Letter

Moments of parton distribution functions of any order from lattice QCD

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We describe a procedure to determine moments of parton distribution functions of any order in lattice quantum chromodynamics (QCD). The procedure is based on the gradient flow for fermion and gauge fields. The flowed matrix elements of twist-2 operators renormalize multiplicatively, and the matching with the physical matrix elements can be obtained using continuum symmetries and the irreducible representations of Euclidean 4-dimensional rotations. We calculate the matching coefficients at one-loop in perturbation theory for moments of any order in the flavor nonsinglet case. We also give specific examples of operators that could be used in lattice QCD computations. It turns out that it is possible to choose operators with identical Lorentz indices and still have a multiplicative matching. One can thus use twist-2 operators exclusively with temporal indices, thus substantially improving the signal-to-noise ratio in the computation of the hadronic matrix elements.

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Introduction. Quantum chromodynamics (QCD) describes quark-gluon interactions over a wide energy range, posing nonperturbative challenges at lower energies. Hadron structure studies depend on parton distribution functions (PDFs), with PDF determinations relying on precise QCD calculations, influencing investigations at the LHC and future projects (EIC, HL-LHC, LHeC). The quest for new physics hinges on accurate experimental-theoretical comparisons, emphasizing PDF precision.

QCD factorization in collider processes, beyond PDFs, involves generalized parton distributions (GPDs) and transverse-momentum-dependent distributions (TMDs). While not directly measurable, these distribution functions can be determined by factorizing the process's cross section into hard and soft components.

In deep inelastic scattering (DIS), a key role is played by the structure functions, $F_k(x, Q^2)$ $(k = 1, 2, 3)$, which depend on $Q^2 = -q^2$, where q is the 4-momentum transferred to the nucleon, and $x = Q^2/2(pq)$ represents the fraction of nucleon momentum carried by the parton. Here, p denotes the nucleon momentum. For Q^2 much larger than the nucleon mass, the factorization theorem [1] relates the structure functions to the convolution of the scale-dependent parton distribution functions $f_a(x, \mu^2)$ with the Wilson coefficients $C_k^a(Q^2/\mu^2)$ (summed over quark flavors and gluons, a). The Wilson coefficients are calculable in perturbation theory, and the evolution of the PDFs in μ is governed by the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equations [2–5]. The DGLAP equations can be solved once we specify an initial condition at a certain scale $\mu = \mu_0$. The evolution of the PDFs is calculable in perturbation theory, but the precision of experimental data always necessitates higher-order calculations. The standard choice for renormalization and factorization scheme is the $\overline{\text{MS}}$ scheme. PDFs can be determined experimentally by analyzing a set of hard-scattering processes involving nucleons. There is a long-standing community effort to determine PDFs using well-defined parametrizations for their x-dependence and starting the evolution around $\mu_0^2 = 1$–4 GeV.

Lattice QCD can offer a theoretical input for the determination of the PDFs and their evolution. The direct calculation of PDFs using lattice QCD poses particular challenges due to the Euclidean geometry and the light-cone dominance of the kinematics. The connection between PDFs and hadronic matrix elements, which are calculable in lattice QCD, is established through the moments of the PDFs, denoted as $\langle x^n \rangle$ [6,7]. Lattice QCD calculations of the moments of the PDFs, pioneered in Refs. [8–10], provide, in principle, a means for the complete reconstruction of the PDFs. This possibility has remained impractical due to the theoretical and numerical challenges associated with computing high moments. Alternative approaches have been developed to determine

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the x-dependence of the PDFs, including the auxiliary scalar field [11] heavy-quarks OPE [12], the quasi-PDF [13], the psuedo-PDF [14,15], the OPE-based method of [16], the current-current approach [17], the good lattice cross-sections [18] and the hadron tensor method of [19]. It is worth noticing that these approaches allow in principle an indirect determination of the moments of PDF of nucleons [20,21] and pions [22,23].

Different high-energy scatterings are associated with different PDFs: unpolarized, helicity, and transversity. Certain kinematical regions, as well as specific PDFs, pose challenges for extraction from experimental data. This emphasizes the significance of reconstructing the PDFs from lattice QCD simulation data.

In this work, we revisit the possibility of calculating moments of PDFs, which, if successful, would provide an important and complementary approach to determine distribution functions from QCD. We describe a method that addresses both the theoretical and numerical challenges faced in the past, which hindered the calculation of moments of any order from lattice QCD.

