Available online at www.sciencedirect.com

ScienceDirect

Journal of Approximation Theory 305 (2025) 106099

Journal of Approximation Theory

www.elsevier.com/locate/jat

Full Length Article

Dual spaces vs. Haar measures of polynomial hypergroups

Stefan Kahler[a,b,*,1], Ryszard Szwarc[c]

[a] Fachgruppe Mathematik, RWTH Aachen University, Pontdriesch 14-16, 52062 Aachen, Germany

[b] Lehrstuhl A für Mathematik, RWTH Aachen University, 52056 Aachen, Germany

[c] Institute of Mathematics, Wrocław University, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

Received 27 December 2023; received in revised form 30 August 2024; accepted 3 September 2024
Available online 7 September 2024

Communicated by W. Van Assche

Abstract

Many symmetric orthogonal polynomials $(P_n(x))_{n\in\mathbb{N}_0}$ induce a hypergroup structure on $\mathbb{N}_0$. The Haar measure is the counting measure weighted with $h(n) := 1/\int_{\mathbb{R}} P_n^2(x)\,\mathrm{d}\mu(x) \geq 1$, where μ denotes the orthogonalization measure. We observed that many naturally occurring examples satisfy the remarkable property $h(n) \geq 2$ $(n \in \mathbb{N})$. We give sufficient criteria and particularly show that $h(n) \geq 2$ $(n \in \mathbb{N})$ if the (Hermitian) dual space $\widehat{\mathbb{N}_0}$ equals the full interval $[-1, 1]$, which is fulfilled by an abundance of examples. We also study the role of nonnegative linearization of products (and of the resulting harmonic and functional analysis). Moreover, we construct two example types with $h(1) < 2$. To our knowledge, these are the first such examples. The first type is based on Karlin–McGregor polynomials, and $\widehat{\mathbb{N}_0}$ consists of two intervals and can be chosen "maximal" in some sense; h is of quadratic growth. The second type relies on hypergroups of strong compact type; h grows exponentially, and $\widehat{\mathbb{N}_0}$ is discrete.

MSC: primary 43A62; secondary 28C10, 33C47, 46E30

Keywords: Hypergroups; Orthogonal polynomials; Haar measure; Dual space; Polynomial expansions; Polynomial estimates

* Corresponding author at: Fachgruppe Mathematik, RWTH Aachen University, Pontdriesch 14-16, 52062 Aachen, Germany.

E-mail addresses: kahler@mathematik.rwth-aachen.de (S. Kahler), ryszard.szwarc@uwr.edu.pl (R. Szwarc).

1 The work was started while the first author was visiting the Institute of Mathematics of the University of Wrocław.

https://doi.org/10.1016/j.jat.2024.106099

1. Introduction

1.1. Basic setting and observation

Let $(P_n(x))_{n\in\mathbb{N}_0} \subseteq \mathbb{R}[x]$ with $\deg P_n(x) = n$ be given by some recurrence relation $P_0(x) = 1$, $P_1(x) = x$,

$$x P_n(x) = a_n P_{n+1}(x) + c_n P_{n-1}(x) \ (n \in \mathbb{N}), \tag{1.1}$$

where $(c_n)_{n\in\mathbb{N}} \subseteq (0, 1)$ and $a_n \equiv 1 - c_n$; to avoid case differentiations, we additionally define $a_0 := 1$. Obviously, the resulting polynomials are symmetric and normalized by $P_n(1) \equiv 1$. It is well-known from the theory of orthogonal polynomials[2] that $(P_n(x))_{n\in\mathbb{N}_0}$ is orthogonal w.r.t. a unique probability (Borel) measure μ on $\mathbb{R}$ which satisfies $|\text{supp}\ \mu| = \infty$ and $\text{supp}\ \mu \subseteq [-1, 1]$ (Favard's theorem). Moreover, it is well-known that the zeros of the polynomials are real, simple and located in the interior of the convex hull of supp μ. In particular, all P_n are strictly positive at the right end point of supp μ. We are interested in sequences which satisfy the additional 'nonnegative linearization of products' property

$$P_m(x)P_n(x) = \sum_{k=0}^{m+n} \underbrace{g(m, n; k)}_{\overset{!}{\geq} 0} P_k(x) \ (m, n \in \mathbb{N}_0), \tag{1.2}$$

i.e., the product of any two polynomials $P_m(x)$, $P_n(x)$ is a convex combination w.r.t. the basis $\{P_k(x) : k \in \mathbb{N}_0\}$. Due to orthogonality, one has $g(m, n; |m - n|), g(m, n; m + n) \neq 0$ and $g(m, n; k) = 0$ for $k < |m - n|$, so the summation in (1.2) starts with $k = |m - n|$ (and (1.2) can be regarded as an extension of the recurrence (1.1)). The nonnegativity of the linearization coefficients $g(m, n; k)$ gives rise to a commutative discrete hypergroup on $\mathbb{N}_0$, where the convolution $(m, n) \mapsto \sum_{k=|m-n|}^{m+n} g(m, n; k)\delta_k$ maps $\mathbb{N}_0 \times \mathbb{N}_0$ into the probability measures on $\mathbb{N}_0$, the identity on $\mathbb{N}_0$ serves as involution and 0 is the unit element.[3] Such hypergroups are called polynomial hypergroups, were introduced by Lasser in the 1980s and are generally very different from groups or semigroups [11]. There is an abundance of examples, and the individual behavior strongly depends on the underlying polynomials $(P_n(x))_{n\in\mathbb{N}_0}$. We briefly recall some basics [11,13]. The nonnegativity of the $g(m, n; k)$ implies that

$$\{\pm 1\} \cup \text{supp}\ \mu \subseteq \widehat{\mathbb{N}_0} \subseteq [-1, 1], \tag{1.3}$$

where the compact set $\widehat{\mathbb{N}_0}$ is defined by

$$\widehat{\mathbb{N}_0} := \left\{ x \in \mathbb{R} : \max_{n\in\mathbb{N}_0} |P_n(x)| = 1 \right\}. \tag{1.4}$$

If $f : \mathbb{N}_0 \to \mathbb{C}$ is an arbitrary function, then, for every $n \in \mathbb{N}_0$, the translation $T_n f : \mathbb{N}_0 \to \mathbb{C}$ is given by $T_n f(m) = \sum_{k=|m-n|}^{m+n} g(m, n; k) f(k)$; the translation operator $T_n : \mathbb{C}^{\mathbb{N}_0} \to \mathbb{C}^{\mathbb{N}_0}$ is defined by $f \mapsto T_n f$. The corresponding Haar measure, normalized such that $\{0\}$ is mapped to 1, is the counting measure on $\mathbb{N}_0$ weighted by the values of the Haar function $h : \mathbb{N}_0 \to [1, \infty)$,

$$h(n) := \frac{1}{g(n, n; 0)} = \frac{1}{\int_{\mathbb{R}} P_n^2(x)\, \mathrm{d}\mu(x)} = \begin{cases} 1, & n = 0, \\ \prod_{k=1}^{n} \frac{a_{k-1}}{c_k}, & n \in \mathbb{N}. \end{cases} \tag{1.5}$$

[2] Standard results on orthogonal polynomials can be found in [3], for instance.

[3] The full hypergroup axioms can be found in standard literature like [2]. The axioms for the special case of a discrete hypergroup are considerably simpler and can be found in [13].

