{u}\|_\infty &\lesssim \|\nabla^2 u\|_\infty \lesssim (T-t)^{-\frac{2}{3}}\|\psi\|_\infty.
\end{aligned} \tag{A.3}$$

Furthermore recalling $v = -|\nabla|u = \partial_z\bar{u}(x,0)$, we have

$$\|\partial_z\bar{u}\|_\infty \lesssim \|v\|_\infty \lesssim (T-t)^{-\frac{1}{3}}\|\psi\|_\infty, \tag{A.4}$$

$$\|\partial_z\nabla\bar{u}\|_\infty \lesssim \|\nabla v\|_\infty \lesssim (T-t)^{-\frac{2}{3}}\|\psi\|_\infty, \tag{A.5}$$

$$\|\partial_z^2\nabla\bar{u}\|_\infty \lesssim \|\partial_z\nabla v\|_\infty \lesssim (T-t)^{-1}\|\psi\|_\infty. \tag{A.6}$$

Proof. Recall that $\bar{u}$ is the harmonic extension of the function u satisfying

$$\partial_t u + 2|\nabla|\Delta u = 0 \quad \text{in } [0,T)\times\mathbb{R}^d,$$
$$u(T) = \psi \quad \text{in } \mathbb{R}^d.$$

By the maximum principle all estimates for u, v and their derivatives carry over to $\bar{u}$ and $\partial_z\bar{u}$. The estimates for u are a consequence of

$$\|\nabla^m u(t)\|_\infty \le \|\nabla^m G(T-t)\|_1\|\psi\|_\infty, \quad m = 0,1,2,$$

where G is the kernel from (1.23). Bounds for $\|\nabla^m G\|_1$ are derived by combining L^∞ bounds and decay at infinity coming from the Fourier transform. The bounds for v can be derived based on the properties of the Poisson kernel and the estimates for u. □

In the duality proof for large times, we make repeated use of the following fact.

Lemma A.3. *For constants* $0 < a, b < 1$ *such that* $a + b \ge 1$*, there holds*

$$\int_0^T \frac{1}{(T-t)^a}\frac{1}{t^b}\,\mathrm{d}t \lesssim \frac{1}{T^{a+b-1}}. \tag{A.7}$$

Proof. The proof can be obtained by elementary integration after separating the region of integration into $[0, T/2]$ and $[T/2, T]$ and noting that on the first region, the first term in the integrand is bounded by $\lesssim T^{-a}$ and analogously on the second region. □

For the convenience of the reader we include the proof of the following interpolation estimate.

Lemma A.4. *Let* $d \in \mathbb{N}$ *and* $V \in L^6(\mathbb{R}^d)\cap \dot{H}^1(\mathbb{R}^d)$*. Then,* $V^2 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^d)$ *and*

$$\|V^2\|^2_{\dot{H}^{\frac{1}{2}}} \lesssim \|V\|_6^3\|\nabla V\|_2. \tag{A.8}$$

Proof. We start with

$$\begin{aligned}
\|V^2\|^2_{\dot{H}^{\frac{1}{2}}} &= \int_{\mathbb{R}^d} |k|\left(\widehat{V^2}\right)^2(k)\mathrm{d}k = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}\widehat{V^2}(k)\frac{k}{|k|}\int_{\mathbb{R}^d} k e^{-ikx}V^2(x)\mathrm{d}x\,\mathrm{d}k \\
&= -2\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}\widehat{V^2}(k)\frac{ik}{|k|}\int_{\mathbb{R}^d} e^{-ikx}V(x)\nabla V(x)\mathrm{d}x\,\mathrm{d}k \\
&= -2\int_{\mathbb{R}^d}\widehat{V^2}(k)\frac{ik}{|k|}\left(\hat{V} * \widehat{\nabla V}\right)(k)\mathrm{d}k \\
&= -\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{ik}{|k|}\widehat{V^2}(k)\hat{V}(k-k')\widehat{\nabla V}(k')\mathrm{d}k'\,\mathrm{d}k \\
&= \int_{\mathbb{R}^d}\widehat{\nabla V}(k')\int_{\mathbb{R}^d}\frac{-ik}{|k|}\widehat{V^2}(k)\hat{V}(k-k')\mathrm{d}k\,\mathrm{d}k'.
\end{aligned}$$