Twist-2 operators. QCD factorization relates the moments of the structure functions and the moments of the parton distribution functions. Taking the example of F_1, the relation is given by e.g.

$$\int_0^1 dx\, x^{n-1} F_1(x, Q^2) = \sum_a C^a_{1,n}(Q^2/\mu^2) A^{a/h}_n(\mu), \quad (1)$$

where $A^{a/h}_n(\mu)$ represents the reduced matrix element of certain local operators, denoted by O_n, between the hadron states, denoted as h, and $C^a_{1,n}(Q^2/\mu^2)$ are the corresponding Wilson coefficients, calculable in perturbation theory. In the context of DIS kinematics, the leading contribution, given in Eq. (1), arises from local operators with the lowest twist, $\tau = d_O - n = 2$, where d_O is the physical dimension of the local operator and n is its spin. In the parton model the reduced matrix element is related to the moments of the PDFs, $A^{a/h}_n = \langle x^{n-1} \rangle_{a/h}$ [6,7].

The form of the local operators O_n can be constrained by requiring that they belong to irreducible representations of the Lorentz group. This guarantees that the renormalization procedure does not generate mixing with operators of the same or lower dimensions. Lattice QCD simulations are performed in a Euclidean setup though, and the symmetry group in a Euclidean lattice is the hyper-cubic group H(4). The reduced symmetry does not protect the operators from a complicated mixing pattern, that gets more complicated for larger n. Once the continuum limit has been performed, one recovers O(4) symmetry.

The construction of irreducible representations of O(4) symmetry group is standard material (see, for example, Ref. [24]). For the purpose of this work, the only relevant result is that traceless and symmetrized rank-n tensors are an irreducible representation of O(4).

We restrict ourselves to unpolarized structure functions, meaning that we consider hadronic matrix elements averaged over the hadron polarization. Other moments of different distributions can be studied using the same procedure described here. We further limit our consideration to flavor non-singlet operators to avoid mixing with gluonic operators, but the same method is applicable to singlet operators.

We consider the following set of operators

$$O^{rs}_n(x) = O^{rs}_{\mu_1 \cdots \mu_n}(x) = \bar{\psi}^r(x) \gamma_{\{\mu_1} \overleftrightarrow{D}_{\mu_2} \cdots \overleftrightarrow{D}_{\mu_n\}} \psi^s(x), \quad (2)$$

with quark fields, ψ^r, of different flavors, $r \neq s$, and the covariant derivative $D_\mu = \partial_\mu + G_\mu$, where G_μ represent the gluon field. The symmetrization of the covariant derivative $\overleftrightarrow{D}_\mu = \frac{1}{2}(\vec{D}_\mu - \overleftarrow{D}_\mu)$ guarantees a definite transformation under charge conjugation.

In the continuum, traceless non-singlet twist-2 operators, with symmetrized Lorentz indices,[1] renormalize multiplicatively. For example, in the MS scheme, the bare operator $\hat{O}^{rs}_{n,\mathrm{B}}$ renormalizes as follows

$$\hat{O}^{rs}_n(x) = Z^{\mathrm{MS}}_n \hat{O}^{rs}_{n,\mathrm{B}}(x), \quad (3)$$

with renormalization constant Z^{MS}_n. The intrinsic non-perturbative nature of the hadronic matrix elements suggests the use of lattice QCD. However, the breaking of the rotational group symmetry into the hypercubic group H(4) adds complications to the renormalization.

On an hypercubic lattice, the Lorentz indices of the twist-2 operators must be chosen while considering the reduced hypercubic symmetry group H(4). Irreducible representations of O(4) generally become reducible representations of H(4) inducing unwanted mixings under renormalization [8,9,25,26]. The irreducible representations of H(4) allow mixings with lower-dimensional operators, and, additionally, complicate the mixing with operators of the same dimensions. A notable example of this complication occurs with the operator O_3. In the continuum, after symmetrizing over the Lorentz indices and subtracting the traces, the renormalization is multiplicative (e.g. in the MS scheme). However, on the lattice, terms like $1/a^2 \delta_{\mu_i \mu_j} \cos(a p_{\mu_j})$ are allowed by H(4) symmetry and cannot be removed even after subtracting the trace [8]. This pertains to the fact that O_3 belongs to different irreducible representations of H(4) depending on the choice of the Lorentz indices. The operator O_3 with all the indices equal, i.e. $\mu_1 = \mu_2 = \mu_3$, mixes with the lower-dimensional vector current, causing a power divergence