A more precise formulation of this fact can be found in Eq. (1.6). The orthonormal polynomials (with positive leading coefficients) $(p_n(x))_{n\in\mathbb{N}_0}$ which correspond to $(P_n(x))_{n\in\mathbb{N}_0}$ satisfy $p_n(x) = \sqrt{h(n)}P_n(x)$ $(n \in \mathbb{N}_0)$ and are given by the recurrence relation $p_0(x) = 1$, $p_1(x) = x/\sqrt{c_1}$, $xp_n(x) = \alpha_{n+1}p_{n+1}(x) + \alpha_n p_{n-1}(x)$ $(n \in \mathbb{N})$, where $\alpha_1 = \sqrt{c_1}$ and $\alpha_n = \sqrt{c_n a_{n-1}} = \sqrt{c_n(1-c_{n-1})}$ for $n \geq 2$. If $f \in \ell^1(h) := \{f : \mathbb{N}_0 \to \mathbb{C} : \|f\|_1 < \infty\}$, $\|f\|_1 := \sum_{k=0}^{\infty} |f(k)|h(k)$, then $T_n f \in \ell^1(h)$ and

$$\sum_{k=0}^{\infty} T_n f(k)h(k) = \sum_{k=0}^{\infty} f(k)h(k) \tag{1.6}$$

for every $n \in \mathbb{N}_0$. The norm $\|.\|_1$, the convolution $(f, g) \mapsto f * g$, $f * g(n) := \sum_{k=0}^{\infty} T_n f(k)g(k)h(k)$ and complex conjugation make $\ell^1(h)$ a semisimple commutative unital Banach $*$-algebra, so the polynomials $(P_n(x))_{n\in\mathbb{N}_0}$ can be studied via methods coming from Gelfand's theory. In particular, the important property (1.3) is a consequence of functional analysis like Gelfand's theory. Recent publications deal with amenability properties of $\ell^1(h)$ [7,8]. Polynomial hypergroups are accompanied by a sophisticated harmonic analysis and Fourier analysis. The orthogonalization measure μ serves as Plancherel measure, and $\widehat{\mathbb{N}_0}$ has an important interpretation as a dual object: let

$$\mathcal{X}^b(\mathbb{N}_0) := \left\{ z \in \mathbb{C} : \max_{n\in\mathbb{N}_0} |P_n(z)| = 1 \right\}.$$

Via the homeomorphism $\mathcal{X}^b(\mathbb{N}_0) \to \Delta(\ell^1(h))$, $z \mapsto \varphi_z$ with

$$\varphi_z(f) := \sum_{k=0}^{\infty} f(k)\overline{P_k(z)}h(k) \; (f \in \ell^1(h)),$$

the compact set $\mathcal{X}^b(\mathbb{N}_0)$ can be identified with $\Delta(\ell^1(h))$, and $\widehat{\mathbb{N}_0} = \mathcal{X}^b(\mathbb{N}_0) \cap \mathbb{R}$ can be identified with the Hermitian structure space $\Delta_s(\ell^1(h))$. The following result [18,21,22] is essential:

Theorem 1.1. *If h is of subexponential growth (i.e., for all $\epsilon > 0$ there is some $M > 0$ such that $h(n) \leq M(1+\epsilon)^n$ for all $n \in \mathbb{N}_0$), then* supp μ, *$\widehat{\mathbb{N}_0}$ and $\mathcal{X}^b(\mathbb{N}_0)$ coincide.*

Since $g(n, n; 0)$ and $g(n, n; 2n)$ are nonzero and $\sum_{k=0}^{2n} g(n, n; k) = 1$, nonnegative linearization of products always implies that $h(n) = 1/g(n, n; 0) > 1$ for all $n \in \mathbb{N}$. Studying various examples, we observed that all of them satisfied the stronger property $h(n) \geq 2$ $(n \in \mathbb{N})$. The paper is devoted to questions concerning this eye-catching observation.

1.2. Motivation and outline of the paper

To start with, we give an additional and more detailed motivation for the problem: we are not aware of any convenient characterization of the crucial nonnegative linearization of products property (in terms of the recurrence coefficients $(a_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$, in terms of the orthogonalization measure μ etc.). However, there are several sufficient criteria, starting with results of Askey [1] and continued by Szwarc et al. in a series of papers. One of these criteria [17, Theorem 1 p. 966] reads as follows:

Theorem 1.2. *If $(c_n)_{n\in\mathbb{N}}$ is bounded from above by $1/2$ and both $(c_{2n-1})_{n\in\mathbb{N}}$ and $(c_{2n})_{n\in\mathbb{N}}$ are nondecreasing, then nonnegative linearization of products is satisfied.*

Now if $(c_n)_{n\in\mathbb{N}}$ is bounded from above by $1/2$ (and thus $(a_n)_{n\in\mathbb{N}}$ is bounded from below by $1/2$) like in Theorem 1.2, then it is clear that indeed $h(n) \geq 2$ for all $n \in \mathbb{N}$ (recall that $a_0 = 1$). Therefore, it is at least not surprising that many examples satisfy this property (particularly all examples constructed via Theorem 1.2). Recently, Kahler successfully applied Theorem 1.2 to the large class of associated symmetric Pollaczek polynomials (with monotonicity of the whole sequence $(c_n)_{n\in\mathbb{N}}$) [7].

In [6], Kahler recently found the following example which, for certain choices of the parameters, satisfies nonnegative linearization of products without fulfilling the conditions of Theorem 1.2: for any $\alpha, \beta > -1$, let the sequence of generalized Chebyshev polynomials $(T_n^{(\alpha,\beta)}(x))_{n\in\mathbb{N}_0} \subseteq \mathbb{R}[x]$ be given by $c_{2n-1} = (n+\beta)/(2n+\alpha+\beta)$, $c_{2n} = n/(2n+\alpha+\beta+1)$. These polynomials are the quadratic transformations of the Jacobi polynomials, and one has $[-1, 1] = \operatorname{supp}\mu = \widehat{\mathbb{N}_0}$ [3, Chapter V 2 (G)] [11, 3 (f)]. The generalized Chebyshev polynomials are of particular interest concerning product formulas and duality structures [10,11]. In [6, Theorem 3.2], Kahler showed that $(T_n^{(\alpha,\beta)}(x))_{n\in\mathbb{N}_0}$ satisfies nonnegative linearization of products if and only if (α, β) is an element of the set $V \subseteq [-1/2, \infty) \times (-1, \infty)$ given by

$$V := \left\{(\alpha, \beta) \in (-1, \infty)^2 : \alpha \geq \beta, a(a + 5)(a + 3)^2 \geq (a^2 - 7a - 24)b^2\right\},$$

where $a := \alpha + \beta + 1$ and $b := \alpha - \beta$.[4] The progress of this contribution compared to older results concerns the case $(\alpha, \beta) \in V$ with $\alpha+\beta+1 < 0$ because the conditions of Theorem 1.2 are satisfied if and only if $(\alpha, \beta) \in V$ with $\alpha+\beta+1 \geq 0$. If $(\alpha, \beta) \in V$ but $\alpha+\beta+1 < 0$, then $(c_{2n})_{n\in\mathbb{N}}$ is strictly decreasing and always greater than $1/2$. Nevertheless, elementary calculus and explicit formulas for h [11, 3 (f)] imply that still $h(n) \geq 2$ for all $n \in \mathbb{N}$.

These observations yield the questions whether $h(n) \geq 2$ $(n \in \mathbb{N})$ is true for *every* sequence $(P_n(x))_{n\in\mathbb{N}_0}$ which satisfies nonnegative linearization of products and whether maximal dual spaces $\widehat{\mathbb{N}_0} = [-1, 1]$ (as satisfied by the generalized Chebyshev polynomials) play a more general role. In Section 2, we give sufficient criteria which cover many naturally occurring examples, including the generalized Chebyshev polynomials (also those with $\alpha + \beta + 1 < 0$ considered above). Concerning these criteria, we will discuss the role of nonnegative linearization of products, and we will consider the example of Grinspun polynomials. Moreover, in Section 3 we show that there are also counterexamples. To our knowledge, these are the first examples with $h(1) < 2$. For every $\epsilon \in (0, 1)$, we will construct two types of polynomial hypergroups with $h(1) = 1 + \epsilon$. The problem under consideration is also interesting for the following reason: for the well-known Chebyshev polynomials of the first kind, which play a fundamental role in asymptotics and optimization, $h(n)$ *equals* 2 for all $n \in \mathbb{N}$. Hence, our results show that under a large class of naturally occurring examples the Chebyshev polynomials of the first kind are optimal w.r.t. minimizing the Haar function—however, they are not optimal among all possible examples. Finally, Section 4 is devoted to some open problems.

We remark that we used computer algebra systems (Maple) to find suitable decompositions of long expressions, find explicit formulas, get conjectures and so on. The final proofs can be understood without any computer usage, however.