By the Plancherel theorem, (A.8) follows from the estimate

$$\|G\|_2^2 := \|\mathcal{F}^{-1}\Big(\int_{\mathbb{R}^d} \frac{-ik}{|k|}\widehat{V^2}(k)\hat{V}(k-k')\mathrm{d}k\Big)\|_2^2 \lesssim \|V\|_6^6, \tag{A.9}$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform. To prove (A.9) we write

$$G = \int_{\mathbb{R}^d} A[V^2](x)V(-x),$$

where A is the operator associated to the Fourier-multiplier $\frac{-ik}{|k|}$. This implies

$$\|G\|_2^2 = \int_{\mathbb{R}^d} A[V^2]^2(x)V^2(x)\mathrm{d}x \le \|V\|_6^2\|A[V^2]\|_3^2,$$

from which (A.9) follows by an application of the Hörmander–Mikhlin theorem ($\|A[V^2]\|_3 \lesssim \|V^2\|_3$). □

Finally, we collect without proof the interpolation estimates that are more or less direct consequences of the Gagliardo–Nirenberg–Sobolev inequalities.

Lemma A.5. *The following interpolation inequalities hold.*

(i) *Let $g \in L^2(\mathbb{R}^2)$ with $\nabla g \in L^p(\mathbb{R}^2)$ for some $p > 2$. Then $g \in L^\infty(\mathbb{R}^2)$ and*

$$\|g\|_\infty \lesssim \|g\|_2^{\frac{p-2}{2(p-1)}}\|\nabla g\|_p^{\frac{p}{2(p-1)}}.$$

For $h \in \dot{H}^1(\mathbb{R}^2)$ with $\nabla^2 h \in L^p(\mathbb{R}^2)$ this entails in particular

$$\|\nabla h\|_\infty \lesssim \|\nabla h\|_2^{\frac{p-2}{2(p-1)}}\|\nabla^2 h\|_p^{\frac{p}{2(p-1)}}.$$

(ii) *Let $h \in L^1(\mathbb{R}^d)$ with $\nabla h \in L^2(\mathbb{R}^d)$. Then $h \in L^2(\mathbb{R}^d)$ and*

$$\|h\|_2 \lesssim \|h\|_1^{\frac{2}{d+2}}\|\nabla h\|_2^{\frac{d}{d+2}}.$$

(iii) *Let $h \in L^1(\mathbb{R}^d)$ with $\nabla^2 h \in L^2(\mathbb{R}^d)$. Then $\nabla h \in L^2(\mathbb{R}^d)$ and*

$$\|\nabla h\|_2 \lesssim \|h\|_1^{\frac{2}{d+4}}\|\nabla^2 h\|_2^{\frac{d+2}{d+4}}.$$

(iv) *Let $h \in L^1(\mathbb{R}^2)$ with $\nabla h \in L^\infty(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$. Then $h \in L^\infty(\mathbb{R}^2)$ and*

$$\|h\|_\infty \lesssim \|h\|_1^{\frac{1}{5}}\|\nabla h\|_2^{\frac{2}{5}}\|\nabla h\|_\infty^{\frac{2}{5}}.$$

If $h \in L^1(\mathbb{R})$ with $h_x \in L^2(\mathbb{R})$ then $h \in L^\infty(\mathbb{R})$ and

$$\|h\|_\infty \lesssim \|h\|_1^{\frac{1}{3}}\|h_x\|_2^{\frac{2}{3}}.$$

(v) *Let $q \in (2,\infty)$. If $g \in L^2(\mathbb{R}^2)$ with $\nabla g \in L^2(\mathbb{R}^2)$, then $g \in L^q(\mathbb{R}^2)$ and*

$$\|g\|_q \lesssim \|g\|_2^{\frac{2}{q}}\|\nabla g\|_2^{\frac{q-2}{q}}.$$