[1]We denote the traceless operators with $\hat{O}_n$.

proportional to $1/a^2$. For the operators with $\mu_1 \neq \mu_2 = \mu_3$, such as the irreducible representation $O_{411} - O_{433}$, the renormalization is multiplicative, up to a small mixing with a different operator of the same dimension.[2] We note that while O_{411} is also affected by power divergences, the subtraction $O_{411} - O_{433}$ guarantees their removal. Another choice is $\mu_1 \neq \mu_2 \neq \mu_3$. In this case, the operator belongs to another irreducible representation of H(4) and renormalizes multiplicatively without introducing power divergences. However, this last example, while optimal in terms of renormalization, is far from optimal when calculating hadronic matrix elements[3]

$$\langle h(p)|\hat{O}_n|h(p)\rangle = 2p_{\mu_1}\cdots p_{\mu_n}A_n^h(\mu), \tag{4}$$

because it requires nonvanishing external momentum in at least 2 spatial directions. This substantially degrades the quality of the signal-to-noise ratio that is realized in numerical lattice QCD simulations. For $O_{411} - O_{433}$, one needs a nonvanishing external momentum in at least 1 spatial directions, and despite the subtraction of power divergences, one might hope to achieve a better signal-to-noise ratio. The operator with all the same indices has a power divergence that requires a complicated nonperturbative procedure to be subtracted and is not a viable solution. This is unfortunate because the matrix element of O_{444}, for example, does not require any external spatial momentum and should be optimal in terms of the signal-to-noise ratio.

The mixing pattern becomes more cumbersome for higher dimensional operators, $n \geq 4$. For $n = 4$, choosing all 4 indices differently results in the only 2 irreducible representations: the totally symmetric and antisymmetric representations. One then needs a nonvanishing external momentum in all 3 spatial directions to calculate the hadronic matrix elements [25]. For $n > 4$, mixings are unavoidable. The complicated renormalization pattern just described, together with the need of an external momentum to calculate the hadronic matrix elements, has, *de facto*, prevented the lattice calculation of the moments of parton distribution functions for $n > 4$ and rendered the calculations for $n = 3$, 4 challenging.

A very interesting idea to circumvent the renormalization issue, based on the recovery of the continuum rotational symmetry, was proposed in Ref. [27]. Even though the method was not further pursued numerically, it represents a first important step toward resolving the theoretical challenges encountered when calculating moments of PDFs.

[2]The operator $O_{411} - O_{433}$ belongs to an irreducible representation of H(4) that appears more than once. Therefore, a mixing between operators belonging to these equivalent representations can occur [25]. In perturbation theory the mixing seems to be numerically small.

[3]In Eq. (4) for simplicity we omit the flavor indices.

Gradient flow. The gradient flow (GF) for gauge and fermion fields [28–30] provides a different way to regulate short-distance singularities of the twist-2 operators. The connection with the physical renormalized matrix elements at vanishing flow time, $t = 0$, is obtained with a short-flow-time expansion (SFTX) after performing the continuum limit of the hadronic matrix elements at a fixed physical value of the flow time t [31].

The flowed twist-2 operators, written in terms of flowed fermions, $\chi^r(x,t)$ and $\bar{\chi}^r(x,t)$, and gauge fields, $B_\mu(x,t)$, are given by

$$O_n^{rs}(x,t) = \bar{\chi}^r(x,t)\gamma_{\{\mu_1}\overleftrightarrow{D}_{\mu_2}\cdots\overleftrightarrow{D}_{\mu_n\}}\chi^s(x,t), \tag{5}$$

where $\{\cdots\}$ denotes symmetrization over the included indices. These operators are particularly advantageous because they renormalize multiplicatively also on the lattice with a renormalization factor that depends only on the fermion content of the operator,