2. Sufficient criteria for $h(n) \geq 2$ $(n \in \mathbb{N})$, the role of the dual space and the role of nonnegative linearization of products

In this section, we give some sufficient criteria for $h(n) \geq 2$ $(n \in \mathbb{N})$. They do not rely on boundedness properties of $(c_n)_{n\in\mathbb{N}}$, and they particularly cover examples where $(c_n)_{n\in\mathbb{N}}$

[4] This is the analogue to a well-known result of Gasper on the (nonsymmetric) class of Jacobi polynomials [5, Theorem 1].

exceeds 1/2 as considered in Section 1. The dual space $\widehat{\mathbb{N}_0}$ will play a crucial role. Our approach is based on the connection coefficients to the Chebyshev polynomials of the first kind $(T_n(x))_{n\in\mathbb{N}_0}$ (which, in terms of the generalized Chebyshev polynomials recalled above, are just $(T_n^{(-1/2,-1/2)}(x))_{n\in\mathbb{N}_0}$): given an orthogonal polynomial sequence $(P_n(x))_{n\in\mathbb{N}_0}$ as in Section 1, let $C_n(0),\ldots,C_n(n)$ be defined by the expansions

$$P_n(x)=\sum_{k=0}^{n}C_n(k)T_k(x).$$

It is clear that $C_n(n)\neq 0$. We need the following classical estimation result from Chebyshev theory [20, Theorem (3.1)]:

Lemma 2.1. *Let $P(x)\in\mathbb{R}[x]$ be a polynomial of degree $n\in\mathbb{N}$ with leading coefficient* 1. *Then* $\max_{x\in[-1,1]}|P(x)|\geq 1/2^{n-1}$*, and equality holds if and only if* $P(x)=T_n(x)/2^{n-1}$.

In the following, we always assume that $(P_n(x))_{n\in\mathbb{N}_0}$ satisfies nonnegative linearization of products. The following theorem is the central result of this section.

Theorem 2.1. *Let the dual space $\widehat{\mathbb{N}_0}$ coincide with the full interval* $[-1,1]$. *Then $h(n)\geq 2$ for all $n\in\mathbb{N}$.*

Proof. Let $n\in\mathbb{N}\backslash\{1\}$ and expand $P_n(x)=\sum_{k=0}^{n}C_n(k)T_k(x)$. Since $\widehat{\mathbb{N}_0}=[-1,1]$, by Lemma 2.1 we have

$$1=\max_{x\in[-1,1]}|P_n(x)|=C_n(n)\max_{x\in[-1,1]}\left|\sum_{k=0}^{n}\frac{C_n(k)}{C_n(n)}T_k(x)\right|\geq C_n(n).$$

Since the leading coefficient of $P_n(x)$ is $1/\prod_{k=1}^{n-1}a_k$ and the leading coefficient of $T_n(x)$ is 2^{n-1}, we get $1/\prod_{k=1}^{n-1}a_k=C_n(n)\cdot 2^{n-1}\leq 2^{n-1}$ and consequently

$$4^{n-1}\prod_{k=1}^{n-1}a_k^2\geq 1.$$

Moreover, by (1.5) we have

$$h(n)=\frac{1}{c_1}\prod_{k=2}^{n}\frac{a_{k-1}}{c_k}=\frac{1}{c_n}\prod_{k=1}^{n-1}\frac{a_k}{c_k}=\frac{1}{c_n}\prod_{k=1}^{n-1}\frac{a_k^2}{c_k(1-c_k)}.$$

Since $c_k(1-c_k)\leq 1/4$ for all $k\in\{1,\ldots,n-1\}$, we now obtain $h(n)\geq 1/c_n\cdot 4^{n-1}\prod_{k=1}^{n-1}a_k^2\geq 1/c_n$. Therefore, for every $n\in\mathbb{N}$ we have both $1\leq c_nh(n)$ (with equality for $n=1$) and $1\leq c_{n+1}h(n+1)=a_nh(n)$ (the latter equality follows from (1.5)), so $2\leq c_nh(n)+a_nh(n)=h(n)$. $\square$

Concerning the applicability of Theorem 2.1, we mention that the condition $\widehat{\mathbb{N}_0}=[-1,1]$ is fulfilled by an abundance of examples (see [2,11–13], for instance). We now give several corollaries.

Corollary 2.1. *If all connection coefficients $C_n(0),\ldots,C_n(n)$ are nonnegative, then $h(n)\geq 2$ for all $n\in\mathbb{N}$.*

Proof. As the connection coefficients $C_n(0), \ldots, C_n(n)$ sum up to 1, the presumed nonnegativity allows to conclude in two ways: either obtain that $\widehat{\mathbb{N}_0} = [-1, 1]$ as an immediate consequence and apply Theorem 2.1, or just use that the assumption particularly yields $C_n(n) \leq 1$ and proceed as in the proof of Theorem 2.1; the latter way avoids Lemma 2.1. □

Corollary 2.2. *If there exists a function* $g : [-1, 1] \to [0, \infty)$ *such that* $|P_n(x)| \leq g(x)$ *for all* $x \in [-1, 1]$ *and for all* $n \in \mathbb{N}_0$*, then* $h(n) \geq 2$ *for all* $n \in \mathbb{N}$.

Proof. It is a general result on polynomial hypergroups and their harmonic/functional analysis that the existence of such a function g implies that $(P_n(x))_{n\in\mathbb{N}_0}$ is uniformly bounded on $[-1, 1]$ by ± 1; one always has

$$\left\{x \in \mathbb{R} : \sup_{n\in\mathbb{N}_0} |P_n(x)| < \infty\right\} = \widehat{\mathbb{N}_0} \tag{2.1}$$

[13]. Now Theorem 2.1 yields the assertion. □

Corollary 2.3. *If* $\operatorname{supp} \mu = [-a, a]$ *for some* $a \in (0, 1]$*, then* $h(n) \geq 2$ *for all* $n \in \mathbb{N}$.

Proof. If $\operatorname{supp} \mu = [-a, a]$ for some $a \in (0, 1]$, then $[-a, a] \subseteq \widehat{\mathbb{N}_0}$ due to (1.3). Since $P_n(1) \equiv 1$ and since the zeros of the polynomials $(P_n(x))_{n\in\mathbb{N}_0}$ are real, simple and located in $(-a, a)$, every $P_n(x)$ is positive and nondecreasing on $[a, 1]$. This shows that also $(a, 1] \subseteq \widehat{\mathbb{N}_0}$. Finally, by symmetry we can conclude that $\widehat{\mathbb{N}_0} = [-1, 1]$. Hence, the assertion follows from Theorem 2.1. □

Corollary 2.4. *If* $(c_n)_{n\in\mathbb{N}}$ *is convergent, then* $h(n) \geq 2$ *for all* $n \in \mathbb{N}$.

Proof. If $(c_n)_{n\in\mathbb{N}}$ is convergent, then the limit c is an element of $(0, 1/2]$ and $\operatorname{supp} \mu = [-2\sqrt{c(1-c)}, 2\sqrt{c(1-c)}]$ due to [12, Theorem (2.2)], so the assertion follows from Corollary 2.3. Alternatively, one can obtain the result from Corollary 2.1: by [12, Theorem (2.6)] or [14, Corollary 2], all connection coefficients $C_n(0), \ldots, C_n(n)$ are nonnegative. □

Note that the formal definitions of $\widehat{\mathbb{N}_0}$ (1.4) and h (1.5) also make sense if nonnegative linearization of products is not satisfied (and hence without the underlying hypergroup structure). With regard to (1.3), we note that it is obvious that still $\{\pm 1\} \subseteq \widehat{\mathbb{N}_0} \subseteq [-1, 1]$. However, the property $\operatorname{supp} \mu \subseteq \widehat{\mathbb{N}_0}$, which is a consequence of harmonic/functional analysis on polynomial hypergroups, does no longer have to be satisfied. Furthermore, h can now map into the larger codomain $(0, \infty)$. The rest of the section is devoted to the question which of our results remain true under these more general conditions. The proof of Theorem 2.1 remains fully true if the nonnegative linearization of products condition is dropped, as well as the proof of Corollary 2.1. The following example shows that the further corollaries do not extend if the nonnegative linearization of products condition is dropped (and therefore the tool of harmonic/functional analysis on polynomial hypergroups is no longer available), however.