$$O_n^{rs}(t) = Z_n O_{n,B}^{rs}(t), \qquad Z_n = Z_\chi, \tag{6}$$

where $Z_\chi^{1/2}$ is the renormalization constant of the flowed fermion fields [30]. For correlation functions containing flowed twist-2 operators, the flow-time t provides a regulator for short-distance singularities. Beside the renormalization of the bare parameters of the theory, the flowed fermion fields, and any local field at $t = 0$, correlation functions do not require any additional renormalization. For lattice QCD applications, it is convenient to define Z_χ in a scheme that is regularization independent, as then the matching and the renormalization can be performed in the same scheme. The choice of the specific scheme is not relevant for the general discussion, but it becomes important when calculating the matching coefficients. In the calculations of the matching coefficients described below, we renormalize the twist-2 operators using the so-called ringed fields [32], defined by the $\mathrm{SU}(N_c)$ gauge invariant and regularization independent condition

$$\langle \mathring{\bar{\chi}}^r(x,t)\overleftrightarrow{\not{D}}\mathring{\chi}^r(x,t)\rangle = -\frac{N_c}{(4\pi)^2 t^2}. \tag{7}$$

Imposing this condition in dimensional regularization, with $D = 4 - 2\epsilon$, leads to a finite renormalization between the ringed fields and renormalized fields in the MS scheme

$$\begin{aligned}\chi^r(x,t) &= (8\pi t)^{\epsilon/2}\zeta_\chi^{1/2}\mathring{\chi}^r(x,t)\\ \bar{\chi}^r(x,t) &= (8\pi t)^{\epsilon/2}\zeta_\chi^{1/2}\mathring{\bar{\chi}}^r(x,t).\end{aligned} \tag{8}$$

The 1-loop result for the finite renormalization is given by [32]

$$\zeta_\chi = 1 - \frac{\bar{g}^2(\mu)}{(4\pi)^2}C_F\big(3\log(8\pi\mu^2 t) - \log(432)\big), \tag{9}$$

where $\bar{g}$ is the strong coupling renormalized in the MS scheme at the renormalization scale μ, and $C_F = N_c^2 - 1/2N_c$. This result has been extended to $O(\bar{g}^4)$ in Refs. [33,34]. Results obtained in the MS scheme can be converted to the $\overline{\text{MS}}$ scheme by replacing the renormalization scale μ with $\bar{\mu}$ related by $\log \mu^2 = \log \bar{\mu}^2 + \gamma_E - \log 4\pi$.

O(a) improvement. The lattice QCD calculation of hadronic matrix elements of twist-2 operators is affected by cutoff effects.

For nonperturbatively improved clover fermions, the hadronic matrix elements of the standard twist-2 operators (with $t = 0$) are still affected by $O(a)$ cutoff effects that could be removed by adopting an $O(a)$ improved definition of the twist-2 operators. Excluding the case $n = 2$, where the corresponding improvement coefficients have been determined at 1-loop in perturbation theory [35], the continuum limit is reached with $O(a)$ discretization effects.

If we consider flowed twist-2 operators for a generic n, the hadronic matrix elements are affected only by $O(am)$ cutoff effects, where m is the quark mass. Additional improvement coefficients specific to the twist-2 operators are not needed. In the discussion of the method's applications below, we also consider ratios of flowed matrix elements where both Z_χ and the $O(am)$ cutoff effects are eliminated. In summary, regardless of the lattice setup adopted, the use of flowed twist-2 operators not only provides a significant simplification in the renormalization pattern, but also improves the parametric scaling in the lattice spacing towards the continuum limit.

Matching coefficients. After performing the continuum limit of the hadronic matrix element for a fixed $t > 0$ value of the flow time, the connection with physical renormalized matrix element at $t = 0$ is obtained with a SFTX [31]. The SFTX of the twist-2 operators $O_n(x,t)$, in general contains power divergences with terms like $1/t^m \delta_{\mu_i \mu_j}$, where $m = (d_n - d_p)/2$ depends on the dimensions of O_n, d_n, and on the dimension of the lower-dimensional operator, d_p. These terms, though, are classified using the continuum O(4) symmetry and, as a result, the SFTX of the symmetrized and traceless operators, $\hat{O}_n$, receives contributions only from the corresponding traceless operators at $t = 0$.

We renormalize the flowed fermion fields adopting ringed fields, introduced in the previous section, and define the ringed twist-2 operators

$$\mathring{\hat{O}}_n^{rs}(x,t) = \mathring{\bar{\chi}}^r(x,t)\gamma_{\{\mu_1}\overleftrightarrow{D}_{\mu_2}\cdots\overleftrightarrow{D}_{\mu_n\}}\mathring{\chi}^s(x,t) - \text{terms with } \delta_{\mu_i\mu_j}, \tag{10}$$

where the subtraction guarantees that the resulting operator has vanishing traces. The SFTX of $\mathring{\hat{O}}_n^{rs}$ is

$$\mathring{\hat{O}}_n^{rs}(t) = c_n(t,\mu)\hat{O}^{rs}_{n,\text{MS}}(\mu) + \cdots, \tag{11}$$

where we make explicit only the dependence on the flow time and the renormalization scale. The neglected terms are higher-order contributions of positive powers of t.

The matching coefficients are determined considering off-shell amputated one-particle irreducible (1PI) Green's functions containing the flowed operators $\mathring{\hat{O}}_n^{rs}(t)$. The matching equations with massless fermions

$$\langle \psi^r \mathring{\hat{O}}_n^{rs}(t)\bar{\psi}^s\rangle = c_n(t,\mu)\langle \psi^r \hat{O}^{rs}_{n,\text{MS}}(t=0,\mu)\bar{\psi}^s\rangle \tag{12}$$

are solved in $D = 4 - 2\epsilon$ at 1-loop in perturbation theory with

$$c_n(t,\mu) = 1 + \frac{\bar{g}^2(\mu)}{(4\pi)^2} c_n^{(1)}(t,\mu) + O(\bar{g}^4). \tag{13}$$

The matching coefficients $c_n(t,\mu)$ depend on the renormalized coupling and are independent of the soft scales the calculation of the matching coefficients. In the Supplemental Material [36] we describe their calculation and, based on the observation that they satisfy a renormalization group equation, we also provide a resummed expression at next-to-leading log (NLL).

The final result for the matching coefficients is

$$c_n^{(1)}(t,\mu) = C_F[\gamma_n \log(8\pi\mu^2 t) + B_n], \tag{14}$$

where $\gamma_n = 1 + 4\sum_{j=2}^n \frac{1}{j} - \frac{2}{n(n+1)}$, and

$$\begin{aligned} B_n = {} & \frac{4}{n(n+1)} + 4\frac{n-1}{n}\log 2 + \frac{2-4n^2}{n(n+1)}\gamma_E \\ & - \frac{2}{n(n+1)}\psi(n+2) + \frac{4}{n}\psi(n+1) - 4\psi(2) \\ & - 4\sum_{j=2}^{n}\frac{1}{j(j-1)}\frac{1}{2^j}\phi(1/2,1,j) - \log(432). \end{aligned} \tag{15}$$

In this expression $\gamma_E = 0.57221\ldots$ is the Euler's constant, $\psi(z)$ is the digamma function, and $\phi(z,s,a)$ is the Lerch transcendent defined as $\phi(z,s,a) = \sum_{k=0}^\infty \frac{z^k}{(k+a)^s}$.

The expression for γ_n matches the 1-loop anomalous dimension of the twist-2 operators [47] and provides a welcome check of the calculation. For $n > 2$ the finite part, B_n, in Eq. (15) provides a new result. The result with $n = 2$ was already obtained in Ref. [32] in the analysis of the energy-momentum tensor, and we reproduce their result.

The fact that the matching of the flowed matrix elements is so simple opens the possibility of computing moments of the parton distribution functions of any order. A possible operational procedure is the following: After fixing n, one constructs the symmetric and traceless operator $\hat{O}_n^{rs}$.

One proceeds calculating the matrix elements between hadron states of $\hat{\hat{O}}_n^{rs}(t)$ using standard techniques based on the spectral decomposition. To apply the decomposition in the lattice calculation of the 3-point function, one needs to ensure that the physical distance between the interpolators of the hadron states and of the flowed operators is much larger than the flow-time radius $\sqrt{8t}$. After performing the continuum limit, the renormalized matrix element in the MS scheme is calculated simply by

$$A_n^{\rm MS}(\mu) = c_n(t,\mu)^{-1} A_n(t). \tag{16}$$

This matching is correct only if the flowed fields are defined as ringed fields. If the flowed fields are renormalized in a different scheme, the finite part of the matching coefficient has to be modified.

If the matching is successful, the l.h.s of Eq. (16) should be independent of the flow time. Violations stem from higher dimensional operators as linear terms in t, and from higher order terms in the perturbative expansion of the matching coefficients. To mitigate these possible systematics it is important to extend to $O(\bar{g}^4)$ the calculation presented here and conduct a thorough numerical study of the residual flow-time dependence of the hadronic matrix element after the matching.