Example 2.1 (*Grinspun Polynomials*)**.** Let $c_1 \in (1/2, 1)$ be arbitrary, and let $c_n = 1/2$ for every $n \geq 2$. The resulting polynomials $(P_n(x))_{n\in\mathbb{N}_0}$ are the Grinspun polynomials and orthogonal w.r.t. a measure μ with $\operatorname{supp} \mu = [-1, 1]$ [3, Chapter VI 13 (C) (iv)]. Via induction and the recurrence relation of the Chebyshev polynomials of the first kind, it is easy to see that

$$P_n(x) = \frac{1}{2-2c_1} T_n(x) + \frac{1-2c_1}{2-2c_1} T_{n-2}(x) \; (n \geq 2) \tag{2.2}$$

and therefore

$$P_n(x) = T_n(x) + \frac{2c_1 - 1}{2 - 2c_1}(T_n(x) - T_{n-2}(x)) \ (n \geq 2) \tag{2.3}$$

(cf. also [3, VI-(13.9)] and [23, Section 3.2]). The expansions (2.2) imply that $(P_n(x))_{n\in\mathbb{N}_0}$ is uniformly bounded on $[-1, 1]$ by $\pm c_1/(1-c_1)$, but $h(1) = 1/c_1 < 2$ and $h(n) = 2\cdot(1-c_1)/c_1 < 2$ $(n \geq 2)$ by (1.5). This shows that Corollary 2.2 is not valid without nonnegative linearization of products; if $c_1 > 2/3$, then h does not even map to $[1, \infty)$. Reconsidering the proof of Corollary 2.2, we see that (2.1) made use of nonnegative linearization of products (and of the resulting harmonic/functional analysis due to the hypergroup aspect). Clearly, the example also shows that Corollaries 2.3 and 2.4 are not valid without nonnegative linearization of products. It is already clear from the preceding considerations that neither Corollary 2.1 nor Theorem 2.1 can apply, and one can see from (2.2) and (2.3) at which stages an application exactly fails: (2.2) yields that $C_n(n-2) < 0$ for all $n \geq 2$. Moreover, one has $\widehat{\mathbb{N}_0} = \{\pm 1\}$, which can be seen as follows: let $x \in (-1, 1)$ and $\varphi \in (0, \pi)$ with $x = \cos(\varphi)$. Then, by (2.3),

$$\begin{aligned} P_n(x) &= T_n(\cos(\varphi)) + \frac{2c_1 - 1}{2 - 2c_1}(T_n(\cos(\varphi)) - T_{n-2}(\cos(\varphi))) \\ &= \cos(n\varphi) + \frac{2c_1 - 1}{2 - 2c_1}(\cos(n\varphi) - \cos((n-2)\varphi)) \\ &= \cos(n\varphi) + \frac{2c_1 - 1}{2 - 2c_1}(1 - \cos(2\varphi))\cos(n\varphi) - \frac{2c_1 - 1}{2 - 2c_1}\sin(2\varphi)\sin(n\varphi) \end{aligned}$$

for every $n \geq 2$. Now let $(n_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}\backslash\{1\}$ be a sequence with $\lim_{k\to\infty}\cos(n_k\varphi) = 1$ (and consequently $\lim_{k\to\infty}\sin(n_k\varphi) = 0$). Then $\lim_{k\to\infty} P_{n_k}(x) = 1 + (2c_1 - 1)/(2 - 2c_1)\cdot(1 - \cos(2\varphi)) > 1$ and we can conclude that $x \notin \widehat{\mathbb{N}_0}$.

3. Two types of examples which do not satisfy $h(n) \geq 2$ ($n \in \mathbb{N}$) and properties of their dual spaces

Having seen sufficient criteria for $h(n) \geq 2$ ($n \in \mathbb{N}$) in the previous section, we now construct examples where nonnegative linearization of products is satisfied but $h(1) < 2$. Moreover, we deal with a necessary criterion concerning the latter property:

Proposition 3.1. *If $h(1) = 1 + \epsilon$ with $\epsilon \in (0, 1)$, then*

$$\widehat{\mathbb{N}_0} \subseteq \left[-1, -\sqrt{\frac{1-\epsilon}{1+\epsilon}}\right] \cup \left[\sqrt{\frac{1-\epsilon}{1+\epsilon}}, 1\right]. \tag{3.1}$$

Proof. If $h(1) = 1 + \epsilon$ with $\epsilon \in (0, 1)$, then, by (1.5),

$$P_2(x) = \frac{x^2 - c_1}{1 - c_1} = \frac{h(1)x^2 - 1}{h(1) - 1} = \frac{(1+\epsilon)x^2 - 1}{\epsilon},$$

so $P_2(\pm\sqrt{(1-\epsilon)/(1+\epsilon)}) = -1$. Therefore, we have $P_2(x) < -1$ for $x \in (-\sqrt{(1-\epsilon)/(1+\epsilon)}, \sqrt{(1-\epsilon)/(1+\epsilon)})$, which yields the assertion. □

Remark 3.1. As a much less trivial result, in Theorem 3.1 we will obtain that there are examples which satisfy (3.1) with *equality*; so the estimation provided by Proposition 3.1 cannot be improved.

We use the Karlin–McGregor polynomials as a starting point in order to construct polynomial hypergroups with $h(1) < 2$. For $\alpha, \beta \geq 2$, the Karlin–McGregor polynomials $(K_n^{(\alpha,\beta)}(x))_{n\in\mathbb{N}_0} \subseteq \mathbb{R}[x]$ are given by $c_{2n-1} = 1/\alpha$ and $c_{2n} = 1/\beta$ [13, Sect. 6]. For any choice of $\alpha, \beta \geq 2$, $(K_n^{(\alpha,\beta)}(x))_{n\in\mathbb{N}_0}$ fulfills the conditions of Theorem 1.2, so nonnegative linearization of products is always satisfied and $h(n) \geq 2$ $(n \in \mathbb{N})$. Nevertheless, a modification of the Karlin–McGregor polynomials will yield examples which fulfill the desired property $h(1) < 2$ (see Theorem 3.1). We first recall some basics about the Karlin–McGregor polynomials. One has

$$\operatorname{supp} \mu = \begin{cases} [-\gamma_1, -\gamma_2] \cup [\gamma_2, \gamma_1], & \alpha \leq \beta, \\ [-\gamma_1, -\gamma_2] \cup \{0\} \cup [\gamma_2, \gamma_1], & \alpha > \beta \end{cases}$$

with

$$\gamma_1 := \frac{1}{\sqrt{\alpha\beta}}(\sqrt{\alpha - 1} + \sqrt{\beta - 1}),\ \gamma_2 := \frac{1}{\sqrt{\alpha\beta}}|\sqrt{\alpha - 1} - \sqrt{\beta - 1}|$$

[9]. It is obvious from (1.5) that the Haar weights are given by $h(0) = 1$ and

$$h(2n - 1) = \alpha(\alpha - 1)^{n-1}(\beta - 1)^{n-1} \tag{3.2}$$

and

$$h(2n) = \beta(\alpha - 1)^n(\beta - 1)^{n-1} \tag{3.3}$$

for $n \in \mathbb{N}$ [13, Sect. 6]. Moreover, it is easy to see via induction that

$$K_{2n}^{(\alpha,\beta)}(\gamma_1) = \frac{(\alpha - 2)\sqrt{\beta - 1} + (\beta - 2)\sqrt{\alpha - 1}}{\beta(\alpha - 1)^{\frac{n+1}{2}}(\beta - 1)^{\frac{n}{2}}} \cdot n + \frac{1}{(\alpha - 1)^{\frac{n}{2}}(\beta - 1)^{\frac{n}{2}}} \tag{3.4}$$

and

$$K_{2n+1}^{(\alpha,\beta)}(\gamma_1) = \frac{(\alpha - 2)\sqrt{\beta - 1} + (\beta - 2)\sqrt{\alpha - 1}}{\sqrt{\alpha\beta}(\alpha - 1)^{\frac{n+1}{2}}(\beta - 1)^{\frac{n+1}{2}}} \cdot n + \frac{\sqrt{\alpha - 1} + \sqrt{\beta - 1}}{\sqrt{\alpha\beta}(\alpha - 1)^{\frac{n}{2}}(\beta - 1)^{\frac{n}{2}}} \tag{3.5}$$

for all $n \in \mathbb{N}_0$.