Applications. The simplicity of the matching is guaranteed once the operators are symmetrized on the Lorentz indices, and all the traces have been subtracted. Traceless operators for any n are not difficult to write down. For example, for $n = 4$, we have

$$\hat{O}_{4444} = O_{4444} - \frac{3}{4} O_{\{\alpha\alpha 44\}} + \frac{1}{16} O_{\{\alpha\alpha\beta\beta\}}. \tag{17}$$

The hadronic matrix element of $\hat{O}_{4444}(t)$ is calculable with no external spatial momentum, and the matching with the MS scheme at renormalization scale μ is obtained using Eq. (16).

The continuum limit of the reduced hadronic matrix element of $\hat{\hat{O}}_n$, denoted as $A_n^h(t) = \langle x^{n-1}\rangle^h(t)$, would require a determination of Z_χ from lattice QCD simulations. However, this can be avoided using the second moment $A_2^{h,\rm MS}(\mu) = \langle x\rangle^h_{\rm MS}(\mu)$ as an observable input already computed with any method of choice. Then all the other moments for $n > 2$ are calculable from

$$\langle x^{n-1}\rangle^h_{\rm MS}(\mu) = \langle x^{n-2}\rangle^h_{\rm MS}(\mu) \frac{c_{n-1}(t,\mu)}{c_n(t,\mu)} R_n^h(t), \tag{18}$$

where $c_n(t,\mu)$ is given by Eqs. (13)–(15). The ratios

$$R_n^h(t) = \frac{\langle x^{n-1}\rangle^h(t)}{\langle x^{n-2}\rangle^h(t)}, \quad n > 2, \tag{19}$$

have a finite continuum limit, and there is no need to renormalize the flowed fields. Moreover, these ratios exhibit $O(a^2)$ scaling violations once the lattice theory has been nonpertubatively improved. If convenient numerically, it is also possible to use different hadrons for the calculation of the moment matrix elements in Eq. (18) because the matching coefficients are independent on the hadron for which we calculate the matrix element.

The method proposed in this paper can be also generalized to study distribution amplitudes, but care should be taken when studying the short flow time expansion due to the presence of operators with vanishing contributions for forward matrix elements [48,49].

The gradient flow has previously been employed to propose a locally smeared operator product expansion in scalar field theory [50] and to define quasi-PDF in [51], with the 1-loop matching calculated in [52]. For a more realistic application of the gradient flow in quasi-PDF calculations see Ref. [53]. While, in principle, the determination of moments of PDFs should allow a precise reconstruction of the PDFs, it would be intriguing to explore whether the determination of the PDF moments outlined in this work can be combined with quasi-PDF calculations.

Summary. We have described a method that resolves both the theoretical and numerical challenges faced when calculating moments of parton distribution functions from lattice QCD. A lattice regulator breaks the Euclidean rotational symmetry into the hypercubic group. As a result, the renormalization of twist-2 operators on the lattice is a substantial hurdle to determine moments of parton distribution functions $\langle x^{n-1}\rangle$. While for $n > 4$, power divergences in the lattice spacing are unavoidable, for $n = 3, 4$ the cancellation of power divergences is still possible by choosing suitable irreducible representations of the hypercubic group. The price to pay is that the calculation of the hadronic matrix elements necessitates the use of a spatial external momentum, worsening the signal-to-noise ratio of the correlation functions. At finite flow time t, the correlation functions of twist-2 operators are finite once the bare parameters of the theory and the flowed fermion fields are renormalized. After performing the continuum limit, the matching with the physical matrix element can be done at short flow time using the continuum symmetries of the theory. Using traceless and symmetrized operators guarantees that the matching is multiplicative, and we have calculated, at 1-loop in perturbation theory, the matching coefficients.

It would be also very interesting to investigate how the approach described in this work can complement recent efforts aimed at the direct determination of PDFs from lattice QCD correlators.

While, the method proposed in this paper is applicable, at least in principle, to calculate any moments of parton

distribution functions, in practice lattice QCD calculations of hadronic matrix elements of the flowed twist-2 operators in Eq. (10) are subject to several systematic uncertainties.

In the Supplemental Material [36] there is a discussion of the systematics, but only a thorough numerical investigation provides the final response about the range of applicability, and whether this result opens the way to determine moments of any order of the parton distribution functions.

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