We now can find examples which satisfy $h(1) < 2$ or even $h(1) = 1 + \epsilon$ with $\epsilon \in (0, 1)$ (and nonnegative linearization of products). We start with the Karlin–McGregor polynomials $(K_n^{(\alpha,\beta)}(x))_{n\in\mathbb{N}_0}$, $\alpha, \beta \geq 2$. Next, we rescale them in such a way that the right endpoint of the support of the measure becomes 1. Finally, we renormalize the resulting polynomials in such a way that $P_n(1) \equiv 1$ again. This procedure ends up in the sequence $(P_n(x))_{n\in\mathbb{N}_0} \subseteq \mathbb{R}[x]$ of modified Karlin–McGregor polynomials given by

$$P_n(x) = \frac{K_n^{(\alpha,\beta)}(\gamma_1 x)}{K_n^{(\alpha,\beta)}(\gamma_1)},$$

and $(P_n(x))_{n\in\mathbb{N}_0}$ still satisfies nonnegative linearization of products. The above-mentioned examples with $h(1) = 1 + \epsilon$ will be obtained below for suitable choices of α and β. We first study the polynomials $(P_n(x))_{n\in\mathbb{N}_0}$ in detail and compute the associated Haar measures and recurrence coefficients. By construction, the Haar weights corresponding to the modified polynomials $(P_n(x))_{n\in\mathbb{N}_0}$ and the Haar weights corresponding to the original Karlin–McGregor polynomials $(K_n^{(\alpha,\beta)}(x))_{n\in\mathbb{N}_0}$ are linked to each other by multiplication with $(K_n^{(\alpha,\beta)}(\gamma_1))^2$. Using (3.2) to (3.5), we obtain that the Haar weights associated with $(P_n(x))_{n\in\mathbb{N}_0}$ satisfy $h(0) = 1$

and

$$h(2n-1)=\frac{1}{\beta}\left[\left(\frac{\alpha-2}{\sqrt{\alpha-1}}+\frac{\beta-2}{\sqrt{\beta-1}}\right)\cdot(n-1)+\sqrt{\alpha-1}+\sqrt{\beta-1}\right]^2 \tag{3.6}$$

and

$$h(2n)=\frac{1}{\beta}\left[\left(\frac{\alpha-2}{\sqrt{\alpha-1}}+\frac{\beta-2}{\sqrt{\beta-1}}\right)\cdot n+\frac{\beta}{\sqrt{\beta-1}}\right]^2 \tag{3.7}$$

for $n\in\mathbb{N}$. Observe that h is always of quadratic (and therefore subexponential) growth, which particularly implies that

$$\widehat{\mathbb{N}_0}=\operatorname{supp}\mu=\begin{cases}\left[-1,-\frac{\gamma_2}{\gamma_1}\right]\cup\left[\frac{\gamma_2}{\gamma_1},1\right], & \alpha\leq\beta,\\ \left[-1,-\frac{\gamma_2}{\gamma_1}\right]\cup\{0\}\cup\left[\frac{\gamma_2}{\gamma_1},1\right], & \alpha>\beta\end{cases} \tag{3.8}$$

as a consequence of Theorem 1.1. Via (1.5), (3.6) and (3.7), we can recursively compute the recurrence coefficients $(c_n)_{n\in\mathbb{N}}$ which correspond to the modified polynomials $(P_n(x))_{n\in\mathbb{N}_0}$. Alternatively, one can compute $(c_n)_{n\in\mathbb{N}}$ from (3.4) and (3.5) because the recurrence coefficients are linked to those belonging to $(K_n^{(\alpha,\beta)}(x))_{n\in\mathbb{N}_0}$ by multiplication with $K_{n-1}^{(\alpha,\beta)}(\gamma_1)/(\gamma_1K_n^{(\alpha,\beta)}(\gamma_1))$. We obtain

$$\begin{aligned}&\frac{\sqrt{\alpha-1}+\sqrt{\beta-1}}{\sqrt{\beta-1}}c_{2n-1}\\&=1-\sqrt{\alpha-1}\cdot\frac{\sqrt{\alpha-1}\sqrt{\beta-1}-1}{((\alpha-2)\sqrt{\beta-1}+(\beta-2)\sqrt{\alpha-1})\cdot n+\sqrt{\alpha-1}+\sqrt{\beta-1}},\\&\frac{\sqrt{\alpha-1}+\sqrt{\beta-1}}{\sqrt{\alpha-1}}c_{2n}\\&=1-\sqrt{\beta-1}\cdot\frac{\sqrt{\alpha-1}\sqrt{\beta-1}-1}{((\alpha-2)\sqrt{\beta-1}+(\beta-2)\sqrt{\alpha-1})\cdot n+\beta\sqrt{\alpha-1}}.\end{aligned}$$

For every $n\in\mathbb{N}$, we compute

$$\alpha_n=\begin{cases}\frac{\sqrt{\beta}}{\sqrt{\alpha-1}+\sqrt{\beta-1}}, & n=1,\\ \frac{\sqrt{\alpha-1}}{\sqrt{\alpha-1}+\sqrt{\beta-1}}, & n \text{ even},\\ \frac{\sqrt{\beta-1}}{\sqrt{\alpha-1}+\sqrt{\beta-1}}, & \text{else},\end{cases}$$

so the coefficients in the orthonormal normalization become periodic. This shows that $(P_n(x))_{n\in\mathbb{N}_0}$ belongs to the class of Geronimus polynomials [15] and that nonnegative linearization of products also follows directly from a general criterion in [19] (without using nonnegative linearization of products for the Karlin–McGregor polynomials): if $\alpha\leq\beta$, then [19, Theorem 3 (i)] can be applied, and if $\alpha>\beta$, then [19, Theorem 3 (ii) combined with Remark 3] works. For the special case $\alpha=\beta$, nonnegative linearization of products also follows from [11, 3 (g) (i)]. Coming back to the problem "$h(1)<2$", from (3.6) we have

$$h(1)=\alpha\gamma_1^2=\frac{1}{\beta}(\sqrt{\alpha-1}+\sqrt{\beta-1})^2. \tag{3.9}$$

In particular, one has $h(1)<2$ if and only if $\alpha<3\beta-2\sqrt{2\beta^2-2\beta}$.

Theorem 3.1. *Let $\alpha,\beta\geq 2$, and let $P_n(x)=K_n^{(\alpha,\beta)}(\gamma_1x)/K_n^{(\alpha,\beta)}(\gamma_1)$ ($n\in\mathbb{N}_0$).*

(i) For every $\epsilon \in (0, 1)$, there exists a polynomial hypergroup on $\mathbb{N}_0$ such that $h(1) = 1+\epsilon$. More precisely, for any choice of α the parameter β can be chosen in such a way that the hypergroup induced by the sequence $(P_n(x))_{n\in\mathbb{N}_0}$ has the desired property; for this example type, the dual space $\widehat{\mathbb{N}_0}$ is of the form $[-1, -1+\delta] \cup [1-\delta, 1]$ with $\delta \in (0, 1-\sqrt{(1-\epsilon)/(1+\epsilon)}]$. Furthermore, for $\alpha = 2$ and $\beta = (2+2\sqrt{1-\epsilon^2})/\epsilon^2$ one additionally has that $\widehat{\mathbb{N}_0}$ equals the maximal possible set $[-1, -\sqrt{(1-\epsilon)/(1+\epsilon)}] \cup [\sqrt{(1-\epsilon)/(1+\epsilon)}, 1]$ (cf. Proposition 3.1 and Remark 3.1).

(ii) For any choice of α, β, the polynomial hypergroup induced by $(P_n(x))_{n\in\mathbb{N}_0}$ satisfies $h(n) \geq 2$ for all $n \geq 2$. Moreover, h is nondecreasing and of quadratic growth.

Proof.

(i) Let $\epsilon \in (0, 1)$, and let $\alpha \geq 2$ be arbitrary. Then, by (3.9), $h(1) \to 1$ ($\beta \to \infty$). This yields that β can be chosen such that $h(1) = 1+\epsilon$. By (3.8) and Proposition 3.1, $\widehat{\mathbb{N}_0} = \operatorname{supp}\mu = [-1, -1+\delta] \cup [1-\delta, 1]$ with $\delta \in (0, 1-\sqrt{(1-\epsilon)/(1+\epsilon)}]$. Now let $\alpha = 2$ and $\beta = (2+2\sqrt{1-\epsilon^2})/\epsilon^2$. Then $h(1) = 1+\epsilon$ by (3.9) and, by (3.8),

$$\widehat{\mathbb{N}_0} = \operatorname{supp}\mu = \left[-1, -\frac{\gamma_2}{\gamma_1}\right] \cup \left[\frac{\gamma_2}{\gamma_1}, 1\right] = \left[-1, -\sqrt{\frac{1-\epsilon}{1+\epsilon}}\right] \cup \left[\sqrt{\frac{1-\epsilon}{1+\epsilon}}, 1\right].$$

(ii) For every $n \in \mathbb{N}$, the explicit formulas (3.6) and (3.7) for h yield

$$\sqrt{\frac{h(2n)}{h(2n-1)}} - 1 = \sqrt{\beta-1} \cdot \frac{\sqrt{\alpha-1}\sqrt{\beta-1} - 1}{((\alpha-2)\sqrt{\beta-1} + (\beta-2)\sqrt{\alpha-1}) \cdot n + \sqrt{\alpha-1} + \sqrt{\beta-1}}$$

and

$$\sqrt{\frac{h(2n+1)}{h(2n)}} - 1 = \sqrt{\alpha-1} \cdot \frac{\sqrt{\alpha-1}\sqrt{\beta-1} - 1}{((\alpha-2)\sqrt{\beta-1} + (\beta-2)\sqrt{\alpha-1}) \cdot n + \beta\sqrt{\alpha-1}}$$

for every $n \in \mathbb{N}$. Since the right-hand sides are nonnegative, this shows that h is nondecreasing. Hence, by (3.7) we have

$$h(n) \geq h(2) \geq \frac{1}{\beta}\left[\frac{\beta-2}{\sqrt{\beta-1}} + \frac{\beta}{\sqrt{\beta-1}}\right]^2 = 4 \cdot \frac{\beta-1}{\beta} \geq 2$$

for all $n \geq 2$. We have already observed that h is of quadratic growth. □

Corollary 3.1. *The converse of Theorem 2.1 and the converses of Corollary 2.1 to Corollary 2.4 are not true.*

Proof. Let $\beta \geq 2$ and $\alpha \geq 3\beta - 2\sqrt{2\beta^2 - 2\beta}$ with $\alpha \neq \beta$, and let $P_n(x) = K_n^{(\alpha,\beta)}(\gamma_1 x)/K_n^{(\alpha,\beta)}(\gamma_1)$ ($n \in \mathbb{N}_0$). Then, as consequence of the preceding observations and (3.8), we have $h(n) \geq 2$ for all $n \in \mathbb{N}$ but $\widehat{\mathbb{N}_0} \neq [-1, 1]$ (in fact, for $\alpha \in [3\beta - 2\sqrt{2\beta^2 - 2\beta}, \beta)$ we do not even have $0 \in \widehat{\mathbb{N}_0}$). This shows that the converse of Theorem 2.1 is not true. The latter implies that the converses of Corollary 2.1 to Corollary 2.4 are also not true. □

We finally construct another type of polynomial hypergroups with $h(1) < 2$ (and even "$h(1) = 1 + \epsilon$"). It does not rely on the Karlin–McGregor polynomials; the dual space $\widehat{\mathbb{N}_0}$ is discrete, and h is of exponential growth now.

Theorem 3.2. *Let $(\lambda_n)_{n\in\mathbb{N}_0} \subseteq (0,1)$ with $\lim_{n\to\infty} \lambda_{2n} = 1$ satisfy both $\lambda_{2n-2}+\lambda_{2n-1} \le \lambda_{2n}$ and $\lambda_{2n-1} + \lambda_{2n} \le \lambda_{2n+2}$ for every $n \in \mathbb{N}$, and let $(Q_n(x))_{n\in\mathbb{N}_0} \subseteq \mathbb{R}[x]$ be given by the recurrence relation $Q_0(x) = 1$, $Q_1(x) = x/\lambda_0$, $xQ_n(x) = \lambda_n Q_{n+1}(x) + \lambda_{n-1}Q_{n-1}(x)$ $(n \in \mathbb{N})$. The following hold:*

(i) The sequence $(Q_n(1))_{n\in\mathbb{N}_0}$ is strictly positive and strictly increasing.
(ii) The sequence $(P_n(x))_{n\in\mathbb{N}_0}$ defined by $P_n(x) := Q_n(x)/Q_n(1)$ $(n \in \mathbb{N}_0)$ satisfies nonnegative linearization of products, and $(Q_n(x))_{n\in\mathbb{N}_0}$ are the orthonormal polynomials which correspond to $(P_n(x))_{n\in\mathbb{N}_0}$.
(iii) The dual space $\widehat{\mathbb{N}_0}$ satisfies $\widehat{\mathbb{N}_0} = \operatorname{supp}\mu = \{\pm 1\} \cup \{\pm x_n : n \in \mathbb{N}\}$ with a strictly increasing sequence $(x_n)_{n\in\mathbb{N}} \subseteq (0,1)$ with $\lim_{n\to\infty} x_n = 1$.
(iv) For every $\epsilon \in (0,1)$, the sequence $(\lambda_n)_{n\in\mathbb{N}_0}$ can be chosen in such a way that the polynomial hypergroup induced by $(P_n(x))_{n\in\mathbb{N}_0}$ fulfills $h(1) = 1+\epsilon$; in that case, one has $(x_n)_{n\in\mathbb{N}} \subseteq [\sqrt{(1-\epsilon)/(1+\epsilon)}, 1)$. An explicit construction is as follows: let $(s_n)_{n\in\mathbb{N}_0} \subseteq (0,1)$ be any null sequence which is convex (i.e., $s_{n+1} \le (s_n + s_{n+2})/2$ for all $n \in \mathbb{N}_0$). Then the sequence $(\lambda_n)_{n\in\mathbb{N}_0} \subseteq \mathbb{R}$ given by

$$\lambda_n := \begin{cases} 1 - s_{\frac{n}{2}}, & n \text{ even}, \\ s_{\frac{n+1}{2}} - s_{\frac{n+3}{2}}, & n \text{ odd} \end{cases}$$

satisfies the conditions above, and if $s_0 = 1 - 1/\sqrt{1+\epsilon}$, then $h(1) = 1+\epsilon$.[5]
(v) For any choice of $(\lambda_n)_{n\in\mathbb{N}_0}$, the polynomial hypergroup induced by $(P_n(x))_{n\in\mathbb{N}_0}$ satisfies $h(n) > 4$ for all $n \ge 2$. Moreover, h is strictly increasing and of exponential growth.

Proof.

(i) For every $n \in \mathbb{N}$, we compute

$$Q_{n+1}(1) = Q_n(1) + \underbrace{\frac{1-\lambda_{n-1}-\lambda_n}{\lambda_n}}_{>0} Q_n(1) + \underbrace{\frac{\lambda_{n-1}}{\lambda_n}}_{>0}(Q_n(1) - Q_{n-1}(1)).$$

Since $Q_0(1) = 1$ and $Q_1(1) = 1/\lambda_0 > 1$, this yields the assertion.

(ii) As a consequence of (i), $(P_n(x))_{n\in\mathbb{N}_0}$ is well-defined. By [16, Corollary 2 (ii)], $(Q_n(x))_{n\in\mathbb{N}_0}$ satisfies nonnegative linearization of products. Hence, (i) implies that $(P_n(x))_{n\in\mathbb{N}_0}$ satisfies nonnegative linearization of products, too. It is clear from the recurrence relations that $(Q_n(x))_{n\in\mathbb{N}_0}$ are the orthonormal polynomials which correspond to $(P_n(x))_{n\in\mathbb{N}_0}$.

(iii) As a consequence of [16, Remark 2 p. 427], there exists a strictly increasing sequence $(x_n)_{n\in\mathbb{N}} \subseteq [0,1)$ with $\lim_{n\to\infty} x_n = 1$ and

$$\operatorname{supp}\mu = \{\pm 1\} \cup \{\pm x_n : n \in \mathbb{N}\}.$$

It remains to show that $\widehat{\mathbb{N}_0} = \operatorname{supp}\mu$ and $0 \notin \widehat{\mathbb{N}_0}$. This can be seen as follows: let $(R_n(x))_{n\in\mathbb{N}_0} \subseteq \mathbb{R}[x]$ be defined by $R_n(x^2) = P_{2n}(x)$ (this approach is motivated

[5] Our construction is motivated by similar (but less general) ideas in [16, Remark 1 p. 427].

by [16, Section 6]). Then $(R_n(x))_{n\in\mathbb{N}_0}$ satisfies the recurrence relation $R_0(x) = 1$, $R_1(x) = (x - c_1)/a_1$, $R_1(x)R_n(x) = a_n^R R_{n+1}(x) + b_n^R R_n(x) + c_n^R R_{n-1}(x)$ $(n \in \mathbb{N})$, where

$$a_n^R := \frac{a_{2n}a_{2n+1}}{a_1}, b_n^R := \frac{a_{2n}c_{2n+1} + c_{2n}a_{2n-1} - c_1}{a_1}, c_n^R := \frac{c_{2n}c_{2n-1}}{a_1} \in (0, 1)$$

(as usual, $(c_n)_{n\in\mathbb{N}} \subseteq (0, 1)$ and $a_n \equiv 1 - c_n$ shall denote the recurrence coefficients which belong to the sequence $(P_n(x))_{n\in\mathbb{N}_0}$). The estimation $b_n^R > 0$ follows from the monotonicity behavior of $(\lambda_{2n})_{n\in\mathbb{N}_0}$ because

$$a_{2n}c_{2n+1} + c_{2n}a_{2n-1} = \lambda_{2n}^2 + \lambda_{2n-1}^2 > \lambda_{2n}^2 > \lambda_0^2 = c_1$$

for every $n \in \mathbb{N}$; the remaining estimations are obvious from $a_n^R + b_n^R + c_n^R = 1$ $(n \in \mathbb{N})$. Since, for every $n \geq 2$,

$$1 > c_{2n-1} = \frac{\lambda_{2n-2}^2}{a_{2n-2}} > \lambda_{2n-2}^2 \to 1\ (n \to \infty),$$

we have $\lim_{n\to\infty} c_{2n-1} = 1$ (hence $\lim_{n\to\infty} a_{2n-1} = 0$) and consequently

$$c_{2n} = 1 - \frac{\lambda_{2n}^2}{c_{2n+1}} \to 0\ (n \to \infty)$$

(hence $\lim_{n\to\infty} a_{2n} = 1$). Therefore, we have $\lim_{n\to\infty} a_n^R = 0$, $\lim_{n\to\infty} b_n^R = 1$, $\lim_{n\to\infty} c_n^R = 0$. Let μ_R denote the orthogonalization (probability) measure of $(R_n(x))_{n\in\mathbb{N}_0}$. μ_R can be regarded as pushforward measure of μ. It is clear from the construction that

$$\text{supp}\ \mu_R = \{1\} \cup \{x_n^2 : n \in \mathbb{N}\}.$$

By [4, Proposition 4], the behavior of the sequences $(a_n^R)_{n\in\mathbb{N}}$, $(b_n^R)_{n\in\mathbb{N}}$ and $(c_n^R)_{n\in\mathbb{N}}$ as obtained above implies that $(R_n(x))_{n\in\mathbb{N}_0}$ induces a polynomial hypergroup of 'strong compact type' on $\mathbb{N}_0$, which means that the operator $T_n - \text{id}$ is compact on $\ell^1(h)$ for every $n \in \mathbb{N}_0$.[6] By [4, Theorem 2], this yields

$$\left\{x \in \mathbb{R} : \max_{n\in\mathbb{N}_0} |R_n(x)| = 1\right\} = \text{supp}\ \mu_R.$$

Therefore, we obtain that

$$\widehat{\mathbb{N}_0} \subseteq \left\{x \in \mathbb{R} : \max_{n\in\mathbb{N}_0} |\underbrace{P_{2n}(x)}_{=R_n(x^2)}| = 1\right\} = \{\pm 1\} \cup \{\pm x_n : n \in \mathbb{N}\}.$$

Since, however, $\{\pm 1\} \cup \{\pm x_n : n \in \mathbb{N}\} = \text{supp}\ \mu \subseteq \widehat{\mathbb{N}_0}$ due to (1.3), we obtain equality. Moreover, we have $0 \notin \widehat{\mathbb{N}_0}$ because $|P_{2n}(0)| = \prod_{k=0}^{n-1} (c_{2k+1}/a_{2k+1}) \to \infty$ $(n \to \infty)$ by the limiting behavior of $(c_{2n-1})_{n\in\mathbb{N}}$ and $(a_{2n-1})_{n\in\mathbb{N}}$.

(iv) We first show that our explicit construction works. Since the convexity of the null sequence $(s_n)_{n\in\mathbb{N}_0} \subseteq (0, 1)$ implies that $(s_n)_{n\in\mathbb{N}_0}$ is strictly decreasing, we have $(\lambda_n)_{n\in\mathbb{N}_0} \subseteq (0, 1)$. Moreover, it is clear that $\lim_{n\to\infty} \lambda_{2n} = 1$. Finally, by convexity we have

$$\lambda_0 + \lambda_1 = 1 - s_0 + s_1 - s_2 \leq 1 - s_1 = \lambda_2$$

[6] Note that the sequence $(R_n(x))_{n\in\mathbb{N}_0}$ is not symmetric. The definitions of a polynomial hypergroup, the translation operator T_n, the Haar function h and the space $\ell^1(h)$ are the same as for the symmetric case (as recalled in Section 1), however (just for the sake of completeness, we mention that the last equation in (1.5) must be slightly modified).

and, for every $n \in \mathbb{N}$,

$$\lambda_{2n+1} = s_{n+1} - s_{n+2} \leq s_n - s_{n+1} = \lambda_{2n-1}$$

and

$$\lambda_{2n-1} + \lambda_{2n} = s_n - s_{n+1} + 1 - s_n = 1 - s_{n+1} = \lambda_{2n+2}.$$

If $s_0 = 1 - 1/\sqrt{1+\epsilon}$, then (ii) yields

$$h(1) = Q_1^2(1) = \frac{1}{\lambda_0^2} = \frac{1}{(1-s_0)^2} = 1 + \epsilon.$$

It remains to show that $h(1) = 1 + \epsilon$ implies that $(x_n)_{n\in\mathbb{N}} \subseteq [\sqrt{(1-\epsilon)/(1+\epsilon)}, 1)$. This is an immediate consequence of Proposition 3.1, however.

(v) As a consequence of (i) and (ii), the sequence $(h(n))_{n\in\mathbb{N}_0}$ coincides with $(Q_n^2(1))_{n\in\mathbb{N}_0}$ and is therefore strictly increasing. Hence, we have

$$h(n) \geq h(2) = Q_2^2(1) = \left(\frac{1-\lambda_0^2}{\lambda_0\lambda_1}\right)^2 > \left(\frac{1-\lambda_0^2}{\lambda_0(1-\lambda_0)}\right)^2 = \left(1 + \frac{1}{\lambda_0}\right)^2 > 4$$

for all $n \geq 2$. It remains to prove that h is of exponential growth. Let $n \in \mathbb{N}$. Then $xQ_{2n}(x) = \lambda_{2n}Q_{2n+1}(x) + \lambda_{2n-1}Q_{2n-1}(x)$ and consequently

$$\begin{aligned} &x^2 Q_{2n}(x) \\ &= \lambda_{2n} x Q_{2n+1}(x) + \lambda_{2n-1} x Q_{2n-1}(x) \\ &= \lambda_{2n}\lambda_{2n+1} Q_{2n+2}(x) + (\lambda_{2n}^2 + \lambda_{2n-1}^2) Q_{2n}(x) + \lambda_{2n-1}\lambda_{2n-2} Q_{2n-2}(x), \end{aligned}$$

so

$$Q_{2n}(1) = \lambda_{2n}\lambda_{2n+1} Q_{2n+2}(1) + (\lambda_{2n}^2 + \lambda_{2n-1}^2) Q_{2n}(1) + \lambda_{2n-1}\lambda_{2n-2} Q_{2n-2}(1).$$

The latter yields

$$Q_{2n+2}(1) = \frac{1 - \lambda_{2n}^2 - \lambda_{2n-1}^2}{\lambda_{2n}\lambda_{2n+1}} Q_{2n}(1) - \frac{\lambda_{2n-1}\lambda_{2n-2}}{\lambda_{2n}\lambda_{2n+1}} Q_{2n-2}(1).$$

Since $Q_{2n}(1) > Q_{2n-2}(1)$ by (i), we get

$$\begin{aligned} Q_{2n+2}(1) &> \frac{1 - \lambda_{2n}^2 - \lambda_{2n-1}^2 - \lambda_{2n-1}\lambda_{2n-2}}{\lambda_{2n}\lambda_{2n+1}} Q_{2n}(1) \\ &= \frac{1 - \lambda_{2n}^2 - \lambda_{2n-1}(\lambda_{2n-1} + \lambda_{2n-2})}{\lambda_{2n}\lambda_{2n+1}} Q_{2n}(1) \\ &\geq \frac{1 - \lambda_{2n}^2 - \lambda_{2n-1}\lambda_{2n}}{\lambda_{2n}\lambda_{2n+1}} Q_{2n}(1) \\ &= \frac{1 - \lambda_{2n}(\lambda_{2n} + \lambda_{2n-1})}{\lambda_{2n}\lambda_{2n+1}} Q_{2n}(1) \\ &\geq \frac{1 - \lambda_{2n}\lambda_{2n+2}}{\lambda_{2n}\lambda_{2n+1}} Q_{2n}(1) \\ &> \frac{1 - \lambda_{2n}}{\lambda_{2n+1}} Q_{2n}(1) \\ &\geq \frac{1 + \lambda_{2n+1} - \lambda_{2n+2}}{\lambda_{2n+1}} Q_{2n}(1) \end{aligned}$$

$$\geq \frac{1+\lambda_{2n+1}+\lambda_{2n+1}-\lambda_{2n+4}}{\lambda_{2n+1}} Q_{2n}(1)$$
$$> 2Q_{2n}(1).$$

This shows that $(Q_{2n}(1))_{n\in\mathbb{N}_0}$ is of exponential growth. Therefore, we obtain that $(h(2n))_{n\in\mathbb{N}_0} = (Q_{2n}^2(1))_{n\in\mathbb{N}_0}$ (and hence h) is of exponential growth. □

Remark 3.2. Reconsider the explicit construction studied in the proof of Theorem 3.2 (iv), now for $s_0 \geq 1-\sqrt{2}/2$. In this case, one has $h(1) \geq 2$ (and consequently $h(n) \geq 2$ for all $n \in \mathbb{N}$ by Theorem 3.2 (v)). However, the dual space $\widehat{\mathbb{N}_0}$ is a discrete subset of $[-1,1]$ by Theorem 3.2 (iii). Therefore, the example provides an alternative proof of Corollary 3.1.

Remark 3.3. If $(P_n(x))_{n\in\mathbb{N}_0}$ is as in Theorem 3.1 or Theorem 3.2, then $\widehat{\mathbb{N}_0} = \mathcal{X}^b(\mathbb{N}_0)$. The first case is clear from the quadratic (hence subexponential) growth of h and Theorem 1.1. The second case can be obtained by a slight modification of the proof of Theorem 3.2 (iii), based on [4, Theorem 2].

4. Open problems

We finish our paper with a collection of some open problems:

(i) Is $h(2) \geq 2$ always true?
(ii) Is $\liminf_{n\to\infty} h(n) \geq 2$ always true?
(iii) Is $h(n) \geq 2$ ($n \in \mathbb{N}\backslash\{1\}$) always true?
(iv) Is $h(1) \geq 2$ sufficient for $h(2) \geq 2$, $\liminf_{n\to\infty} h(n) \geq 2$ or $h(n) \geq 2$ ($n \in \mathbb{N}$)?
(v) Is $0 \in \widehat{\mathbb{N}_0}$ sufficient for $h(2) \geq 2$, $\liminf_{n\to\infty} h(n) \geq 2$ or $h(n) \geq 2$ ($n \in \mathbb{N}$)?

The questions (i), (ii) and (iii) are motivated by our observations made in Theorem 3.1 (ii) and Theorem 3.2 (v). Concerning (iv) and (v), recall that $0 \in \widehat{\mathbb{N}_0}$ implies at least $h(1) \geq 2$, which is a consequence of Proposition 3.1.

Data availability

No data was used for the research described in the article.

Acknowledgments

The joint work was started while the first author was visiting the Institute of Mathematics of the University of Wrocław, whose hospitality is greatly acknowledged. The authors thank Rupert Lasser who had a look at the manuscript and gave the feedback that he is not aware of further previous results which should be cited. The authors also thank the referee for carefully reading the manuscript, as well as for the valuable comments.

References

[1] R. Askey, Linearization of the product of orthogonal polynomials, in: Problems in Analysis (Sympos. in Honor of Salomon Bochner, Princeton Univ., Princeton, N.J., 1969), Princeton Univ. Press, Princeton, NJ, 1970, pp. 131–138.

[2] W.R. Bloom, H. Heyer, Harmonic analysis of probability measures on hypergroups, in: De Gruyter Studies in Mathematics, vol. 20, Walter de Gruyter & Co., Berlin, 1995, p. vi+601, http://dx.doi.org/10.1515/9783110877595.

[3] T.S. Chihara, An Introduction to Orthogonal Polynomials, Gordon and Breach Science Publishers, New York-London-Paris, 1978, p. xii+249, Mathematics and its Applications, Vol. 13.
[4] F. Filbir, R. Lasser, R. Szwarc, Hypergroups of compact type, J. Comput. Appl. Math. 178 (1–2) (2005) 205–214, http://dx.doi.org/10.1016/j.cam.2004.03.028.
[5] G. Gasper, Linearization of the product of Jacobi polynomials. II, Canad. J. Math. 22 (1970) 582–593, http://dx.doi.org/10.4153/CJM-1970-065-4.
[6] S. Kahler, Nonnegative and strictly positive linearization of Jacobi and generalized Chebyshev polynomials, Constr. Approx. 54 (2) (2021) 207–236, http://dx.doi.org/10.1007/s00365-021-09552-3.
[7] S. Kahler, Turán's inequality, nonnegative linearization and amenability properties for associated symmetric Pollaczek polynomials, J. Approx. Theory 268 (2021) http://dx.doi.org/10.1016/j.jat.2021.105580, Paper No. 105580, 33.
[8] S. Kahler, Harmonic analysis of little q-Legendre polynomials, J. Approx. Theory 295 (2023) http://dx.doi.org/10.1016/j.jat.2023.105946, Paper No. 105946, 21.
[9] S. Karlin, J. McGregor, Random walks, Illinois J. Math. 3 (1959) 66–81.
[10] T.P. Laine, The product formula and convolution structure for the generalized Chebyshev polynomials, SIAM J. Math. Anal. 11 (1) (1980) 133–146, http://dx.doi.org/10.1137/0511012.
[11] R. Lasser, Orthogonal polynomials and hypergroups, Rend. Mat. (7) 3 (2) (1983) 185–209.
[12] R. Lasser, Orthogonal polynomials and hypergroups. II. The symmetric case, Trans. Amer. Math. Soc. 341 (2) (1994) 749–770, http://dx.doi.org/10.2307/2154581.
[13] R. Lasser, Discrete commutative hypergroups, in: Inzell Lectures on Orthogonal Polynomials, in: Adv. Theory Spec. Funct. Orthogonal Polynomials, vol. 2, Nova Sci. Publ., Hauppauge, NY, 2005, pp. 55–102.
[14] R. Lasser, M. Rösler, A note on property (T) of orthogonal polynomials, Arch. Math. (Basel) 60 (5) (1993) 459–463, http://dx.doi.org/10.1007/BF01202312.
[15] V.I. Lebedev, On the solution of inverse problems and trigonometric forms for the Geronimus polynomials. Application to the theory of iterative methods, Russian J. Numer. Anal. Math. Modelling 15 (1) (2000) 73–93, http://dx.doi.org/10.1515/rnam.2000.15.1.73.
[16] W. Młotkowski, R. Szwarc, Nonnegative linearization for polynomials orthogonal with respect to discrete measures, Constr. Approx. 17 (3) (2001) 413–429, http://dx.doi.org/10.1007/s003650010039.
[17] R. Szwarc, Orthogonal polynomials and a discrete boundary value problem. I, II, SIAM J. Math. Anal. 23 (4) (1992) 959–964, 965–969, http://dx.doi.org/10.1137/0523053.
[18] R. Szwarc, A lower bound for orthogonal polynomials with an application to polynomial hypergroups, J. Approx. Theory 81 (1) (1995) 145–150, http://dx.doi.org/10.1006/jath.1995.1040.
[19] R. Szwarc, Nonnegative linearization of the associated q-ultraspherical polynomials, Methods Appl. Anal. 2 (4) (1995) 399–407, http://dx.doi.org/10.4310/MAA.1995.v2.n4.a2.
[20] J. Todd, Introduction to the Constructive Theory of Functions, Academic Press, Inc., New York, 1963, p. 127.
[21] M. Vogel, Spectral synthesis on algebras of orthogonal polynomial series, Math. Z. 194 (1) (1987) 99–116, http://dx.doi.org/10.1007/BF01168009.
[22] M. Voit, Positive characters on commutative hypergroups and some applications, Math. Z. 198 (3) (1988) 405–421, http://dx.doi.org/10.1007/BF01184674.
[23] M. Voit, A formula of Hilb's type for orthogonal polynomials, J. Comput. Appl. Math. 49 (1–3) (1993) 339–348, http://dx.doi.org/10.1016/0377-0427(93)90167-A.